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A vectorization for nonconvex set-valued optimization

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Abstract: Vectorization is a technique that replaces a set-valued optimization problem with a vector optimization problem. In this work, by using an extension of the Gerstewitz function, a vectorizing function is defined to replace a given set-valued optimization problem with respect to the set less order relation. Some properties of this function are studied. Moreover, relationships between a set-valued optimization problem and a vector optimization problem, derived via vectorization of this set-valued optimization problem, are examined. Furthermore, necessary and sufficient optimality conditions are presented without any convexity assumption.

Key words: Set-valued optimization, nonconvex optimization, vectorization, optimality conditions

1. Introduction

Set-valued optimization, a generalization of vector optimization, has become a popular subject since it has many applications in game theory, engineering, control theory, finance, etc. [2, 4, 8, 13, 14, 23, 25, 26]. Optimal solutions of set-valued optimization problems can be given with a vector or set optimization approach. In the vector approach, the solutions that give efficient (minimal or maximal) points of image set of the objective set-valued map are looked for. Some studies related to the vector approach are [1, 3, 5, 9, 10, 13, 16, 18, 24, 27]. In the set optimization approach [9, 11, 13, 20, 22], the solutions that give efficient sets of the image family of the objective set-valued map are looked for. We consider the set optimization approach in this work.

Kuroiwa et al. [22] presented six order relations for sets. Then the set optimization approach was introduced by Kuroiwa [21]. Later, Jahn and Ha defined new order relations and examined some of their properties [12].

Scalarization and vectorization are some basic tools for solving set-valued optimization problems with respect to the set optimization approach. Recently, some scalarization techniques obtained via the Gerstewitz function have been widely used [6, 7, 9, 13].

Hernández and Rodríguez-Marín examined relationships between solution concepts for the vector approach and the set optimization approach with respect to the lower set less order relation. Moreover, they defined an extension of the Gerstewitz function and obtained a nonconvex scalarization and optimality conditions for set-valued optimization problems with respect to the lower set less order relation [9]. Köbis and

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Köbis obtained nonconvex scalarizations with respect to several well-known set order relations [15]. Xu and Li presented a scalarization via oriented distance function and obtained optimality conditions for set-valued optimization problems with respect to the upper set less order relation [28].

Vectorization is a tool for solving set-valued optimization problems by using vector-valued functions. This method replaces a set-valued optimization problem with a vector optimization problem that can be solved by using known methods such as numerical methods and scalarization [1, 5, 10, 24]. Solutions obtained via these methods are also solutions of the set-valued optimization problem.

Vectorization based on total ordering cones was first introduced by Küçük et al. [17, 19]. They showed that a set-valued optimization problem can be represented as a vector-valued problem. They defined a vectorizing function via existence and uniqueness of a minimal element of cone-closed and cone-bounded sets with respect to a total ordering cone. The value of this function at a point is a unique minimal element of the value of the set-valued map at this point with respect to the total ordering cone.

Another vectorization technique was given by Jahn [11]. Jahn used linear approximations to define a vectorizing function for the set-valued optimization problem with respect to the set less order relation [11]. Under some convexity assumptions he gave optimality conditions for set-valued optimization problems with respect to the set less order relation.

Hernández and Rodríguez-Marín [9] introduced the function $G_e^\ell(\cdot,\cdot)$ to scalarize set-valued optimization problems with respect to the lower set-less order relation $\preceq^\ell$. By using this function, a scalarization can be obtained for set-valued optimization problems with respect to the upper set-less order relation $\preceq^u$ since there is a relation between upper set-less and lower set-less orders as $A \preceq^\ell B \Leftrightarrow -B \preceq^u -A$. In the present study, we define a vectorizing function $w_e(\cdot,\cdot)$ by means of $G_e^\ell(\cdot,\cdot)$ since we consider set-valued optimization problems with respect to the set less order relation $\preceq^s$ that involves upper and lower set-less orders. The details are given in Section 3. The main advantage of the vectorization $w_e(\cdot,\cdot)$ is that it is simple to compute; we demonstrate this in Example 3.7. Using arguments parallel to those given in [9, Section 4] we present necessary and sufficient optimality conditions for minimal and weak minimal solutions in terms of the vectorization $w_e(\cdot,\cdot)$ for set-valued optimization problems with respect to set-less order. Similar to [9], given results on optimality conditions do not require convexity assumptions. The single variable Gerstewitz function is used for the first time in our study while the two variable Gerstewitz function was used in [9]. Furthermore, some examples for convex and nonconvex cases are used to demonstrate the usage of Gerstewitz vectorization.

The paper is organized as follows. In section 2, we recall basic concepts of the theory of vector optimization and set-valued optimization. In section 3, we introduce the Gerstewitz vectorizing function and examine some properties of this function. Moreover, relationships between this function and the set less order relation are studied. In the last section, some optimality conditions are presented via Gerstewitz vectorization.

2. Preliminaries

Throughout this paper, X is any nonempty set and Y denotes a real topological linear space ordered by a convex, closed, and pointed cone $C \subset Y$ with nonempty interior. $\mathcal{P}_0(Y)$ is the notation of the family of all nonempty subsets of Y. Given any set $A \in \mathcal{P}_0(Y)$, $int(A)$ and $cl(A)$ are topological interior and the closure of A, respectively. $\mathbb{R}^2$ is partially ordered by cone $\mathbb{R}^2_+$.

It is known that the cone C induces the following ordering relations on Y for $y, y' \in Y$:

$$y \leq_C y' \iff y' - y \in C$$
$$y <_C y' \iff y' - y \in int(C).$$

Let $A \subset Y$ and $a_0 \in A$. a_0 is a minimal (maximal) point of A with respect to cone C if $A \cap (a_0 - C) = \{a_0\}$ ($A \cap (a_0 + C) = \{a_0\}$). The set of all minimal (maximal) points of A is denoted by $\min A$ ($\max A$). Similarly, a_0 is a weak minimal (weak maximal) point of A with respect to cone C if $A \cap (a_0 - int(C)) = \emptyset$ ($A \cap (a_0 + int(C)) = \emptyset$) and the set of all weak minimal (weak maximal) points of A is denoted by $W \min A$ ($W \max A$).

A vector optimization problem is defined by

$$(VOP) \begin{cases} \min(\max) f(x) \\ s.t. \ x \in X, \end{cases}$$

where $f : X \to Y$ is a vector valued function.

Definition 2.1 *[10] An element $\bar{x} \in X$ is called a minimal (maximal) solution of (VOP) with respect to cone C iff there is not any $x \in X$ such that*

$$f(x) \leq_C f(\bar{x}) \quad (f(\bar{x}) \leq_C f(x)) \text{ and } f(x) \neq f(\bar{x}).$$

Definition 2.2 *[10] An element $\bar{x} \in X$ is called a minimal (maximal) strongly solution of (VOP) with respect to cone C iff*

$$f(\bar{x}) \leq_C f(x) \quad (f(x) \leq_C f(\bar{x})) \quad \text{for all } x \in X.$$

If $\bar{x} \in X$ is a strongly minimal (maximal) solution of (VOP), it is also a minimal (maximal) solution of (VOP) [10].

It is said that A is C-closed iff $A + C$ is a closed set; C-bounded iff for each neighborhood U of zero in Y, there exists a positive real number t such that $A \subset tU + C$; C-compact iff any cover of the form $\{U_\alpha + C \mid U_\alpha \text{ are open }, \alpha \in I\}$ admits a finite subcover. Every C-compact set is C-closed and C-bounded [24].

A is called $\mp C$-bounded if $A \in \mathcal{P}_0(Y)$ is C-bounded and $-C$-bounded in Y; if A is C-closed and $-C$-closed, A is called $\mp C$-closed; if A is C-compact and $-C$-compact, A is called $\mp C$-compact set. A set $A \in \mathcal{P}_0(Y)$ is called C-proper iff $A + C \neq Y$ and we denote by $\mathcal{P}_{0C}(Y)$ the family of all C-proper subsets of Y [9]. A set $A \in \mathcal{P}_0(Y)$ is called $-C$-proper iff $A - C \neq Y$ and we denote by $\mathcal{P}_{0-C}(Y)$ the family of all $-C$-proper subsets of Y [28]. $\mathcal{P}^0_{\mp C}(Y)$ denotes the family of C-proper and $-C$-proper subsets of Y, namely, $\mathcal{P}^0_{\mp C}(Y) := \mathcal{P}_{0C}(Y) \cap \mathcal{P}_{0-C}(Y)$.

Let $F : X \rightrightarrows Y$ be a set-valued map and "N" denotes some property of a set in Y. F is called N valued on X if $F(x)$ has the property "N" for every $x \in X$. For example, if $F(x)$ is closed for all $x \in X$, we say that F is closed valued on X.

Let $F : X \rightrightarrows Y$ be a set-valued map and $F(x) \neq \emptyset$ for all $x \in X$. The set-valued optimization problem is defined by

$$(SOP) \begin{cases} \min(\max) F(x) \\ s.t. \ x \in X. \end{cases}$$

According to the vector approach, we are looking for efficient points of the set $F(X)=\bigcup_{x\in X} F(x)$ to solve (SOP), that is, $x_0 \in X$ is a solution of the set-valued optimization problem if

$$F(x_0)\cap \min \bigcup_{x\in X} F(x)\neq \emptyset \quad \Big(F(x_0)\cap \max \bigcup_{x\in X} F(x)\neq \emptyset\Big).$$

When (SOP) is considered according to the vector approach, we denote the problem by $(v-SOP)$. Similarly, $x_0 \in X$ is a weak solution of $(v-SOP)$ if

$$F(x_0)\cap W\min \bigcup_{x\in X} F(x)\neq \emptyset \quad \Big(F(x_0)\cap W\max \bigcup_{x\in X} F(x)\neq \emptyset\Big).$$

The set optimization approach is based on a comparison among the values of set-valued map [21]. That is, we are looking for efficient sets of the family $\mathcal{F}(X)=\{F(x)\mid x\in X\}$ to solve (SOP).

Definition 2.3 *[9, 12, 22, 28] Let $A,B\in\mathcal{P}_0(Y)$.*

(i) lower set less order relation ($\preceq^{\ell}$) is defined by

$$A\preceq^{\ell} B \Longleftrightarrow B\subset A+C$$

(ii) strict lower set less order relation ($\prec^{\ell}$) is defined by

$$A\prec^{\ell} B \Longleftrightarrow B\subset A+int(C)$$

(iii) upper set less order relation ($\preceq^{u}$) is defined by

$$A\preceq^{u} B \Longleftrightarrow A\subset B-C$$

(iv) strict upper set less order relation ($\prec^{u}$) is defined by

$$A\prec^{u} B \Longleftrightarrow A\subset B-int(C)$$

(v) set less order relation ($\preceq^{s}$) is defined by

$$A\preceq^{s} B \Longleftrightarrow A\preceq^{\ell} B \text{ and } A\preceq^{u} B$$

(vi) strict set less order relation ($\prec^{s}$) is defined by

$$A\prec^{s} B \Longleftrightarrow A\prec^{\ell} B \text{ and } A\prec^{u} B.$$

Note that $\preceq^{\ell}$, $\preceq^{u}$, and $\preceq^{s}$ order relations are reflexive and transitive on $\mathcal{P}_0(Y)$. There is a relationship between $\preceq^{\ell}$ and $\preceq^{u}$: Let $A,B\in\mathcal{P}_0(Y)$ and we have

$$A\preceq^{\ell} B \Longleftrightarrow -B\preceq^{u} -A. \tag{1}$$

Let $\sharp \in \{\ell, u, s\}$. $\sim^\sharp$ relation defined by

$$A \sim^\sharp B \Longleftrightarrow A \preceq^\sharp B \text{ and } B \preceq^\sharp A$$

is an equivalence relation on $\mathcal{P}_0(Y)$. $[A]^\sharp$ denotes the equivalence class of A with respect to $\sim^\sharp$, where $A \in \mathcal{P}_0(Y)$ [9, 12].

Note that

$$A \in [B]^\ell \Longleftrightarrow -A \in [-B]^u \tag{2}$$

where $A, B \in \mathcal{P}_0(Y)$.

Now we recall the minimal, maximal, weak minimal, and weak maximal set of a family with respect to order relations $\preceq^\ell$, $\preceq^u$, and $\preceq^s$.

Definition 2.4 *[9, 11] Let $\mathcal{S} \subset \mathcal{P}_0(Y)$, $A \in \mathcal{S}$, and $\sharp \in \{\ell, u, s\}$ be given.*

(i) A is said to be a $\sharp$-minimal set of $\mathcal{S}$ iff for any $B \in \mathcal{S}$ such that $B \preceq^\sharp A$ implies $A \preceq^\sharp B$. The family of $\sharp$-minimal sets of $\mathcal{S}$ is denoted by $\sharp - \min \mathcal{S}$.

(ii) A is said to be a $\sharp$-maximal set of $\mathcal{S}$ iff for any $B \in \mathcal{S}$ such that $A \preceq^\sharp B$ implies $B \preceq^\sharp A$. The family of $\sharp$-maximal sets of $\mathcal{S}$ is denoted by $\sharp - \max \mathcal{S}$.

Definition 2.5 *[9, 11] Let $\mathcal{S} \subset \mathcal{P}_0(Y)$, $A \in \mathcal{S}$, and $\sharp \in \{\ell, u, s\}$ be given.*

(i) A is called a weak $\sharp$-minimal set of $\mathcal{S}$ iff for any $B \in \mathcal{S}$ such that $B \prec^\sharp A$ implies $A \prec^\sharp B$. The family of weak $\sharp$-minimal sets of $\mathcal{S}$ is denoted by $\sharp - W \min \mathcal{S}$.

(ii) A is called a weak $\sharp$-maximal set of $\mathcal{S}$ iff for any $B \in \mathcal{S}$ such that $A \prec^\sharp B$ implies $B \prec^\sharp A$. The family of weak $\sharp$-maximal sets of $\mathcal{S}$ is denoted by $\sharp - W \max \mathcal{S}$.

Let $\sharp \in \{\ell, u, s\}$ and (SOP) be given. According to the set optimization approach, if $F(x_0)$ is a $\sharp$-minimal ($\sharp$-maximal) set of $\mathcal{F}(X)$, then x_0 is called a solution of (SOP) with respect to $\preceq^\sharp$. When (SOP) is considered with respect to $\preceq^\sharp$, we denote it by $(\sharp - SOP)$. Similarly, if $F(x_0)$ is a weak $\sharp$-minimal (weak $\sharp$-maximal) set of $\mathcal{F}(X)$, then x_0 is called a weak solution of $(\sharp - SOP)$.

Note that if F is a vector-valued function, solution(s) of $(v - SOP)$ coincides with solution(s) of $(\sharp - SOP)$.

The following definition is related tot he monotonicity of a real valued function defined on $\mathcal{P}_0(Y)$.

Definition 2.6 *[9, 28] Let $\sharp \in \{\ell, u\}$ and $\mathcal{S} \subset \mathcal{P}_0(Y)$. A function $T : \mathcal{P}_0(Y) \to \mathbb{R}$ is called*

(i) $\sharp$-decreasing ($\sharp$-increasing) on $\mathcal{S}$ if $A, B \in \mathcal{S}$ and $A \preceq^\sharp B$ implies $T(B) \leq T(A)$ $(T(A) \leq T(B))$,

(ii) strictly $\sharp$-decreasing (strictly $\sharp$-increasing) on $\mathcal{S}$ if $A, B \in \mathcal{S}$ and $A \prec^\sharp B$ implies $T(B) < T(A)$ $(T(A) < T(B))$.

Hernández and Rodríguez-Marín generalized the Gerstewitz function as

$$G_e(A,B) = \sup_{b\in B}\{\phi_{e,A}(b)\} \tag{3}$$

where $e \in -int(C)$ and $\phi_{e,A}(y) = \inf\{t \in \mathbb{R} \mid y \in te + A + C\}$ [9] and examined some properties of this function and obtained scalarization and optimality conditions for $(\ell - SOP)$. Throughout this paper, we use notation $G_e^\ell(\cdot,\cdot)$ instead of $G_e(\cdot,\cdot)$.

If we consider the nonconvex scalarization function $\phi^u_{e,A}(y) = \sup\{t \in \mathbb{R} \mid y \in te + A - C\}$, then one can obtain optimality conditions for $(u - SOP)$ similar to the conditions given by Hernández and Rodríguez-Marín in [9]. In this function taking $A = \{0\}$ and $k = -e$ the equality $\phi^u_{-k,\{0\}}(y) = -z^{C,k}(y)$ is obtained, where $z^{C,k}$ is used to present nonconvex scalarization and some optimality conditions with respect to $\preceq^u$, $\preceq^\ell$, and $\preceq^s$ by Köbis and Köbis in [15].

3. Gerstewitz vectorizing function

In this section, a vectorizing function is defined to replace a $(s - SOP)$ with (VOP) using the generalized Gerstewitz function (3). Some properties including the monotonicity of this function are studied. Furthermore, relationships between this function and the set less order relation are examined.

Now we give definition of monotonicity of a function from $\mathcal{P}^0_{\mp C}(Y)$ to $\mathbb{R}^2$.

Definition 3.1 *Let $\mathcal{A} \subset \mathcal{P}^0_{\mp C}(Y)$. A function $T : \mathcal{P}^0_{\mp C}(Y) \to \mathbb{R}^2$ is called*

(i) s-increasing (s-decreasing) on $\mathcal{A}$ if $A, B \in \mathcal{A}$ and $A \preceq^s B$ implies $T(A) \leq_{\mathbb{R}^2_+} T(B)$ ($T(B) \leq_{\mathbb{R}^2_+} T(A)$),

(ii) strictly s-increasing (strictly s-decreasing) on $\mathcal{A}$ if $A, B \in \mathcal{A}$ and $A \prec^s B$ implies $T(A) <_{\mathbb{R}^2_+} T(B)$ ($T(B) <_{\mathbb{R}^2_+} T(A)$).

Now we introduce a vectorizing function that is the main tool to present a new vectorization.

Definition 3.2 *Let $A, B \in \mathcal{P}^0_{\mp C}(Y)$ and $e \in -int(C)$. The vectorizing function $w_e : \mathcal{P}^0_{\mp C}(Y) \times \mathcal{P}^0_{\mp C}(Y) \to \overline{\mathbb{R}}^2$ defined by*

$$w_e(A,B) = \big(-G_e^\ell(A,B), -G_e^\ell(-B,-A)\big) \tag{4}$$

is called the Gerstewitz vectorizing function.

Throughout this paper, in order to emphasize that the scalarization is adapted for $\preceq^u$ we simply use the notation $G_e^u(B,A)$ instead of $-G_e^\ell(-B,-A)$ where $A, B \in \mathcal{P}^0_{\mp C}(Y)$. Then

$$w_e(A,B) = \big(-G_e^\ell(A,B), G_e^u(B,A)\big)$$

for all $A, B \in \mathcal{P}^0_{\mp C}(Y)$.

Here some properties of $w_e(\cdot,\cdot)$ are stated.

Theorem 3.3 *Let $A, B \in \mathcal{P}^0_{\mp C}(Y)$. Then the following statements are true:*

(i) *If* A, B *are* $\mp C$*-bounded, then* $w_e(A,B) \in \mathbb{R}^2$,

(ii) *If* $A \in [B]^s$, *then* $w_e(A,\cdot) = w_e(B,\cdot)$ *and* $w_e(\cdot,A) = w_e(\cdot,B)$,

(iii) *If* $A \in [B]^s$, *then* $w_e(A,B) = w_e(B,A)$,

(iv) $w_e(\cdot,A)$ *is* s*-decreasing on* $\mathcal{P}^0_{\mp C}(Y)$,

(v) $w_e(A,\cdot)$ *is* s*-increasing on* $\mathcal{P}^0_{\mp C}(Y)$.

Proof

(i) Since A and B are C-bounded and $-C$-bounded, we have $-G^\ell_e(A,B) \in \mathbb{R}$ and $G^u_e(B,A) \in \mathbb{R}$ from Theorem 3.6 of [9]. Therefore, we obtain $w_e(A,B) \in \mathbb{R}^2$.

(ii) Since $A \in [B]^\ell$ and $A \in [B]^u$, we have $-G^\ell_e(A,\cdot) = -G^\ell_e(B,\cdot)$ and $G^u_e(\cdot,A) = G^u_e(\cdot,B)$ from Theorem 3.8 (i) and (iii) of [9], respectively. Therefore, we obtain $w_e(A,\cdot) = w_e(B,\cdot)$. Similarly, we get $w_e(\cdot,A) = w_e(\cdot,B)$ by using Theorem 3.8 (i) and (iii) of [9].

(iii) Since $A \in [B]^\ell$ and $A \in [B]^u$, we have $-G^\ell_e(A,B) = -G^\ell_e(B,A)$ and $G^u_e(A,B) = G^u_e(B,A)$ from Theorem 3.8 (iv) of [9], respectively. Then we obtain $w_e(A,B) = w_e(B,A)$.

(iv) Assume that $B, D \in \mathcal{P}_{0\mp C}(Y)$ and $B \preceq^s D$. Then $B \preceq^\ell D$ and $B \preceq^u D$. We have $-G^\ell_e(D,A) \leq -G^\ell_e(B,A)$ and $G^u_e(A,D) \leq G^u_e(A,B)$ from Theorem 3.8 (v) and (ii) of [9], respectively. Then we have $w_e(D,A) \leq_{\mathbb{R}^2_+} w_e(B,A)$. Therefore $w_e(\cdot,A)$ is s-decreasing on $\mathcal{P}^0_{\mp C}(Y)$.

(v) Assume that $B, D \in \mathcal{P}_{0\mp C}(Y)$ and $B \preceq^s D$. Then $B \preceq^\ell D$ and $B \preceq^u D$. We have $-G^\ell_e(A,B) \leq -G^\ell_e(A,D)$ and $G^u_e(B,A) \leq G^u_e(D,A)$ from Theorem 3.8 (ii) and (v) of [9], respectively. Then we have $w_e(A,B) \leq_{\mathbb{R}^2_+} w_e(A,D)$. Therefore, $w_e(A,\cdot)$ is s-increasing on $\mathcal{P}^0_{\mp C}(Y)$.

□

Theorem 3.4 *Let* $A \in \mathcal{P}^0_{\mp C}(Y)$ *be a* $\mp C$*-compact set. Then the following statements are true:*

(i) $w_e(\cdot,A)$ *is strictly* s*-decreasing on the family of* $\mp C$*-compact sets,*

(ii) $w_e(A,\cdot)$ *is strictly* s*-increasing on the family of* $\mp C$*-compact sets.*

Proof

(i) Assume that $B, D \in \mathcal{P}^0_{\mp C}(Y)$ are $\mp C$-compact sets and $B \prec^s D$. Then $B \prec^\ell D$ and $B \prec^u D$. We have $-G^\ell_e(D,A) < -G^\ell_e(B,A)$ and $G^u_e(A,D) < G^u_e(A,B)$ from Theorem 3.9 (ii) and (i) of [9], respectively. Hence, we obtain $w_e(D,A) <_{\mathbb{R}^2_+} w_e(B,A)$. Therefore, $w_e(\cdot,A)$ is strictly s-decreasing on the family of $\mp C$-compact sets.

(ii) This statement can be proved similar to (i) by using Theorem 3.9 (i) and (ii) of [9].

□

Now we examine relationships between the set less order relation and Gerstewitz vectorizing function.

Under different assumptions a necessary and sufficient condition similar to (iii) of Theorem 3.5 was given by means of $z^{C,k}$ in [15]. These results are similar because if $k=-e$ then $G_e^{\ell}(A,B)=\sup\limits_{b\in B}\inf\limits_{a\in A} z^{C,k}(a-b)$.

Theorem 3.5 *Let $A\in\mathcal{P}^0_{\mp C}(Y)$ be a $\mp C$-closed set. Then the following statements are true:*

(i) $w_e(A,A)=(0,0)$,

(ii) *If $A\in[B]^s$, then* $w_e(A,B)=w_e(B,A)=(0,0)$,

(iii) $A\preceq^s B$ *if and only if* $(0,0)\leq_{\mathbb{R}^2_+} w_e(A,B)$.

Proof

(i) From Theorem 3.10 (i) of [9] we have $-G_e^{\ell}(A,A)=0$ and $G_e^{u}(A,A)=0$. Hence, we obtain

$$w_e(A,A)=(-G_e^{\ell}(A,A),G_e^{u}(A,A))=(0,0).$$

(ii) Since $A\in[B]^{\ell}$ and $A\in[B]^{u}$, we have $-G_e^{\ell}(A,B)=-G_e^{\ell}(B,A)=0$ and $G_e^{u}(A,B)=G_e^{u}(B,A)=0$ from Theorem 3.10 (ii) of [9], respectively. Therefore, $w_e(A,B)=w_e(B,A)=(-G_e^{\ell}(A,B),G_e^{u}(B,A))=(0,0)$.

(iii) ($\Longrightarrow$) Let $A\preceq^s B$. Then $A\preceq^{\ell} B$ and $A\preceq^{u} B$. Since $A\preceq^{\ell} B$ and $A\preceq^{u} B$, we have $-G_e^{\ell}(A,B)\geq 0$ and $G_e^{u}(B,A)\geq 0$ from Theorem 3.10 (iii) of [9], respectively. Thus, we obtain $(0,0)\leq_{\mathbb{R}^2_+} w_e(A,B)$.

($\Longleftarrow$) Let $(0,0)\leq_{\mathbb{R}^2_+} w_e(A,B)$. Then we have $G_e^{\ell}(A,B)\leq 0$ and $G_e^{u}(B,A)\geq 0$. Hence, $A\preceq^{\ell} B$ and $A\preceq^{u} B$ from Theorem 3.10 (iii) of [9], respectively. Therefore, $A\preceq^s B$.

□

Theorem 3.6 *Let $A,B\in\mathcal{P}^0_{\mp C}(Y)$ be $\mp C$-compact sets. Then*

$$(0,0)<_{\mathbb{R}^2_+} w_e(A,B)\Longleftrightarrow A\prec^s B.$$

Proof ($\Longrightarrow$) Let $(0,0)<_{\mathbb{R}^2_+} w_e(A,B)$. Then $G_e^{\ell}(A,B)<0$ and $G_e^{u}(B,A)>0$. Since $G_e^{\ell}(A,B)<0$ and $G_e^{u}(B,A)>0$, we have $A\prec^{\ell} B$ and $A\prec^{u} B$ from Corollary 3.11 (i) of [9], respectively. Hence, $A\prec^s B$.

($\Longleftarrow$) Let $A\prec^s B$. Then $A\prec^{\ell} B$ and $A\prec^{u} B$. As $A\prec^{\ell} B$ and $A\prec^{u} B$, we have $G_e^{\ell}(A,B)<0$ and $G_e^{u}(B,A)>0$ from Corollary 3.11 (i) of [9], respectively. Therefore, we obtain $(0,0)<_{\mathbb{R}^2_+} w_e(A,B)$. □

We define a vectorizing function $v_e:\mathcal{P}^0_{\mp C}(Y)\to\overline{\mathbb{R}}^2$ as

$$v_e(A):=w_e(\{0\},A)=\left(-G_e^{\ell}(\{0\},A),G_e^{u}(A,\{0\})\right),$$

where $e\in -int(C)$ in order to use the advantage of computation of a single variable function.

It can be seen that $v_e(\cdot)$ is s-increasing on $\mathcal{P}^0_{\mp C}(Y)$ and strictly s-increasing on the family of $\mp C$-compact sets. Let $A,B\in\mathcal{P}^0_{\mp C}(Y)$. If $A\in[B]^s$, then $v_e(A)=v_e(B)$.

By taking [11, Example 3.1] we demonstrate the calculations of $w_e(\cdot,\cdot)$ and $v_e(\cdot)$.

Example 3.7 *Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : [-1,1] \rightrightarrows Y$ be defined as*

$$F(x) := \left\{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1 - 2x^2)^2 + (y_2 - 2x^2)^2 \leq (x^2+1)^2\right\}$$

for all $x \in [-1,1]$ (Figure 1).

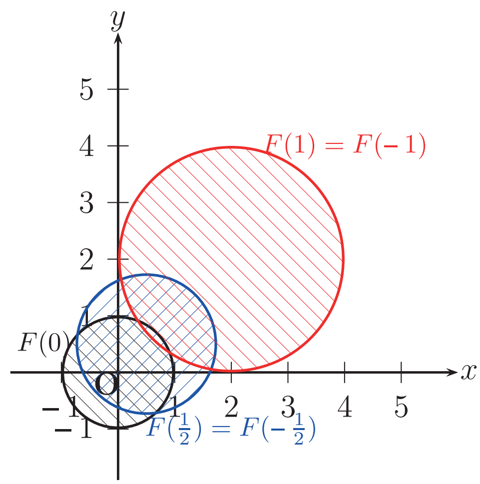
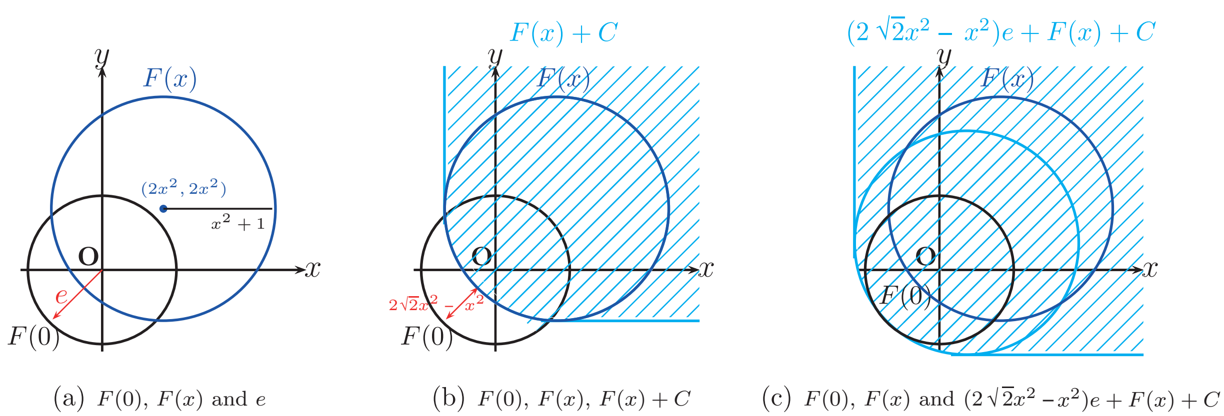


Figure 1. Some image sets of F.

First, we choose $e = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ to find $w_e(F(x), F(0))$, and $v_e(F(x))$ for an arbitrary $x \in [-1,1]$. Thus, we have to calculate $G^\ell_e(F(x), F(0))$, and $G^u_e(F(0), F(x))$. To find the value of

$$G^\ell_e(F(x), F(0)) = \min\{t \in \mathbb{R} \mid F(0) \subset te + F(x) + C\}$$

we should evaluate the smallest t that allows $te + F(x) + C$ to cover $F(0)$. This value means how long at least $F(x) + C$ should move along the direction e to cover the set $F(0)$. We achieve this smallest value clearly by subtracting the difference of radii of $F(x)$ and $F(0)$ from the distance between the centers of these balls as seen in Figure 2. Hence we get $G^\ell_e(F(x), F(0)) = 2\sqrt{2}x^2 - x^2$.



(a) $F(0)$, $F(x)$ and e (b) $F(0)$, $F(x)$, $F(x)+C$ (c) $F(0)$, $F(x)$ and $(2\sqrt{2}x^2 - x^2)e + F(x) + C$

Figure 2. The calculation of $G^\ell_e(F(x), F(0))$ geometrically.

In a similar manner, to find

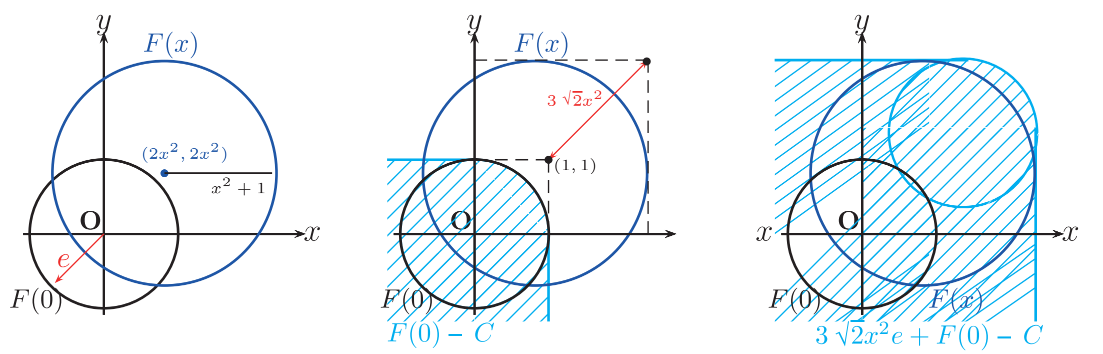
$$G_e^u(F(0), F(x)) = \max\{t \in \mathbb{R} \mid F(x) \subset te + F(0) - C\}$$

we calculate the largest value of t that allows $F(x)$ to be covered by $te + F(0) - C$. This largest value is clearly negative of the distance between the vectors $(3x^2+1, 3x^2+1)$ and $(1,1)$ as seen in Figure 3. Thus, we have

$$G_e^u(F(0), F(x)) = -3\sqrt{2}x^2.$$

Finally, we have

$$\begin{aligned} w_e(F(x), F(0)) &= \left(-G_e^\ell(F(x), F(0)), G_e^u(F(0), F(x))\right) \\ &= \left((1-2\sqrt{2})x^2, -3\sqrt{2}x^2\right). \end{aligned} \tag{5}$$



(a) $F(0)$, $F(x)$ and e (b) $F(0)$, $F(x)$, $F(0) - C$ (c) $F(0)$, $F(x)$ and $3\sqrt{2}x^2e + F(0) - C$

Figure 3. The calculation of $G_e^u(F(0), F(x))$ geometrically.

Now we will find $v_e(F(x))$ for all $x \in [-1,1]$. We need to calculate $G_e^\ell(\{0\}, F(x))$ and $G_e^u(F(x), \{0\})$ for all $x \in [-1,1]$. If C is moved along the direction e until it covers $F(x)$, then we get the value of

$$G_e^\ell(\{0\}, F(x)) = \min\{t \in \mathbb{R} \mid F(x) \subset te + C\} = \sqrt{2}(1 - x^2)$$

by subtracting the distance between the center of $F(x)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, and the radius of $F(x)$. As seen in Figure 4, we get $G_e^\ell(\{0\}, F(x)) = \sqrt{2}(1-x^2)$.

To calculate $G_e^u(F(x), \{0\})$, we can use the formula

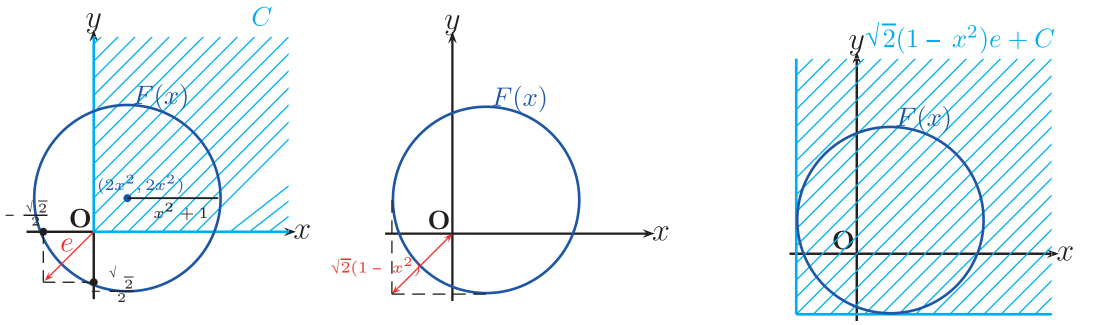
$$G_e^u(F(x), \{0\}) = \max\{t \in \mathbb{R} \mid 0 \in te + F(x) - C\}.$$

As seen in Figure 5, this value can be found by adding distance between the origin and center of $F(x)$, and the radius of $F(x)$. Hence, we get $G_e^u(F(x), \{0\}) = 2\sqrt{2}x^2 + x^2 + 1$.

Finally, we have

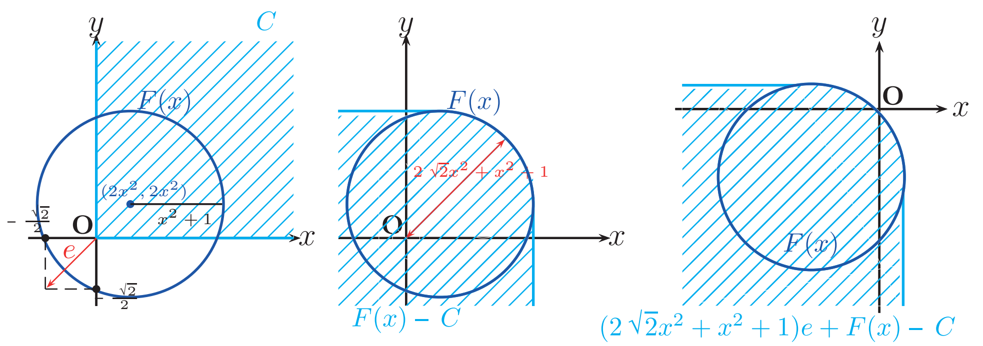
$$\begin{aligned} v_e(F(x)) &= \left(-G_e^\ell(\{0\}, F(x)), G_e^u(F(x), \{0\})\right) \\ &= \left(\sqrt{2}(x^2-1), 2\sqrt{2}x^2 + x^2 + 1\right). \end{aligned} \tag{6}$$

Consequently, the above calculations point out that by choosing a suitable $e \in -int(C)$ we obtained w_e and v_e easily. However, the vectorizing function in [11, Example 3.1] was obtained by considering all vectors of the polar cone of C.



(a) $F(x)$, C and e (b) $F(x)$ (c) $F(x)$ and $\sqrt{2}(1 - x^2)e + C$

Figure 4. The calculation of $G_e^{\ell}(\{0\}, F(x))$ geometrically.



(a) $F(x)$, C and e (b) $F(x)$ and $F(x) - C$ (c) $F(x)$ and $(2\sqrt{2}x^2 + x^2 + 1)e + F(x) - C$

Figure 5. The calculation of $G_e^{u}(F(x), \{0\})$ geometrically.

4. Gerstewitz vectorization and optimality conditions for $(s - SOP)$

$(s - SOP)$ can be replaced by a vector optimization problem using the Gerstewitz vectorizing function. In this section, results of the previous section are employed to give optimality conditions for $(s - SOP)$ without any convexity assumption and relationships between solutions of $(s - SOP)$ and (VOP) derived by the Gerstewitz vectorizing function.

Theorem 4.1 *Let $F : X \rightrightarrows Y$ be $\mp C$-closed and $\mp C$-bounded valued on X. $x_0 \in X$ is an s-maximal (s-minimal) solution of $(s - SOP)$ if and only if there exists an s-increasing (s-decreasing) function $T : \mathcal{P}^0_{\mp C}(Y) \to \mathbb{R}^2$ satisfying the following statements:*

(i) If $x \in X$ and $F(x) \in [F(x_0)]^s$, then $T(F(x)) = (0, 0)$,

(ii) If $x \in X$ and $F(x) \notin [F(x_0)]^s$, then $(0, 0) \not\leq_{\mathbb{R}^2_+} T(F(x))$,

(iii) If $A \in \mathcal{P}^0_{\mp C}(Y)$ and $F(x_0) \preceq^s A$ ($A \preceq^s F(x_0)$), then $(0, 0) \leq_{\mathbb{R}^2_+} T(A)$.

Proof We give the proof for maximality.

($\Longrightarrow$) Suppose that x_0 is an s-maximal solution of $(s-SOP)$. Let us fix any $e \in -int(C)$ and consider the function $T : \mathcal{P}^0_{\mp C}(Y) \to \mathbb{R}^2$ defined as $T(\cdot) = w_e(F(x_0), \cdot) = (-G^\ell_e(F(x_0), \cdot), G^u_e(\cdot, F(x_0)))$. By Theorem 3.3 (v) T is s-increasing on $\mathcal{P}^0_{\mp C}(Y)$. Now we show that T satisfies conditions (i)–(iii).

(i) Since $F(x) \in [F(x_0)]^s$, we have $T(F(x)) = w_e(F(x_0), F(x)) = (0,0)$ from Theorem 3.5 (ii).

(ii) Let $F(x) \notin [F(x_0)]^s$. Since x_0 is an s-maximal solution of $(s-SOP)$, we have $F(x_0) \not\preceq^s F(x)$. Thus, from Theorem 3.5 (iii) we obtain

$$(0,0) \not\leq_{\mathbb{R}^2_+} w_e(F(x_0), F(x)) = T(F(x)).$$

(iii) Assume that $F(x_0) \preceq^s A$. By Theorem 3.5 (iii) we get

$$(0,0) \leq_{\mathbb{R}^2_+} w_e(F(x_0), A) = T(A).$$

($\Longleftarrow$) Let (i)-(iii) be satisfied for some $T : \mathcal{P}^0_{\mp C}(Y) \to \mathbb{R}^2$ which is s-increasing on $\mathcal{P}^0_{\mp C}(Y)$. Assume the contrary that x_0 is not an s-maximal solution of $(s-SOP)$. Then there exists $x' \in X$ such that $F(x_0) \preceq^s F(x')$ and $F(x') \not\preceq^s F(x_0)$. Hence, $F(x') \notin [F(x_0)]^s$. From (ii) we have

$$(0,0) \not\leq_{\mathbb{R}^2_+} T(F(x')). \tag{7}$$

Since $F(x_0) \preceq^s F(x')$, by (iii) we have $(0,0) \leq_{\mathbb{R}^2_+} T(F(x'))$. This contradicts (7). Therefore, x_0 is an s-maximal solution of $(s-SOP)$.

In order to prove the minimality of x_0, it is enough to take the function $T(\cdot) = w_e(\cdot, F(x_0))$. □

Theorem 4.2 *Let $F : X \rightrightarrows Y$ be $\mp C$-compact valued on X. $x_0 \in X$ is a weak s-maximal (weak s-minimal) solution of $(s-SOP)$ if and only if there exists a strictly s-increasing (strictly s-decreasing) function $T : \mathcal{P}^0_{\mp C}(Y) \to \mathbb{R}^2$ satisfying the following statements:*

(i) If $x \in X$ and $F(x) \in [F(x_0)]^s$, then $T(F(x)) = (0,0)$,

(ii) If $x \in X$ and $F(x) \notin [F(x_0)]^s$, then $(0,0) \not<_{\mathbb{R}^2_+} T(F(x))$,

(iii) If $A \in \mathcal{P}^0_{\mp C}(Y)$ is a $\mp C$-compact set and $F(x_0) \prec^s A$ ($A \prec^s F(x_0)$), then $(0,0) <_{\mathbb{R}^2_+} T(A)$.

Proof The proof is similar to the proof of Theorem 4.2. □

Theorem 4.3 *Let $F : X \rightrightarrows Y$ be $\mp C$-closed, $\mp C$-bounded valued on X, and $e \in -int(C)$. $x_0 \in X$ is an s-maximal (s-minimal) solution of $(s-SOP)$ if and only if x_0 is a solution of the problem*

$$(VOP^s_w)\begin{cases} \max w_e(F(x_0), F(x)) \\ s.t.\ x \in X \end{cases} \quad \left(\begin{cases} \max w_e(F(x), F(x_0)) \\ s.t.\ x \in X \end{cases}\right).$$

Proof ($\Longrightarrow$) It is a result of Theorem 4.1.

($\Longleftarrow$) Let x_0 be a solution of (VOP^s_w). Then we have $(0,0) \not\leq_{\mathbb{R}^2_+} w_e(F(x_0), F(x))$ for all $x \in X$, where $F(x) \notin [F(x_0)]^s$. By Theorem 3.5 (iii) $F(x_0) \not\preceq^s F(x)$ for all $x \in X$, where $F(x) \notin [F(x_0)]^s$. Thus, x_0 is a solution of $(s-SOP)$. □

Theorem 4.4 *Let $F : X \rightrightarrows Y$ be $\mp C$-compact valued on X and $e \in -int(C)$. $x_0 \in X$ is a weak s-maximal (weak s-minimal) solution of $(s - SOP)$ if and only if x_0 is a strong solution of the problem*

$$(VOP_w^s)\begin{cases} \max w_e(F(x_0), F(x)) \\ s.t.\ x \in X \end{cases} \quad \left(\begin{cases} \max w_e(F(x), F(x_0)) \\ s.t.\ x \in X \end{cases}\right).$$

Proof

($\Longrightarrow$) It is a result of Theorem 4.2.

($\Longleftarrow$) It can be proved by using Theorem 3.6. □

Corollary 4.5 *Let $F : X \rightrightarrows Y$ be $\mp C$-closed and $\mp C$-bounded valued on X and*

$$F(x) \preceq^s F(y) \text{ or } F(y) \preceq^s F(x) \quad \text{for all } x, y \in X. \tag{8}$$

If $x_0 \in X$ is an s-maximal (s-minimal) solution of $(s - SOP)$, then it is also a strong solution of the problem:

$$(VOP_v^s)\begin{cases} \max v_e(F(x)) \\ s.t.\ x \in X \end{cases} \quad \left(\begin{cases} \min v_e(F(x)) \\ s.t.\ x \in X \end{cases}\right).$$

Proof Let x_0 be an s-maximal solution of $(s - SOP)$. If $F(x) \in [F(x_0)]^s$, then $F(x) \preceq^s F(x_0)$. If $F(x) \notin [F(x_0)]^s$, then from s-maximality of x_0 and (8) we have $F(x) \preceq^s F(x_0)$. Thus, we obtain $F(x) \preceq^s F(x_0)$ for all $x \in X$. Since $v_e(\cdot)$ is s-increasing, we get $v_e(F(x)) \leq_{\mathbb{R}^2_+} v_e(F(x_0))$ for all $x \in X$. Therefore, x_0 is a strong solution of (VOP_v^s). □

The following example shows that the condition (8) is necessary in Corollary 4.5.

Example 4.6 *Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $X = \{1, 2\}$, $A = [1, 2] \times \{1\}$, $B = \{(\frac{3}{2}, 2)\}$, $F : X \rightrightarrows Y$ be defined as $F(1) = A$, $F(2) = B$. Consider the problem*

$$(s - SOP)\begin{cases} \max F(x) \\ s.t.\ x \in \{1, 2\}. \end{cases}$$

As seen in Figure 6, $A \not\preceq^s B$ and $B \not\preceq^s A$, i.e. (8) is not satisfied for this problem. Since $A \not\preceq^s B$ and $B \not\preceq^s A$, solutions of $(s - SOP)$ are 1 and 2.

Let us choose $e = (-1, -1)$. We have

$$v_e(F(1)) = (-G_e^\ell(\{0\}, F(1)), G_e^u(F(1), \{0\})) = (1, 1)$$

and

$$v_e(F(2)) = (-G_e^\ell(\{0\}, F(2)), G_e^u(F(2), \{0\})) = \left(\frac{3}{2}, \frac{3}{2}\right).$$

However, the unique solution of the problem

$$(VOP_v^s)\begin{cases} \max v_e(F(x)) \\ s.t.\ x \in X. \end{cases}$$

is $x_0 = 2$. $x_1 = 1$ is a solution of $(s - SOP)$, but it cannot be obtained by Gerstewitz vectorization.

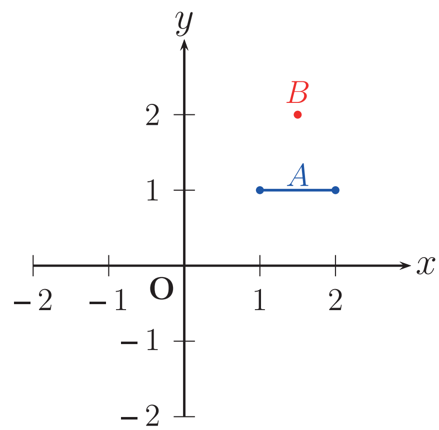


Figure 6. Image sets of F.

Corollary 4.7 *Let $F : X \rightrightarrows Y$ be $\mp C$-compact valued on X and*

$$F(x) \prec^s F(y) \text{ or } F(y) \prec^s F(x) \quad \text{for all } x, y \in X. \tag{9}$$

If $x_0 \in X$ is a weak s-maximal (weak s-minimal) solution of $(s - SOP)$, then it is also a strong solution of the problem

$$(VOP_v^s) \begin{cases} \max v_e(F(x)) \\ s.t\ x \in X \end{cases} \quad \left(\begin{cases} \min v_e(F(x)) \\ s.t.\ x \in X \end{cases} \right).$$

Proof It can be proved using strict monotonicity of $v_e(\cdot)$. □

Theorem 4.8 *Let $F : X \rightrightarrows Y$ be $\mp C$-closed, $\mp C$-bounded valued on X, and $v_e(F(x)) \neq v_e(F(y))$ for all $x, y \in X$, $x \neq y$. If $x_0 \in X$ is a maximal (minimal) solution of (VOP_v^s), then it is also an s-maximal (s-minimal) solution of $(s - SOP)$.*

Proof Let x_0 be a maximal solution of (VOP_v^s). Then we have $v_e(F(x_0)) \not\leq_{\mathbb{R}^2_+} v_e(F(x))$ for all $x \in X$ such that $F(x) \notin [F(x_0)]^s$. As $v_e(\cdot)$ is s-increasing, we get $G_e^\ell(\{0\}, F(x)) \not\leq G_e^\ell(\{0\}, F(x_0))$ or $G_e^u(F(x_0), \{0\}) \not\leq G_e^u(F(x), \{0\})$. Since $G_e^\ell(\{0\}, \cdot)$ and $G_e^u(\cdot, \{0\})$ are ℓ-decreasing, we have $F(x_0) \not\preceq^\ell F(x)$ or $F(x_0) \not\preceq^u F(x)$ from (1), respectively. Hence, we obtain $F(x_0) \not\preceq^s F(x)$ for all $x \in X$ such that $F(x) \notin [F(x_0)]^s$. Therefore, x_0 is an s-maximal solution of $(s - SOP)$. □

We construct Gerstewitz vectorization for $(s - SOP)$ given in [11, Example 3.1] with a convex objective map in the following example.

Example 4.9 *Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : [-1, 1] \rightrightarrows Y$ be defined as*

$$F(x) := \left\{ (y_1, y_2) \in \mathbb{R}^2 \mid (y_1 - 2x^2)^2 + (y_2 - 2x^2)^2 \leq (x^2 + 1)^2 \right\}$$

for all $x \in [-1, 1]$. Consider

$$(s-SOP)\begin{cases}\min F(x)\\ s.t.\ x\in[-1,1].\end{cases}$$

Because $F(x)=F(-x)$ *for all* $x\in[-1,1]$, *we consider the problem*

$$(s-SOP)\begin{cases}\min F(x)\\ s.t.\ x\in[0,1].\end{cases}$$

Let us choose $e=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ *and consider the problem*

$$(VOP_v^s)\begin{cases}\min v_e(F(x))\\ s.t.\ x\in[0,1].\end{cases}$$

As seen in Figure 1 and Figure 7, $F:X\rightrightarrows Y$ *is* $\mp C$*-closed,* $\mp C$*-bounded valued on* X, *and* $v_e(F(x))\neq v_e(F(y))$ *for all* $x\neq y$ *and* $x,y\in[0,1]$. *From (6) we have*

$$v_e(F(x))=\left(\sqrt{2}(x^2-1),2\sqrt{2}x^2+x^2+1\right).$$

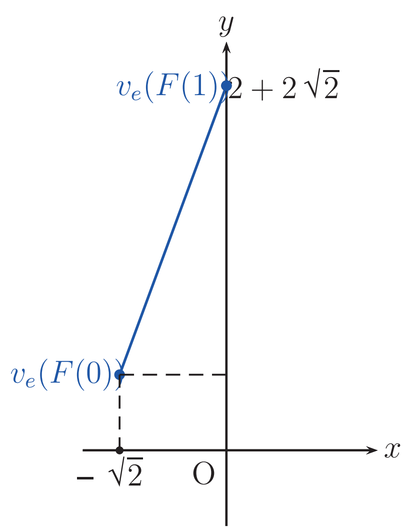


Figure 7. Image set of $v_e(F(x))$.

Moreover, there is not any $x\in(0,1]$ *such that*

$$v_e(F(x))\leq_{\mathbb{R}^2_+}v_e(F(0)).$$

Thus, $x_0=0$ *is the minimal solution of* (VOP_v^s). *Therefore,* $x_0=0$ *is a solution of* $(s-SOP)$ *by Theorem 4.8.*

Now we construct Gerstewitz vectorization for a nonconvex $(s-SOP)$.

Example 4.10 *Let* $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, *and* $F : [0,2] \rightrightarrows Y$ *be defined as*

$$F(x) := \begin{cases} ([x-2,x] \times [x-2,x]) \cup \{(6+x, 6+x)\} & ; x \in [0,2) \\ \operatorname{conv}\{(-5,0),(6,6)\} \cup \operatorname{conv}\{(0,-5),(6,6)\} & ; x = 2. \end{cases}$$

Consider the problem

$$(s-SOP) \begin{cases} \min F(x) \\ s.t.\ x \in [0,2]. \end{cases}$$

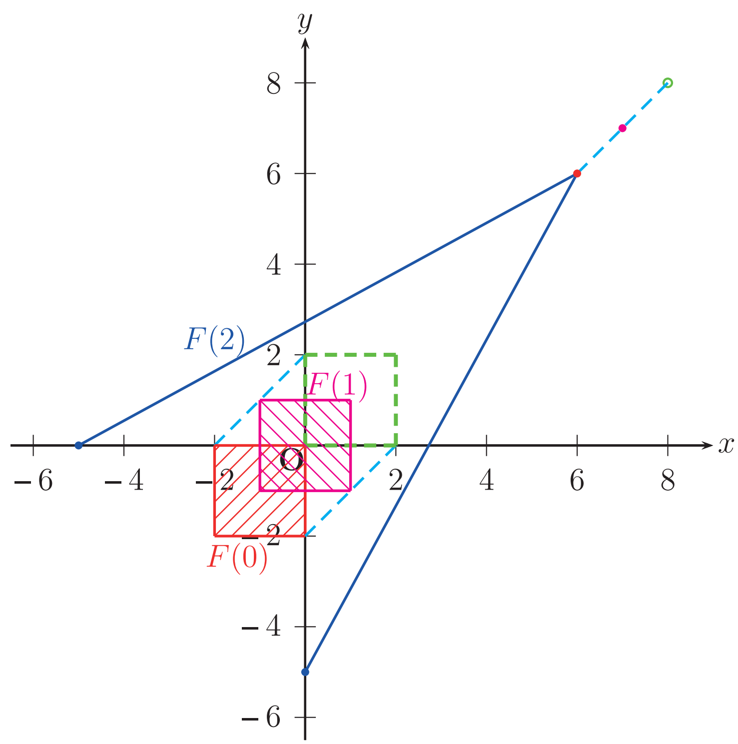


Figure 8. Some image sets of F.

Some image sets of F *are given in Figure 8. Since* $F(x) \npreceq^{\ell} F(2)$, *we have* $F(x) \npreceq^{s} F(2)$ *for all* $x \in [0,2)$. *Then* $x_0 = 2$ *is an* s*-minimal solution of* $(s-SOP)$. *As* $F(x) \npreceq^{\ell} F(0)$, *we get* $F(x) \npreceq^{s} F(0)$ *for all* $x \in (0,2]$. *Hence,* $x_0 = 0$ *is an* s*-minimal solution of* $(s-SOP)$. *Let us choose* $x \in (0,2)$. *We get* $F(0) \preceq^{s} F(x)$ *and* $F(x) \npreceq^{s} F(0)$. *Thus,* x *is not an* s*-minimal solution of* $(s-SOP)$. *Therefore, solutions of* $(s-SOP)$ *are 0 and 2.*

Since $F(2) + C$ *is not convex, vectorization in [11] could not be applied to this problem. However, we can solve this problem via Gerstewitz vectorization.*

Now we show that $x_0 = 2$ *is a solution of this problem by using Theorem 4.3.*

Let us choose $e = (-1,-1)$ *and consider the problem*

$$(VOP^s_w) \begin{cases} \max w_e(F(x), F(2)) \\ s.t.\ x \in [0,2]. \end{cases}$$

We get

$$\begin{aligned} w_e(F(x), F(2)) &= (-G_e^{\ell}(F(x), F(2)), G_e^u(F(2), F(x))) \\ &= \begin{cases} (-3-x, -x) & ; x \neq 2 \\ (0,0) & ; x = 2. \end{cases} \end{aligned}$$

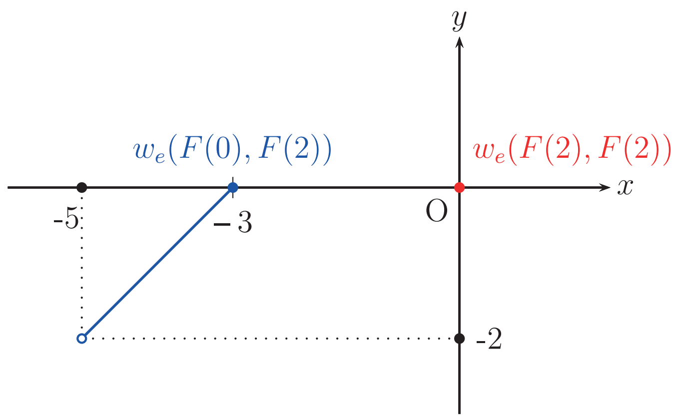


Figure 9. Image set of $w_e(F(x), F(2))$.

As seen in Figure 9 there is not any $x \in [0, 2)$ *such that*

$$w_e(F(2), F(2)) \leq_{\mathbb{R}^2_+} w_e(F(x), F(2)).$$

Thus, $x_0 = 2$ *is the solution of* (VOP_w^s)*. Therefore,* $x_0 = 2$ *is a solution of* $(s - SOP)$ *by Theorem 4.3.*

$x_0 = 0$ *is also a solution of* $(s - SOP)$*. It can be shown similarly via Gerstewitz vectorization.*

5. Conclusion

In this study, our aim is to replace a nonconvex set-valued optimization problem with respect to the set less order relation with a vector optimization problem via the Gerstewitz vectorizing function. This can allow us to use known solution techniques such as scalarization, duality, and derivative in vector optimization to solve nonconvex set-valued optimization problems. For further studies, one can investigate the usage of these techniques in set-valued optimization via different vectorizations.

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