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On a solvable nonlinear difference equation of higher order

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Abstract: In this paper we consider the following higher-order nonlinear difference equation

$$x_n = \alpha x_{n-k} + \frac{\delta x_{n-k} x_{n-(k+l)}}{\beta x_{n-(k+l)} + \gamma x_{n-l}},\ n \in \mathbb{N}_0,$$

where k and l are fixed natural numbers, and the parameters α, β, γ, δ and the initial values x_{-i}, $i = \overline{1, k+l}$, are real numbers such that $\beta^2 + \gamma^2 \neq 0$. We solve the above-mentioned equation in closed form and considerably extend some results in the literature. We also determine the asymptotic behavior of solutions and the forbidden set of the initial values using the obtained formulae for the case $l = 1$.

Key words: Difference equations, solution in closed form, forbidden set, asymptotic behavior

1. Introduction and preliminaries

An interesting class of nonlinear difference equations is the class of solvable difference equations, and one of the interesting problems is to find equations that belong to this class and to solve them in closed form or in explicit form. The formulae found for the solutions of these types of equations can be used easily for description of many features of the solutions of these equations. For this reason, finding a formula for solution of a nonlinear difference equation is worthwhile as well as interesting. A basic prototype for nonlinear difference equations that can be solved is the equation

$$x_n = \frac{a + b x_{n-1}}{c + d x_{n-1}},\ n \in \mathbb{N}_0, \tag{1}$$

with real initial value x_{-1}, which will be used in this study. Eq. (1) is called Riccati difference equation. If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$, then Eq. (1) is trivial such that $x_n = \frac{a}{c}$ for $n \in \mathbb{N}_0$. If $d = 0$, then Eq. (1) reduces the linear equation

$$x_n = \frac{b}{c} x_{n-1} + \frac{a}{c},\ n \in \mathbb{N}_0,$$

which is a degenerate case. If $b + c = 0$, then Eq. (1) can be written as

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$$x_n = \frac{a - cx_{n-1}}{c + dx_{n-1}},\ n \in \mathbb{N}_0,$$

whose every solution is periodic with period two such that $x_{2n-1} = x_{-1}$, $x_{2n} = \frac{a-cx_{-1}}{c+dx_{-1}}$ for $n \in \mathbb{N}_0$. If $d \neq 0 \neq (b+c)$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then, by means of the change of variables

$$x_n = \frac{b+c}{d} y_n - \frac{c}{d},$$

Eq. (1) reduces to the difference equation

$$y_n = \frac{-R + y_{n-1}}{y_{n-1}},\ n \in \mathbb{N}_0, \tag{2}$$

with one parameter, where $R = \frac{bc-ad}{(b+c)^2}$, and it is called the Riccati number. Eq. (2) can be transformed into the second-order linear equation

$$z_{n+1} = z_n - Rz_{n-1},\ n \in \mathbb{N}_0, \tag{3}$$

by means of the change of variables

$$y_n = \frac{z_{n+1}}{z_n},\ n \geq -1.$$

It is easily seen that Eq. (3) has the characteristic equation

$$\lambda^2 - \lambda + R = 0$$

with the roots $\lambda_1 = \frac{1+\sqrt{1-4R}}{2}$ and $\lambda_2 = \frac{1-\sqrt{1-4R}}{2}$. If we choose the initial values $z_{-1} = 1$ and $z_0 = y_{-1}$, then the solution of Eq. (3) is

$$z_n = \begin{cases} \frac{(\lambda_1 y_{-1} - R)\lambda_1^n - (\lambda_2 y_{-1} - R)\lambda_2^n}{\lambda_1 - \lambda_2} & \text{if } R \neq \frac{1}{4}, \\ \left(\frac{2y_{-1} + (2y_{-1}-1)n}{2}\right)\left(\frac{1}{2}\right)^n & \text{if } R = \frac{1}{4}, \end{cases} \quad n \in \mathbb{N}_0, \tag{4}$$

where $R = \frac{bc-ad}{(b+c)^2}$.

In this case, the solution of Eq. (2) is given by

$$y_n = \begin{cases} \frac{(\lambda_1 y_{-1} - R)\lambda_1^{n+1} - (\lambda_2 y_{-1} - R)\lambda_2^{n+1}}{(\lambda_1 y_{-1} - R)\lambda_1^n - (\lambda_2 y_{-1} - R)\lambda_2^n} & \text{if } R \neq \frac{1}{4}, \\ \frac{2y_{-1} + (2y_{-1}-1)(n+1)}{4y_{-1} + (4y_{-1}-2)n} & \text{if } R = \frac{1}{4}, \end{cases} \quad n \in \mathbb{N}_0. \tag{5}$$

Furthermore, the solution of Eq. (1) is given by

$$x_n = \begin{cases} \frac{b+c}{d} \frac{\left(\lambda_1 \frac{dx_{-1}+c}{b+c} - R\right)\lambda_1^{n+1} - \left(\lambda_2 \frac{dx_{-1}+c}{b+c} - R\right)\lambda_2^{n+1}}{\left(\lambda_1 \frac{dx_{-1}+c}{b+c} - R\right)\lambda_1^{n} - \left(\lambda_2 \frac{dx_{-1}+c}{b+c} - R\right)\lambda_2^{n}} - \frac{c}{d} & \text{if } R \neq \frac{1}{4}, \\ \frac{b+c}{d} \frac{2\frac{dx_{-1}+c}{b+c} + \left(2\frac{dx_{-1}+c}{b+c} - 1\right)(n+1)}{4\frac{dx_{-1}+c}{b+c} + \left(4\frac{dx_{-1}+c}{b+c} - 2\right)n} - \frac{c}{d} & \text{if } R = \frac{1}{4}, \end{cases} \quad n \in \mathbb{N}_0. \tag{6}$$

For more details, see [12].

In [7, 8], Elsayed investigated some properties of the solutions of the recursive sequences

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + dx_{n-1}},\ n \in \mathbb{N}_0, \tag{7}$$

and

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1} x_{n-3}}{cx_{n-1} + dx_{n-3}},\ n \in \mathbb{N}_0, \tag{8}$$

and also gave the form of the solution of some special cases of these equations, respectively. McGrath and Teixeira [14] considered the equation

$$x_{n+1} = \frac{ax_{n-1} + bx_n}{ax_{n-1} + bx_n} x_n,\ n \in \mathbb{N}_0, \tag{9}$$

where a, b, c, and d are real numbers, and the initial values are real numbers. The authors reduced Eq. (6) to Eq. (1) and investigated the existence and behavior of the solutions of Eq. (6) by using the known results of Eq. (1). Stević et al. [25] considered the following difference equation:

$$x_n = \frac{x_{n-k} x_{n-l}}{ax_{n-m} + bx_{n-s}},\ n \in \mathbb{N}_0, \tag{10}$$

where k, l, m, and s are fixed natural numbers, a, $b \in \mathbb{R} \setminus \{0\}$, and the initial values x_{-i}, $i = \overline{1,\tau}$, $\tau := \max\{k, l, m, s\}$ are real numbers. The authors showed that Eq. (10) is solvable in closed form, when

(i) $k = m$, $l = s$,

(ii) $k = s$, $l = m$,

(iii) $l = m = k + s$,

and presented formulae for the solutions. They also studied the long-term behavior of the solutions of the equation

$$x_n = \frac{x_{n-k} x_{n-k-s}}{ax_{n-k-s} + bx_{n-s}},\ n \in \mathbb{N}_0, \tag{11}$$

which corresponds to case (iii), by using the formulae for $s = 1$. For more works on the topic, see, for example, [1–13, 15–24, 26–38] and the references therein.

Motivated by the studies of [7, 8, 14, 25], in this paper we deal with the following higher-order nonlinear difference equation

$$x_n = \alpha x_{n-k} + \frac{\delta x_{n-k} x_{n-(k+l)}}{\beta x_{n-(k+l)} + \gamma x_{n-l}},\ n \in \mathbb{N}_0, \tag{12}$$

where k and l are fixed natural numbers, and the parameters α, β, γ, δ and the initial values x_{-i}, $i = \overline{1, k+l}$, are real numbers such that $\beta^2 + \gamma^2 \neq 0$. We solve Eq. (12) in closed form and determine the asymptotic behavior of solutions and the forbidden set of the initial values by using the obtained formulae for the case $l = 1$. Note that Eq. (12) is different from Eq. (11) with the term αx_{n-k} and is a natural extension of Eq. (11). Eq. (12) is also another natural extension of Eq. (7) to Eq. (8). Thus, we considerably extend results in the literature. If we apply the change of variables

$$y_n = \frac{R}{u_n} \tag{13}$$

to Eq. (2), then it becomes

$$u_n = \frac{R}{1-u_{n-1}},\ n \in \mathbb{N}_0, \tag{14}$$

which will be needed in the sequel. By considering (5) and (13), we get

$$u_n = \begin{cases} R\frac{(\lambda_1-u_{-1})\lambda_1^n-(\lambda_2-u_{-1})\lambda_2^n}{(\lambda_1-u_{-1})\lambda_1^{n+1}-(\lambda_2-u_{-1})\lambda_2^{n+1}} & \text{if } R \neq \frac{1}{4}, \\ R\frac{4R+(4R-2u_{-1})n}{2R+(2R-u_{-1})(n+1)} & \text{if } R = \frac{1}{4}, \end{cases} \quad n \in \mathbb{N}_0, \tag{15}$$

where $\lambda_1 = \frac{1+\sqrt{1-4R}}{2}$ and $\lambda_2 = \frac{1-\sqrt{1-4R}}{2}$.

2. Some special cases of Eq. (12)

In this section we consider some special cases of Eq. (12). Note that Eq. (12) is trivial, when $\alpha = \delta = 0$. Eq. (12) with $\beta = a$, $\gamma = b$ and $\delta = 1$ is Eq. (11), which was studied in [25], when $\alpha = 0$. Moreover, Eq. (12) is undefined, when $\beta = \gamma = 0$. Hence we consider defined ones of the remaining cases.

2.1. Case $\delta = 0$

If $\delta = 0$, then Eq. (12) reduces to the following k-order linear difference equation:

$$x_n = \alpha x_{n-k},\ n \in \mathbb{N}_0. \tag{16}$$

By writing $kn+i$ instead of n in (16), we have the equations

$$x_{kn+i} = \alpha x_{k(n-1)+i},\ n \in \mathbb{N}_0,\ i = \overline{0, k-1}, \tag{17}$$

which is a decomposition of (16). The equations in (17) have the solutions

$$x_{k(n-1)+i} = \alpha^n x_{i-k},\ n \in \mathbb{N}_0,\ i = \overline{0, k-1}, \tag{18}$$

whose composition also is the solution of (16).

2.2. Case $\gamma = 0$

If $\gamma = 0$, then Eq. (12) reduces to the following k-order linear difference equation

$$x_n = \left(\alpha + \frac{\delta}{\beta}\right) x_{n-k},\ n \in \mathbb{N}_0, \tag{19}$$

which is essentially Eq. (16). From (18), we directly have that

$$x_{k(n-1)+i} = \left(\alpha + \frac{\delta}{\beta}\right)^n x_{i-k},\ n \in \mathbb{N}_0, \tag{20}$$

where $i = \overline{0, k-1}$.

2.3. Case $\alpha = \gamma = 0$

If $\alpha = \gamma = 0$, then Eq. (12) reduces to the equation

$$x_n = \frac{\delta}{\beta} x_{n-k},\ n \in \mathbb{N}, \tag{21}$$

which is essentially Eq. (16). From (18), we directly have that

$$x_{k(n-1)+i} = \left(\frac{\delta}{\beta}\right)^n x_{i-k},\ n \in \mathbb{N}_0, \tag{22}$$

where $i = \overline{0, k-1}$.

2.4. Case $\alpha = \beta = 0$

If $\alpha = \beta = 0$, then Eq. (12) reduces to the following $(k+l)$-order nonlinear difference equation

$$x_n = \frac{\delta x_{n-k} x_{n-(k+l)}}{\gamma x_{n-l}},\ n \in \mathbb{N}_0. \tag{23}$$

If $x_{-i} \neq 0,\ i = \overline{1, k+l}$, then Eq. (23) can be written in the form of

$$\frac{x_n}{x_{n-k}} = \frac{\delta}{\gamma \frac{x_{n-l}}{x_{n-(k+l)}}},\ n \in \mathbb{N}_0. \tag{24}$$

By the change of variables

$$w_n = \frac{x_n}{x_{n-k}},\ n \geq -l, \tag{25}$$

Eq. (24) becomes

$$w_n = \frac{\delta}{\gamma} \frac{1}{w_{n-l}} = w_{n-2l} = c_j,\ n \geq l,\ j = \overline{1, 2l}, \tag{26}$$

where each c_j is a constant that is dependent on the initial values $x_{-i},\ i = \overline{1, k+l}$. From the change of variables (25), we get the equation

$$x_n = w_n x_{n-k},\ n \geq -l. \tag{27}$$

By applying the decomposition of indices $n \to kn_1 + j_1,\ j_1 = -l, -l+1, \ldots, -l+k-1,\ n_1 \in \mathbb{N}_0$ to (27), it becomes

$$x_{kn_1+j_1} = w_{kn_1+j_1} x_{k(n_1-1)+j_1},\ n_1 \in \mathbb{N}_0, \tag{28}$$

which has the solution

$$x_{k(n_1-1)+j_1} = x_{j_1-k} \prod_{s=0}^{n_1-1} w_{ks+j_1}, n_1 \in \mathbb{N}_0. \tag{29}$$

Consequently, (26) and (29) give the solution in closed form of Eq. (23).

2.5. Case $\beta = 0$

If $\beta = 0$, then Eq. (12) reduces to the following $(k+l)$-order nonlinear difference equation

$$x_n = \alpha x_{n-k} + \frac{\delta x_{n-k} x_{n-(k+l)}}{\gamma x_{n-l}}, \; n \in \mathbb{N}_0. \tag{30}$$

If $x_{-i} \neq 0$, $i = \overline{1, k+l}$, then Eq. (30) can be written in the form of

$$\frac{x_n}{x_{n-k}} = \frac{\frac{\delta}{\gamma} + \alpha \frac{x_{n-l}}{x_{n-(k+l)}}}{\frac{x_{n-l}}{x_{n-(k+l)}}}, \; n \in \mathbb{N}_0. \tag{31}$$

By applying the change of variables (25) to Eq. (31), we have

$$w_n = \frac{\frac{\delta}{\gamma} + \alpha w_{n-l}}{w_{n-l}}, \; n \geq 0. \tag{32}$$

If we apply the decomposition of indices $n \to nl + j$, $j = \overline{0, l-1}$, to (32) for l, then it becomes

$$w_{ln+j} = \frac{\frac{\delta}{\gamma} + \alpha w_{l(n-1)+j}}{w_{l(n-1)+j}}, \; n \geq 0, \tag{33}$$

which are first-order $l-$equations. Let $w_{ln+j} = \alpha y_n^{(j)}$, $n \geq -1$, $j = \overline{0, l-1}$. Then Eq. (33) can be written as the following:

$$y_n^{(j)} = \frac{\frac{\delta}{\gamma \alpha^2} + y_{n-1}^{(j)}}{y_{n-1}^{(j)}}, \; n \geq 0, \tag{34}$$

which is essentially in the form of Eq. (2). Hence, from (5), we can write the solution of (34) by taking $-R = \frac{\delta}{\gamma \alpha^2}$ as follows:

$$y_n^{(j)} = \begin{cases} \frac{(\lambda_1 y_{-l+j} - R)\lambda_1^{n+1} - (\lambda_2 y_{-l+j} - R)\lambda_2^{n+1}}{(\lambda_1 y_{-l+j} - R)\lambda_1^{n} - (\lambda_2 y_{-l+j} - R)\lambda_2^{n}} & \text{if } R \neq \frac{1}{4}, \\ \frac{2y_{-l+j} + (2y_{-l+j} - 1)(n+1)}{4y_{-l+j} + (4y_{-l+j} - 2)n} & \text{if } R = \frac{1}{4}, \end{cases} \; n \geq -1, \; j = \overline{0, l-1}, \tag{35}$$

where $\lambda_1 = \frac{1 + \sqrt{1 + 4\delta/\gamma\alpha^2}}{2}$ and $\lambda_2 = \frac{1 - \sqrt{1 + 4\delta/\gamma\alpha^2}}{2}$. Moreover, we have

$$w_{ln+j} = \begin{cases} \alpha \frac{\left(\lambda_1 \frac{w_{-l+j}}{\alpha} - R\right)\lambda_1^{n+1} - \left(\lambda_2 \frac{w_{-l+j}}{\alpha} - R\right)\lambda_2^{n+1}}{\left(\lambda_1 \frac{w_{-l+j}}{\alpha} - R\right)\lambda_1^{n} - \left(\lambda_2 \frac{w_{-l+j}}{\alpha} - R\right)\lambda_2^{n}} & \text{if } R \neq \frac{1}{4}, \\ \alpha \frac{2\frac{w_{-l+j}}{\alpha} + \left(2\frac{w_{-l+j}}{\alpha} - 1\right)(n+1)}{4\frac{w_{-l+j}}{\alpha} + \left(4\frac{w_{-l+j}}{\alpha} - 2\right)n} & \text{if } R = \frac{1}{4}, \end{cases} \; n \geq -1, \; j = \overline{0, l-1}, \tag{36}$$

Consequently, the solution in closed form of Eq. (30) follows from (29) and (36).

2.6. Case $\alpha\beta\gamma\delta \neq 0$

Here we deal with the case when $\alpha\beta\gamma\delta \neq 0$. Since in this case Eq. (12) can be written in the form of

$$x_n = \alpha x_{n-k} + \frac{x_{n-k} x_{n-(k+l)}}{b x_{n-(k+l)} + c x_{n-l}}, \; n \in \mathbb{N}_0,$$

with $b = \frac{\beta}{\delta}$ and $c = \frac{\gamma}{\delta}$, we may assume that $\delta = 1$. Hence we will consider the equation

$$x_n = \alpha x_{n-k} + \frac{x_{n-k}x_{n-(k+l)}}{\beta x_{n-(k+l)} + \gamma x_{n-l}},\ n \in \mathbb{N}_0, \tag{37}$$

from now on. Moreover, Eq. (37) can be written in the form of

$$\frac{x_n}{x_{n-k}} = \frac{(\alpha\beta + 1) + \alpha\gamma \frac{x_{n-l}}{x_{n-(k+l)}}}{\beta + \gamma \frac{x_{n-l}}{x_{n-(k+l)}}},\ n \in \mathbb{N}_0. \tag{38}$$

Remark 1 *For $\alpha\beta\gamma \neq 0$ in Eq. (12), it is easy to see that there is the degenerate case $\begin{vmatrix} \alpha\beta + \delta & \alpha\gamma \\ \beta & \gamma \end{vmatrix} = 0$ if and only if $\delta = 0$. Hence, we avoid the degenerate case via the assumption $\delta \neq 0$.*

2.6.1. The case $\alpha\gamma + \beta = 0$

If $\alpha\gamma + \beta = 0$, then we get the equation

$$\frac{x_n}{x_{n-k}} = \frac{\left(1 - \frac{\beta^2}{\gamma}\right) - \beta \frac{x_{n-l}}{x_{n-(k+l)}}}{\beta + \gamma \frac{x_{n-l}}{x_{n-(k+l)}}},\ n \in \mathbb{N}_0 \tag{39}$$

from (38). By using the change of variables (25), we get the equation

$$w_n = \frac{\left(1 - \frac{\beta^2}{\gamma}\right) - \beta w_{n-l}}{\beta + \gamma w_{n-l}},\ n \in \mathbb{N}_0,$$

which can be written as

$$\beta + \gamma w_n = \frac{\gamma}{\beta + \gamma w_{n-l}},\ n \in \mathbb{N}_0. \tag{40}$$

Next, by applying the substitution $\beta + \gamma w_n = t_n$ to Eq. (40), we obtain

$$t_n = \frac{\gamma}{t_{n-l}} = t_{n-2l} = c_j,\ n \geq l,\ j = \overline{1, 2l}, \tag{41}$$

where each c_j is a constant that is dependent on the initial values x_{-i}, $i = \overline{1, k+l}$. Consequently, by using $\beta + \gamma w_n = t_n$ and considering (28), we get

$$x_{kn_1+j_1} = \frac{t_{kn_1+j_1} - \beta}{\gamma} x_{k(n_1-1)+j_1},\ n_1 \in \mathbb{N}_0, \tag{42}$$

which has the solution

$$x_{k(n_1-1)+j_1} = x_{j_1-k} \prod_{s=0}^{n_1-1} \frac{t_{ks+j_1} - \beta}{\gamma}, \tag{43}$$

where $n_1 \in \mathbb{N}_0$, $j_1 = -l, -l+1, \ldots, -l+k-1$. Consequently, (43) gives the solution in closed form of Eq. (39).

2.6.2. The case $\alpha\gamma+\beta\neq 0$

If $\alpha\gamma+\beta\neq 0$, then, by using the change of variables

$$\frac{x_n}{x_{n-k}}=\frac{\alpha\gamma+\beta}{\gamma}w_n-\frac{\beta}{\gamma},\ n\geq -l, \tag{44}$$

Eq. (38) is transformed into the following equation:

$$w_n=\frac{-\widetilde{R}+w_{n-l}}{w_{n-l}},\ n\in\mathbb{N}_0, \tag{45}$$

where $-\widetilde{R}=\frac{\gamma}{(\alpha\gamma+\beta)^2}$. If we apply the decomposition of indices $n\to nl+j$, $j=\overline{0,l-1}$ to (45), then it becomes

$$w_{ln+j}=\frac{-\widetilde{R}+w_{l(n-1)+j}}{w_{l(n-1)+j}},\ n\in\mathbb{N}_0,\ j=\overline{0,l-1}. \tag{46}$$

which are first-order l-equations. Let $w_{ln+j}=w_n^{(j)}$, $j=\overline{0,l-1}$. Then Eq. (46) can be written as the following:

$$w_n^{(j)}=\frac{\frac{\gamma}{(\alpha\gamma+\beta)^2}+w_{n-1}^{(j)}}{w_{n-1}^{(j)}},\ n\geq 0, \tag{47}$$

which is essentially in the form of Eq. (2). Hence, from (5), we can write the solution of (46) by taking $-\widetilde{R}=\frac{\gamma}{(\alpha\gamma+\beta)^2}$ as follows:

$$w_{ln+j}=w_n^{(j)}=\begin{cases}\frac{(\lambda_1 w_{-l+j}-\widetilde{R})\lambda_1^{n+1}-(\lambda_2 w_{-l+j}-\widetilde{R})\lambda_2^{n+1}}{(\lambda_1 w_{-l+j}-\widetilde{R})\lambda_1^{n}-(\lambda_2 w_{-l+j}-\widetilde{R})\lambda_2^{n}} & \text{if } \widetilde{R}\neq\frac{1}{4},\\ \frac{2w_{-l+j}+(2w_{-l+j}-1)(n+1)}{4w_{-l+j}+(4w_{-l+j}-2)n} & \text{if } \widetilde{R}=\frac{1}{4},\end{cases}\quad n\geq -1,\ j=\overline{0,l-1}, \tag{48}$$

where $\lambda_1=\frac{1+\sqrt{1-4\widetilde{R}}}{2}$ and $\lambda_2=\frac{1-\sqrt{1-4\widetilde{R}}}{2}$. On the other hand, from (44), we get

$$x_{k(n_1-1)+j_1}=x_{j_1-k}\prod_{s=0}^{n_1-1}\left(\frac{\alpha\gamma+\beta}{\gamma}w_{ks+j_1}-\frac{\beta}{\gamma}\right), \tag{49}$$

where $n_1\in\mathbb{N}_0$, $j_1=-l,-l+1,\dots,-l+k-1$. Consequently, the solution in closed form of Eq. (37) follows from (48) and (49).

3. A study of case $\alpha\beta\gamma\neq 0$ when $l=1$

In this section, we determine the forbidden set of the initial values and the asymptotic behavior of the solutions of Eq. (37) when $l=1$ and $\alpha\beta\gamma\neq 0$. In this case, Eq. (37) becomes

$$x_n=\alpha x_{n-k}+\frac{x_{n-k}x_{n-k-1}}{\beta x_{n-k-1}+\gamma x_{n-1}},\ n\in\mathbb{N}_0, \tag{50}$$

where the initial values x_{-i}, $i=\overline{1,k+1}$, are real numbers. The solution of Eq. (50) is given by

$$x_{k(n-1)+j_1}=x_{j_1-k}\prod_{s=0}^{n-1}\left(\frac{\alpha\gamma+\beta}{\gamma}\frac{\left(\lambda_1\frac{1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\right)\lambda_1^{ks+j_1+1}-\left(\lambda_2\frac{1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\right)\lambda_2^{ks+j_1+1}}{\left(\lambda_1\frac{1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\right)\lambda_1^{ks+j_1}-\left(\lambda_2\frac{1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\right)\lambda_2^{ks+j_1}}-\frac{\beta}{\gamma}\right), \tag{51}$$

where for each $j \in \{-1,0,\ldots,k-2\}$, every $n \in \mathbb{N}_0$ and $\widetilde{R} = \frac{\gamma}{(\alpha\gamma+\beta)^2}$, if $\widetilde{R} \neq \frac{1}{4}$, and

$$x_{k(n-1)+j_1} = x_{j_1-k} \prod_{s=0}^{n-1} \left(\frac{\alpha\gamma+\beta}{\gamma} \frac{\frac{2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right) + \left(\frac{2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-1\right)(ks+j_1+1)}{\frac{4}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right) + \left(\frac{4}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-2\right)(ks+j_1)} - \frac{\beta}{\gamma} \right), \quad (52)$$

where for each $j \in \{-1,0,\ldots,k-2\}$, every $n \in \mathbb{N}_0$ and $\widetilde{R} = \frac{\gamma}{(\alpha\gamma+\beta)^2}$, if $\widetilde{R} = \frac{1}{4}$. In this section, we will also consider the equation

$$w_n = \frac{-\widetilde{R} + w_{n-1}}{w_{n-1}}, \; n \in \mathbb{N}_0, \quad (53)$$

which is obtained from Eq. (46) by taking $l = 1$.

Theorem 2 *The forbidden set of Eq. (53) is the set*

$$F = \left\{ \overrightarrow{X} : x_{-j} = 0, \; j = \overline{1, k+1}, \right\} \bigcup \bigcup_{j=0}^{k} \left\{ \bigcup_{n\in\mathbb{N}_0} \left\{ \overrightarrow{X} : \frac{x_{j-1}}{x_{j-1-k}} = \frac{1}{\alpha\gamma+\beta}\widetilde{s}_n - \frac{\beta}{\gamma} \right\} \right\}, \quad (54)$$

where $\overrightarrow{X} = (x_{-k-1}, x_{-k}, \cdots, x_{-1})$ *and*

$$\widetilde{s}_n = \begin{cases} \widetilde{R}\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1^{n+2}-\lambda_2^{n+2}} & \text{if } \widetilde{R} \neq \frac{1}{4}, \\ \frac{(n+1)}{2(n+2)} & \text{if } \widetilde{R} = \frac{1}{4}, \end{cases}.$$

Proof To prove, we will use Eq. (53) along with Eq. (50). If $x_{-j} = 0$ for some j, $j = \overline{1, k+1}$, then x_n cannot be calculated after a term x_{n_0}, $n_0 \in \mathbb{N}_0$. For example, if $x_{-k} = 0$, then $x_0 = 0$, and so x_1 cannot be calculated. For the other $j = \overline{1, k-1}$, the case is the same. If $x_{-j} \neq 0$, $j = \overline{1, k+1}$, then we assume that, by using Eq. (53), $w_{-1} \neq 0$ but $w_{n_0} = 0$ for $n_0 \in \mathbb{N}_0$. That is, the points $w_0, w_1, \cdots, w_{n_0} = 0$, $n_0 \in \mathbb{N}_0$, can be calculated, and so w_{n_0+1} cannot be calculated. Note that this case is equivalent to the case when $\beta x_{n_0+1-k} + \gamma x_{n_0+1} = 0$, $n_0 \in \mathbb{N}_0$, which can be verified from (44). Now we consider the following equation:

$$u_n = f^{-1}(u_{n-1}), \; f(w) = \frac{-\widetilde{R} + w}{w}, \; u_{-1} = 0, \; n \in \mathbb{N}_0, \quad (55)$$

where f is the function associated with Eq. (53) and f^{-1} is the inverse of f. Now note that difference equation associated with the inverse function f^{-1} is Eq. (14) with $R = \widetilde{R}$. Thus, by applying (15) to (55), when $R = \widetilde{R}$, we get

$$w_{-1} = f^{-n_0-1}(0) = \begin{cases} \widetilde{R}\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1^{n+2}-\lambda_2^{n+2}} & \text{if } \widetilde{R} \neq \frac{1}{4}, \\ \frac{(n+1)}{2(n+2)} & \text{if } \widetilde{R} = \frac{1}{4}, \end{cases} \; n \in \mathbb{N}_0,$$

which implies

$$\frac{x_{-1}}{x_{-1-k}} = \begin{cases} \frac{1}{\alpha\gamma+\beta}\widetilde{R}\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1^{n+2}-\lambda_2^{n+2}} - \frac{\beta}{\gamma} & \text{if } \widetilde{R} \neq \frac{1}{4}, \\ \frac{1}{\alpha\gamma+\beta}\frac{(n+1)}{2(n+2)} - \frac{\beta}{\gamma} & \text{if } \widetilde{R} = \frac{1}{4}, \end{cases} \; n \in \mathbb{N}_0.$$

Hence the forbidden set of Eq. (50) is the same as (54). $\square$

Theorem 3 *The following statements are true:*

(i) If $(1-\alpha)(\beta+\gamma)=1$, *then Eq. (50) has a* $k-$*periodic solution,*

(ii) If $(1+\alpha)(\gamma-\beta)=1$, *then Eq. (50) has a* $2k-$*periodic solution.*

Proof (i)–(ii) Note that Eq. (50) can be written as

$$x_n = x_{n-k} g\left(\frac{x_{n-1}}{x_{n-k-1}}\right), \; n\in\mathbb{N}_0 \tag{56}$$

such that

$$g(u) = \alpha + \frac{1}{\beta+\gamma u}. \tag{57}$$

It is easy to see that Eq. (56) has a $k-$periodic solution if and only if $g(1)=1$. Thus, from (57), we have

$$g(1) = \alpha + \frac{1}{\beta+\gamma} = 1$$

which implies the equality $(1-\alpha)(\beta+\gamma)=1$. Similarly, Eq. (56) has a $2k-$periodic solution if and only if $g(-1)=-1$. Hence, from (57), we have

$$g(-1) = \alpha + \frac{1}{\beta-\gamma} = -1,$$

which implies the equality $(1+\alpha)(\gamma-\beta)=1$. □

Theorem 4 *Suppose that* $\alpha\beta\gamma\neq 0$, $\widetilde{R} = -\frac{\gamma}{(\alpha\gamma+\beta)^2} < \frac{1}{4}$ *and* $x_{-i}\neq 0$, $i=\overline{1,k+1}$. *Then the following statements hold.*

(a) If $|\frac{\alpha\gamma+\beta}{\gamma}\lambda_1 - \frac{\beta}{\gamma}| < 1$ *and* $\frac{\lambda_i}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, *for* $i=1,2$, *then* $x_n\to 0$, *as* $n\to\infty$.

(b) If $|\frac{\alpha\gamma+\beta}{\gamma}\lambda_1 - \frac{\beta}{\gamma}| < 1$, $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$ *and* $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}= 0$, *then* $x_n\to 0$, *as* $n\to\infty$.

(c) If $|\frac{\alpha\gamma+\beta}{\gamma}\lambda_2 - \frac{\beta}{\gamma}| < 1$, $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}= 0$ *and* $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, *then* $x_n\to 0$, *as* $n\to\infty$.

(d) If $|\frac{\alpha\gamma+\beta}{\gamma}\lambda_1 - \frac{\beta}{\gamma}| > 1$ *and* $\frac{\lambda_i}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, *for* $i=1,2$, *then* $|x_n|\to \infty$, *as* $n\to\infty$.

(e) If $|\frac{\alpha\gamma+\beta}{\gamma}\lambda_1 - \frac{\beta}{\gamma}| > 1$, $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$ *and* $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}= 0$, *then* $|x_n|\to \infty$, *as* $n\to\infty$.

(f) If $|\frac{\alpha\gamma+\beta}{\gamma}\lambda_2 - \frac{\beta}{\gamma}| > 1$, $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}= 0$ *and* $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, *then* $|x_n|\to \infty$, *as* $n\to\infty$.

(g) If $\frac{\alpha\gamma+\beta}{\gamma}\lambda_1 - \frac{\beta}{\gamma} = 1$ *and* $\frac{\lambda_i}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, *for* $i=1,2$, *then* x_n *converges to (not necessarily prime)* k*-periodic solution of Eq. (50), as* $n\to\infty$.

(h) If $\frac{\alpha\gamma+\beta}{\gamma}\lambda_1 - \frac{\beta}{\gamma} = 1$, $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$ *and* $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}= 0$, *then* x_n *converges to (not necessarily prime)* k*-periodic solution of Eq. (50), as* $n\to\infty$.

(i) If $\frac{\alpha\gamma+\beta}{\gamma}\lambda_2-\frac{\beta}{\gamma}=1$, $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}=0$ *and* $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, *then* x_n *converges to (not necessarily prime)* k*-periodic solution of Eq. (50), as* $n\to\infty$.

(j) If $\frac{\alpha\gamma+\beta}{\gamma}\lambda_1-\frac{\beta}{\gamma}=-1$ *and* $\frac{\lambda_i}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, *for* $i=1,2$, *then* x_n *converges to (not necessarily prime)* $2k$*-periodic solution of Eq. (50), as* $n\to\infty$.

(k) If $\frac{\alpha\gamma+\beta}{\gamma}\lambda_1-\frac{\beta}{\gamma}=-1$, $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$ *and* $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}=0$ *for* $i=1,2$, *then* x_n *converges to (not necessarily prime)* $2k$*-periodic solution of Eq. (50), as* $n\to\infty$.

(l) If $\frac{\alpha\gamma+\beta}{\gamma}\lambda_2-\frac{\beta}{\gamma}=-1$, $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}=0$ *and* $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, *then* x_n *converges to (not necessarily prime)* $2k$*-periodic solution of Eq.(50), as* $n\to\infty$.

Proof Since $\widetilde{R}=-\frac{\gamma}{(\alpha\gamma+\beta)^2}<\frac{1}{4}$, we have $\lambda_1=\frac{1+\sqrt{1-4\widetilde{R}}}{2}$, $\lambda_2=\frac{1-\sqrt{1-4\widetilde{R}}}{2}\in\mathbb{R}$, $|\lambda_1|>|\lambda_2|$. Let

$$a_s^{(j_1)}:=-\frac{\beta}{\gamma}+\frac{\alpha\gamma+\beta}{\gamma}\frac{\left(\lambda_1\frac{1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\right)\lambda_1^{ks+j_1+1}-\left(\lambda_2\frac{1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\right)\lambda_2^{ks+j_1+1}}{\left(\lambda_1\frac{1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\right)\lambda_1^{ks+j_1}-\left(\lambda_2\frac{1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\right)\lambda_2^{ks+j_1}}\tag{58}$$

for $s\in\mathbb{N}_0$ and $j_1=-1,0,\dots,k-2$. When $\frac{\lambda_i}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, for $i=1,2$, we get

$$\lim_{s\to\infty}a_s^{(j_1)}=\frac{-\beta}{\gamma}+\frac{\alpha\gamma+\beta}{\gamma}\lambda_1\tag{59}$$

for each $j_1\in\{-1,0,\dots,k-2\}$. From (51) and (59), the results follow from the assumptions in (a) and (d). If $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$ and $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}=0$, then we can write

$$a_s^{(j_1)}=\frac{-\beta}{\gamma}+\frac{\alpha\gamma+\beta}{\gamma}\lambda_1,\tag{60}$$

for each $j_1\in\{-1,0,\dots,k-2\}$. From (51) and (60), the results in (b) and (e) can be seen easily. When $\frac{\lambda_1}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}=0$ and $\frac{\lambda_2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-\widetilde{R}\neq 0$, directly we get

$$a_s^{(j_1)}=\frac{-\beta}{\gamma}+\frac{\alpha\gamma+\beta}{\gamma}\lambda_2,\tag{61}$$

for each $j_1\in\{-1,0,\dots,k-2\}$. From (51) and (61), the results in (c) and (f) can be seen easily. For each $j_1\in\{-1,0,\dots,k-2\}$ and sufficiently large s, we can write

$$a_s^{(j_1)}=\frac{-\beta}{\gamma}+\frac{\alpha\gamma+\beta}{\gamma}\lambda_1+\mathcal{O}\left(\left(\frac{|\lambda_2|}{|\lambda_1|}\right)^{ks}\right).\tag{62}$$

From (51), (62), and Theorem 3, the results in (g) and (j) can be seen easily. We easily obtain the statements in (h), (k) and (i), (l) from (51), (60) and (51), (61), respectively. $\square$

Theorem 5 *Suppose that* $\alpha\beta\gamma \neq 0$, $\widetilde{R} = -\frac{\gamma}{(\alpha\gamma+\beta)^2} = \frac{1}{4}$ *and* $x_{-i} \neq 0$, $i = \overline{1, k+1}$. *Then the following statements hold:*

(a) If $|\frac{\alpha\gamma-\beta}{2\gamma}| < 1$, *then* $x_n \to 0$, *as* $n \to \infty$.

(b) If $|\frac{\alpha\gamma-\beta}{2\gamma}| > 1$, *then* $|x_n| \to \infty$, *as* $n \to \infty$.

(c) If $|\frac{\alpha\gamma-\beta}{2\gamma}| = 1$ *and* $\frac{\alpha\gamma-\beta}{\alpha\gamma+\beta} > 0$, *then* $|x_n| \to \infty$, *as* $n \to \infty$.

Proof When $\widetilde{R} = -\frac{\gamma}{(\alpha\gamma+\beta)^2} = \frac{1}{4}$, we have $\lambda_1 = \lambda_2 = \frac{1}{2}$. Let

$$b_s^{(j_1)} := -\frac{\beta}{\gamma} + \frac{\alpha\gamma+\beta}{\gamma}\frac{\frac{2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right) + \left(\frac{2}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-1\right)(ks+j_1+1)}{\frac{4}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right) + \left(\frac{4}{\alpha\gamma+\beta}\left(\gamma\frac{x_{-1}}{x_{-k-1}}+\beta\right)-2\right)(ks+j_1)}, \tag{63}$$

for $s \in \mathbb{N}_0$ and $j_1 = -1, 0, \ldots, k-2$. If $\frac{x_{-1}}{x_{-1-k}} \neq \frac{\alpha\gamma-\beta}{2\gamma}$, then we get

$$\lim_{s\to\infty} b_s^{(j_1)} = \frac{\alpha\gamma-\beta}{2\gamma}, \tag{64}$$

for each $j_1 \in \{-1, 0, \ldots, k-2\}$. Otherwise, when $\frac{x_{-1}}{x_{-1-k}} = \frac{\alpha\gamma-\beta}{2\gamma}$, directly we have

$$b_s^{(j_1)} = \frac{\alpha\gamma-\beta}{2\gamma} \tag{65}$$

for each $j_1 \in \{-1, 0, \ldots, k-2\}$ and $s \in \mathbb{N}_0$. From (52), (64), and (65), the results follow from the assumptions in (a) and (b).

Now we consider the last case (c). For each $j_1 \in \{-1, 0, \ldots, k-2\}$ and sufficiently large s, we obtain

$$\begin{aligned} b_s^{(j_1)} &= \frac{\alpha\gamma-\beta}{2\gamma} + \frac{\alpha\gamma-\beta}{2\gamma\frac{\alpha\gamma-\beta}{\alpha\gamma+\beta}ks} + \mathcal{O}\left(\frac{1}{s^2}\right) \\ &= \pm\left(1 + \frac{1}{\frac{\alpha\gamma-\beta}{\alpha\gamma+\beta}ks} + \mathcal{O}\left(\frac{1}{s^2}\right)\right) \\ &= \pm\exp^{\left(\frac{1}{\frac{\alpha\gamma-\beta}{\alpha\gamma+\beta}ks} + \mathcal{O}\left(\frac{1}{s^2}\right)\right)}. \end{aligned} \tag{66}$$

From (52) and (66) and by using the fact that $\Sigma_{i=1}^{s}(1/i) \to \infty$ as $s \to \infty$, then we easily have $|x_n| \to \infty$, as $n \to \infty$. □

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