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On the J-reflexive operators

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Abstract: A bounded linear operator T on a Banach space X is J-reflexive if every bounded operator on X that leaves invariant the sets $J(T,x)$ for all x is contained in the closure of $orb(T)$ in the strong operator topology. We discuss some properties of J-reflexive operators. We also give and prove some necessary and sufficient conditions under which an operator is J-reflexive. We show that isomorphisms preserve J-reflexivity and some examples are considered. Finally, we extend the J-reflexive property in terms of subsets.

Key words: Orbit of an operator, J-sets, J^{mix}-sets, J-class operator, reflexive operator

1. Introduction

One of the most challenging problems in operator theory is the "invariant subspace problem", which asks whether every operator on a Hilbert space (more generally, a Banach space) admits a nontrivial invariant subspace. Here, "operator" means "continuous linear transformation" and "invariant subspace" means "closed linear manifold such that the operator maps it to itself". A subspace is nontrivial if it is neither the zero subspace nor the whole space. An example constructed by Enflo [7] shows that for some Banach spaces there do exist operators with only trivial invariant subspaces. For a Hilbert space, however, the invariant subspace problem remains open. There is a deep connection between invariant subspaces of an operator and its reflexivity. Reflexive operators are those that can be identified by their nontrivial invariant subspaces and they have been studied for a few decades. For a good source on reflexivity, see [12] by Halmos. An operator T is reflexive if any operator that leaves invariant, T-invariant subspaces belongs to the closure of $\{P(T) : P \text{ is a polynomial}\}$ in the weak operator topology [12, 17]. In [11], the authors introduced orbit-reflexivity: an operator T on a Hilbert space is orbit-reflexive if the only operators that leave invariant every norm closed T-invariant subset are contained in the closure of $orb(T)$ in the strong operator topology. For example, compact operators, normal operators, contractions, and weighted shifts on the Hilbert spaces are orbit-reflexive [11]. Hadwin et al. in [10] also introduced and studied the notion of null-orbit reflexivity, which is a slight perturbation of the notion of orbit-reflexivity. The class of null-orbit reflexive operators includes the classes of hyponormal, algebraic, compact, strictly block-upper (lower) triangular operators, and operators whose spectral radius is not 1. They also proved that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit reflexive. For further references on these topics, see [8–11, 13, 14].

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In this paper, our purpose is to characterize operators on Banach spaces that can be identified by their J-sets. First, we give some preliminaries that we need to give our results.

Let X be a complex Banach space and denote by $B(X)$ the space of all bounded linear operators on X. For $T \in B(X)$, the set $orb(T) = \{T^n : n \geq 0\}$ is called the orbit of T.

If X is a separable Banach space and $orb(T, x) = \{T^n x : n \geq 0\}$ is dense in X for some x, then T is called a hypercyclic operator and we call x a hypercyclic vector.

The J-set of T under x, $J(T, x)$, is defined by:

$$J(T, x) = \{y : \text{ there exists a strictly increasing sequence of positive integers}$$

$$(k_n) \text{ and a sequence } (x_n) \subset X \text{ such that } x_n \to x,\ T^{k_n} x_n \to y\}.$$

The set

$$J^{mix}(T, x) = \{y : \text{there exists a sequence } (x_n) \subset X \text{ such that } x_n \to x \text{ and}$$

$$T^n x_n \to y\}$$

is called the J^{mix}-set of T under x. The sets $J(T, x)$ and $J^{mix}(T, x)$ are closed T-invariant subsets of X. If T is invertible, then all J-sets are T^{-1}-invariant. For more details, see [6].

Recall that we say $T \in B(X)$ is power bounded with a power bound $M > 0$ whenever $\|T^n\| \leq M$ for all positive integers n. In [6, Proposition 2.10] it was shown that if T is power bounded we have the following:

$$J(T, x) = \{y : \text{there exists a strictly increasing sequence of positive integers}$$

$$(k_n) \text{ such that } T^{k_n} x \to y\}.$$

An operator $T \in B(X)$ is called a J-class (J^{mix}-class) operator if $J(T, x) = X$ ($J^{mix}(T, x) = X$) for some $x \in X \backslash \{0\}$. The set of all $x \in X$ satisfying $J(T, x) = X$ ($J^{mix}(T, x) = X$) is denoted by A_T (A_T^{mix}) and its elements are called the J-vectors (J^{mix}-vectors) for T. It is well known that A_T is a closed subset of X and A_T^{mix} is a closed subspace of X. We know that $T \in B(X)$ is hypercyclic if and only if $A_T = X$. For a good source on these topics we refer the reader to [1, 3–6, 15, 16, 18].

Recall that $x \in X \backslash \{0\}$ is called a periodic point for $T \in B(X)$ if $T^n x = x$ for some positive integer n. The smallest such number n is called the period of x.

2. Some properties of J-sets

Here we state and prove some properties of J-sets that will be used in the proof of our main results. The proof of the following lemma is straightforward and so we omit it.

Lemma 2.1 *Suppose that $T \in B(X)$. Then for all $x, y \in X$ we have:*
a) $J(T, x) \subseteq J(T, Tx) \subseteq J(T, T^2 x) \subseteq \cdots$,
b) $J^{mix}(T, x) \subseteq J^{mix}(T, Tx) \subseteq J^{mix}(T, T^2 x) \subseteq \cdots$,
and equality holds if T has a bounded inverse.
c) $J(T^m, x) \subseteq J(T, x)$ for all $m \in \mathbb{N}$.

Lemma 2.2 *Suppose that $T \in B(X)$. We have:*
(a) If $(T^n)_n$ is a Cauchy sequence and it has a subsequence that converges to a bounded operator U in the

strong operator topology, then for every x, *both sets* $J(T,x)$ *and* $J^{mix}(T,x)$ *are equal to the singleton* $\{Ux\}$.
(b) If $(T^n)_n$ *converges to an operator* U *in the strong operator topology, then*

$$2J(T,x_1+x_2)-U(x_1+x_2)\subseteq J(T,x_1)+J(T,x_2)$$

for all $x_1,x_2\in X$. *The same result holds for* J^{mix}*-sets.*

Proof a) Note that T is power bounded. Since for every $x\in X$, $(T^nx)_n$ is a Cauchy sequence that has a subsequence converging to Ux, $T^nx\to Ux$. Hence, $J^{mix}(T,x)=J(T,x)=\{Ux\}$.
b) Suppose that $y\in J(T,x_1+x_2)$. Then there exists a sequence $(w_n)_n$ in X and a strictly increasing sequence of positive integers $(k_n)_n$ such that $w_n\to x_1+x_2$ and $T^{k_n}w_n\to y$. Now we have

$$w_n-x_1\to x_2,\ \ T^{k_n}(w_n-x_1)\to y-Ux_1$$

and

$$w_n-x_2\to x_1,\ \ T^{k_n}(w_n-x_2)\to y-Ux_2.$$

Hence, we get

$$y-Ux_1\in J(T,x_2),\ \ y-Ux_2\in J(T,x_1),$$

and so

$$2y-U(x_1+x_2)\in J(T,x_1)+J(T,x_2),$$

which gives the result. Putting $k_n=n$, by the same method we conclude (b) for J^{mix}-sets. □

Recall that for $T\in B(X)$, by $\{T\}'$ we mean the collection of all bounded operators on X, which commutes with T.

Lemma 2.3 *Suppose that* $T\in B(X)$ *and* $U\in\{T\}'$. *For all positive integers* m *and* $x\in X$, *we have:*
a) $U^m(J(T,x))\subseteq J(T,U^mx)$,
b) $U^m(J^{mix}(T,x))\subseteq J^{mix}(T,U^mx)$,
and equality holds if U *has a bounded inverse. Moreover, if* U *is surjective, then* A_T *is a* U*-invariant subset and* A_T^{mix} *is a* U*-invariant subspace of* X.

Proof See Lemma 2.6 in [2] for parts (a) and (b). If U has a bounded inverse, then for $z\in J(T,U^mx)$, there exists a strictly increasing sequence of positive integers $(k_n)_n$ and a sequence $(u_n)_n$ in X such that $u_n\to U^mx$ and $T^{k_n}u_n\to z$. We therefore get

$$U^{-m}u_n\to x,\ \ T^{k_n}(U^{-m}u_n)\to U^{-m}z.$$

This yields that $U^{-m}z\in J(T,x)$ and hence $z\in U^m(J(T,x))$. Putting $k_n=n$, we can prove it for J^{mix}-sets. Now if $x\in A_T$, then $J(T,x)=X$, and so we have

$$X=U(X)=U(J(T,x))\subseteq J(T,Ux)\subseteq X.$$

Hence, $U(A_T)\subseteq A_T$. Similarly, we can prove that A_T^{mix} is a U-invariant subspace. □

A simple consequence of Lemma 2.11 in [6] yields that if x is a J-vector (J^{mix}-vector) for T, then for any nonzero scalar λ, λx is a J-vector (J^{mix}-vector) for T. If A_T (A_T^{mix}) is nonempty, then it is an infinite set.

3. On the property of J-reflexive operators

In this section, we first define the J-reflexive property, and then some properties of J-reflexive operators are investigated. We also state and prove necessary and sufficient conditions for an operator to be J-reflexive. Finally, we express J-reflexivity in terms of subsets. From now on, for simplicity, we denote the closure of a subset A in the strong operator topology by $\overline{A}^{SOT}$.

Definition 3.1 *We call $T \in B(X)$ a J-reflexive operator if every bounded operator that leaves invariant $J(T,x)$ (for all x in X) is contained in the $\overline{orb(T)}^{SOT}$. J^{mix}-reflexivity can also be defined in a similar way.*

It is clear that every J-reflexive operator on a Hilbert space is orbit-reflexive, but there exists an orbit-reflexive operator that is not J-reflexive. See the following example.

Example 3.2 *Suppose that U is a bilateral shift on $l^2(\mathbb{Z})$, i.e. $Uf(n) = f(n-1)$ where $f \in l^2(\mathbb{Z})$. Then U is orbit-reflexive, since U is a normal operator [11]. We know that $U^{-1}f(n) = f(n+1)$, so U^{-1} does not belong to $\overline{orb(U)}^{SOT}$. By Proposition 2.8 in [6], U^{-1} leaves invariant J-sets of U, so U is not J-reflexive.*

It is easy to see that if $T \in B(X)$, and for all $x \in X$, $J(T,x)$ is an empty set or the whole space X or the singleton $\{0\}$, then T is not J-reflexive. For example, hypercyclic operators are not J-reflexive, since their J-sets are the whole space. However, the identity operator I is the only bounded operator for which $J(I,x) = \{x\}$ for all x. Obviously, I is J-reflexive.

Recall that if T is an operator, then the set $\{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}$ is called the spectrum of T and it is denoted by $\sigma(T)$. Now we investigate some properties of J-reflexive operators.

Theorem 3.3 *Let $T \in B(X)$ be a J-reflexive operator. Then we have the following:*
(a) If T is invertible, then $(T^n)_n$ has a subsequence that converges to I in the strong operator topology.
(b) If T is not identity and $(T^n)_n$ is a Cauchy sequence, then $0 \in \sigma(T)$.
(c) If T is not identity and is invertible, then $(T^n)_n$ diverges.

Proof a) Since by Proposition 2.8 in [6], T^{-1} leaves invariant $J(T,x)$ for all x, by the hypothesis there exists a sequence $(n_k)_k$ of positive integers such that $T^{-1}x = lim_k T^{n_k}x$ for all x. Thus, $T^{n_k+1} \to I$ in the strong operator topology.
b) On the contrary, suppose that $0 \notin \sigma(T)$, so T is invertible, and by part (a), there exists a sequence of positive integers $(n_k)_k$ such that $T^{n_k} \to I$ in the strong operator topology. Since B(X) is a Banach space and $(T^n)_n$ is a Cauchy sequence, it follows that $T^n \to I$ in the strong operator topology. Similarly, since $T^{n_k+1} \to T$ in the strong operator topology, we have $T^n \to T$ in the strong operator topology and so $T = I$, which is a contradiction.
c) If $(T^n)_n$ converges, then by part (b), $0 \in \sigma(T)$, which is a contradiction. □

Lemma 3.4 *If $T \in B(X)$ is a J-reflexive operator, then $\|T\| \geq 1$.*

Proof Suppose that $\|T\| < 1$. Then, since $\|T^n x\| \leq \|T\|^n \|x\|$, we have

$$J^{mix}(T,x) = J(T,x) = \{0\}$$

for all x. Hence, T can not be J-reflexive, which is a contradiction. □

Isomorphisms preserve the J-reflexive property, as follows.

Theorem 3.5 *Suppose that X and Y are Banach spaces and $S : X \to Y$ is an isomorphism. If T is a J-reflexive operator on X, then STS^{-1} is also a J-reflexive operator on Y.*

Proof First, we show that $J(STS^{-1}, y) = S(J(T, S^{-1}y))$. Note that since S is an isomorphism, S and S^{-1} are bounded. If $z \in J(STS^{-1}, y)$, then there exist $(y_n)_n$ in Y and a strictly increasing sequence of positive integers $(k_n)_n$ such that $y_n \to y$ and $\lim_n ST^{k_n}S^{-1}y_n = z$. Since $S^{-1}y_n \to S^{-1}y$, we get $S^{-1}z \in J(T, S^{-1}y)$ and so $z \in S(J(T, S^{-1}y))$. Thus,

$$J(STS^{-1}, y) \subseteq S(J(T, S^{-1}y)).$$

Conversely, if $z \in S(J(T, S^{-1}y))$, then there exist $x \in X$, $(u_n)_n \subseteq X$, and a strictly increasing sequence of positive integers $(k_n)_n$ such that

$$z = Sx,\ u_n \to S^{-1}y \text{ and } \lim_n T^{k_n}u_n = x.$$

Putting $v_n = Su_n$, then $v_n \to y$ and we have

$$z = Sx = \lim_n ST^{k_n}S^{-1}v_n.$$

Hence, $z \in J(STS^{-1}, y)$. Now if $W \in B(Y)$ leaves invariant the sets $J(STS^{-1}, y)$, then for all $y \in Y$, we have

$$WS(J(T, S^{-1}y)) \subseteq S(J(T, S^{-1}y)).$$

Therefore, we get

$$S^{-1}WS(J(T, S^{-1}y)) \subseteq J(T, S^{-1}y).$$

By the J-reflexivity of T, there exists a sequence $(n_k)_k$ of positive integers such that

$$S^{-1}WSx = \lim_k T^{n_k}x$$

for all $x \in X$. Putting $u = Sx$, we have

$$Wu = \lim_k (STS^{-1})^{n_k}u$$

for all $u \in Y$ and the proof is complete. □

In the following theorem, the sufficient conditions for the J-reflexivity of an operator are given.

Theorem 3.6 *Let $T \in B(X)$. If T satisfies one of the following statements, then T is J-reflexive:*
a) T is a power bounded operator that has no periodic point and $x \in J(T, x)$ for all $x \in X$.
b) T is power bounded and invertible, and there exists $m \in \mathbb{N}$ such that $T^m x \in J(T, x)$ for all $x \in X$.
c) T is invertible and there exists positive integer m such that $T^m = I$.

Proof Let T satisfy in (a) and suppose on the contrary that T is not J-reflexive. Then there exists an operator $S \in B(X) \backslash \overline{orb(T)}^{SOT}$ such that

$$S(J(T, x)) \subseteq J(T, x)$$

for all $x \in X$. Therefore, there exist $x_0 \in X$ and $\delta > 0$ such that $B(Sx_0;\delta) \cap orb(T,x_0)$ is a finite set. Since $x_0 \in J(T,x_0)$, $Sx_0 \in J(T,x_0)$, and this means that there exists a strictly increasing sequence of positive integers $(k_n)_n$ such that

$$Sx_0 = \lim_n T^{k_n} x_0.$$

This is possible only if x_0 is a periodic point for T, which is a contradiction. Now suppose that (b) holds. Let $W \in B(X)$ be such that $W(J(T,x)) \subseteq J(T,x)$ for all x. The surjectivity of T yields the surjectivity of T^m and therefore for every $z \in X$ there exists $x \in X$ such that $z = T^m x$. Since for all x, $T^m x \in J(T,x)$, we have $W(T^m x) \in J(T,x)$ and therefore there exists a strictly increasing sequence of positive integers $(k_n)_n$ such that

$$Wz = W(T^m x) = \lim_n T^{k_n} x = \lim_n T^{k_n - m} z.$$

Thus, $W \in \overline{orb(T)}^{SOT}$ and T is J-reflexive. Finally, assume that (c) holds. Since $T^m = I$, $orb(T)$ is a finite set and therefore T is power bounded. On the other hand, for all x, $Tx \in J(T^m, Tx)$, and then by Lemma 2.1 (c), $Tx \in J(T,Tx)$. Eventually, Lemma 2.1 (a) (equality holds) yields that $Tx \in J(T,x)$. Now by using (b), we get the desired result. □

Recall that the space of convergent sequences is usually denoted by c. This is a Banach space over $\mathbb{C}$ or $\mathbb{R}$ under the supremum norm.

Example 3.7 *Define $T : c \to c$ by*

$$T(x_1, x_2, x_3, \cdots) = (x_2, x_1, x_3, \cdots).$$

Clearly T is invertible, and $\|T\| = 1$. Since for any $x \in X$ and all $n \in \mathbb{N}$ we have $T^{2n+1}x = Tx$, thus $(T^2)^n(Tx) = Tx$ and so $Tx \in J(T^2, x)$. Now $Tx \in J(T,x)$ by Lemma 2.1 (c). Indeed, $J(T,x) = \{x, Tx\}$ for all x. Hence, T is J-reflexive by Theorem 3.6 (b).

Example 3.8 *Let $p(A)$ be a permutation p on a finite set A. Define S_{ij} on c by*

$$S_{ij}(x_1, x_2, \cdots) = (x_1, \cdots, x_{i-1}, p\{x_i, \cdots, x_j\}, x_{j+1}, \cdots).$$

If m is the smallest positive integer satisfying $p^{m+1} = p$, then $S_{ij}^{m+1}x = S_{ij}x$. Thus, $S_{ij}x \in J(S_{ij}^m, x)$ and by Lemma 2.1 (c), $S_{ij}x \in J(S_{ij}, x)$. Indeed,

$$J(S_{ij}, x) = \{x, S_{ij}x, \cdots, S_{ij}^{m-1}x\}.$$

Now by Theorem 3.6 (b), it follows that S_{ij} is J-reflexive.

Example 3.9 *Let H be a real Hilbert space and $\Lambda : B(H) \to B(H)$ be defined by $\Lambda(T) = T^*$. Then $\Lambda^2 = I$ and Λ is J-reflexive by Theorem 3.6 (c).*

Next we present some examples of J-reflexive and non-J-reflexive operators on finite dimensional spaces.

Example 3.10 *Let T be defined on the space of complex $n \times n$ matrices by $T(A) = A^t$ where A^t is the transpose of A. Since $T^2 = I$, T is J-reflexive by Theorem 3.6 (c).*

Example 3.11 *Suppose that dim $X = 1$ and T is defined on X by $Tx = (1/2)x$. Then $\|T\| < 1$, and by Lemma 3.4, it follows that T is not J-reflexive.*

The concept of J-reflexivity can be expressed in terms of subsets as follows.

Definition 3.12 *Let T be a bounded linear operator on a Banach space X and M be a subset of X. We call T M-J-reflexive if every $W \in B(X)$ that leaves invariant $J(T,x)$ for all x in M is contained in $\overline{orb(T)}^{SOT}$. M-J^{mix}-reflexivity can also be defined in a similar way.*

It is clear that M-J-reflexivity implies J-reflexivity. The converse is true for power bounded operators and dense subsets.

Theorem 3.13 *If $T \in B(X)$ is J-reflexive and power bounded and M is a dense subset of X, then T is M-J-reflexive.*

Proof Assume that $W \in B(X)$ leaves invariant $J(T,z)$ for all $z \in M$. It is enough to show that W leaves invariant $J(T,x)$ for all $x \in X$. Suppose that $x \in X$ and $y \in J(T,x)$. Then there exists a strictly increasing sequence of positive integers $(k_n)_n$ such that $y = \lim_n T^{k_n}x$. By density of M, we can find sequences $(x_n)_n$ and $(y_n)_n$ in M such that $x_n \to x$ and $y_n \to y$. Thus, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\|x_n - x\| < \epsilon/3L, \ \|y_n - y\| < \epsilon/3, \ \|y - T^{k_n}x\| < \epsilon/3$$

, where L is a power bound of T. Now for $m, n \geq N$, we get

$$\|y_m - T^{k_n}x_m\| \leq \|y_m - y\| + \|y - T^{k_n}x\| + \|T^{k_n}\|\|x - x_m\| < \epsilon.$$

Thus, $y_m \in J(T,x_m)$ and therefore $Wy_m \in J(T,x_m)$ for $m \geq N$. Since Wy_m tends to Wy, by applying the same method that has been used in the proof of Lemma 2.5 in [5], we obtain that $Wy \in J(T,x)$. This completes the proof. □

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