Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1833 – 1844

doi:10.3906/mat-1710-3

On n-absorbing δ-primary ideals

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Received: 02.10.2017 • **Accepted/Published Online:** 29.03.2018 • **Final Version:** 24.07.2018

Abstract: Let R be a commutative ring with nonzero identity and n be a positive integer. In this paper, we study the concepts of n-absorbing δ-primary ideals and weakly n-absorbing δ-primary ideals, which are the generalizations of δ-primary ideals and weakly δ-primary ideals, respectively. We introduce the concepts of n-absorbing δ-primary ideals and weakly n-absorbing δ-primary ideals. Moreover, we give many properties of these new types of ideals and investigate the relations between these structures.

Key words: 2-absorbing ideal, δ-primary ideal, weakly n-absorbing δ-primary ideal

1. Introduction

Throughout this paper, we assume that all rings are commutative with nonzero identity. Let R be a commutative ring and I be an ideal of R. An ideal I is called proper if $I \neq R$. Recall that a proper ideal I is called a 2-absorbing (primary) ideal if $x_1x_2x_3 \in I$ for some $x_1, x_2, x_3 \in R$; then $x_1x_2 \in I$ or $x_2x_3 \in I$ or $x_1x_3 \in I$ ($x_1x_2 \in I$ or $x_2x_3 \in \sqrt{I}$ or $x_1x_3 \in \sqrt{I}$). These concepts were introduced by Badawi, Yetkin, and Tekir in [3] and [6]. Later, many authors studied on this issue. (see [11] and [1]). A proper ideal I of R is said to be weakly 2-absorbing (primary) ideal if $0 \neq x_1x_2x_3 \in I$ for some $x_1, x_2, x_3 \in R$; then $x_1x_2 \in I$ or $x_2x_3 \in I$ or $x_1x_3 \in I$ ($x_1x_2 \in I$ or $x_2x_3 \in \sqrt{I}$ or $x_1x_3 \in \sqrt{I}$). These notions were introduced as generalizations of weakly prime ideals and weakly primary ideals in [4] and [7], respectively. In the same manner, the concepts of n-absorbing (primary) ideals were introduced as other generalizations of prime (primary) ideals in [2]. Afterwards, Darani et al. studied the concept of weakly n-absorbing ideals in [10].

Let $\mathcal{I}(\mathcal{R})$ be the set of all ideals of R and $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ be a function of $\mathcal{I}(\mathcal{R})$. Then δ is called an expansion function of $\mathcal{I}(\mathcal{R})$ if it satisfies the following two conditions: 1. $I \subseteq \delta(I)$, 2. If $I \subseteq J$, then $\delta(I) \subseteq \delta(J)$ for any ideals I, J of R. In [8], Zhao introduced a new concept called δ-primary ideals in commutative rings. This concept is considered to unify prime and primary ideals. Many results of prime and primary ideals are extended to these structures. Recall that a proper ideal I is called a δ-primary ideal if $xy \in I$ for some $x, y \in R$ implies that $x \in I$ or $y \in \delta(I)$. Then Zhao and Fahid introduced the concept of 2-absorbing δ-primary ideal, which is a generalization of δ-primary ideal, that is, the concept of δ-primary ideal has been extended to 2-absorbing δ-primary ideal [9]. Recall that a proper ideal I is called a 2-absorbing δ-primary ideal if $xyz \in I$ for some $x, y, z \in R$ implies that $xy \in I$ or $yz \in \delta(I)$ or $xz \in \delta(I)$. Afterwards, Badawi and Fahid

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2010 *AMS Mathematics Subject Classification:* Primary 05C38, 15A15; Secondary 05A15, 15A18

studied weakly 2-absorbing δ-primary ideals of commutative rings in [5]. Firstly, they introduced the concept of a weakly δ-primary ideal and then gave the concept of a weakly 2-absorbing δ-primary ideal. Additionally, they investigated many properties of these concepts and studied the relations between a δ-primary ideal and a 2-absorbing δ-primary ideal. A proper ideal I is said to be a weakly δ-primary ideal if $0 \neq xy \in I$ for some $x, y \in R$ implies that $x \in I$ or $y \in \delta(I)$. A proper ideal I is called a weakly 2-absorbing δ-primary ideal if $0 \neq xyz \in I$ for some $x, y, z \in R$ implies $xy \in I$ or $yz \in \delta(I)$ or $xz \in \delta(I)$.

In this paper, our aim is to introduce the concepts of n-absorbing δ-primary ideals and weakly n-absorbing δ-primary ideals. These types are two generalizations of the concepts of n-absorbing (primary) ideals and weakly n-absorbing (primary) ideals, respectively. We say a proper ideal I of R is (weakly) n-absorbing δ-primary ideal if whenever $(0 \neq x_1...x_{n+1} \in I)$ $x_1...x_{n+1} \in I$ for some $x_1, ..., x_{n+1} \in R$ implies $x_1...x_n \in I$ or there exists $1 \leq k \leq n$ such that $x_1...\widehat{x_k}...x_{n+1} \in \delta(I)$, where $x_1...\widehat{x_k}...x_{n+1}$ denotes the product of $x_1...x_{k-1}x_{k+1}...x_{n+1}$.

In this paper, we give many specific examples and results of these concepts. Let δ and γ be expansion functions of $\mathcal{I}(\mathcal{R})$. One of the significant results in this paper is that if $\delta(I) \subseteq \gamma(I)$ and I is an (weakly) n-absorbing δ-primary ideal, then I is an (weakly) n-absorbing γ-primary ideal. Then we show that every (weakly) n-absorbing δ-primary ideal is an (weakly) m-absorbing δ-primary ideal for positive integers m, n with $m > n$. It is given that if I is an (weakly) n-absorbing δ-primary ideal with $J \subseteq K \subseteq I$ and $\delta(I) = \delta(J)$ for some ideals J, K of R, then K is (weakly) m-absorbing δ-primary ideal for positive integers $m > n$. We also show that if I is a weakly n-absorbing δ-primary ideal of R but is not an n-absorbing δ-primary ideal, then $I^{n+1} = (0)$. Let S be a multiplicatively closed subset of R and δ_S be an expansion function of $\mathcal{I}(\mathcal{R}_S)$ such that $\delta_S(I_S) = (\delta(I))_S$, where $\mathcal{R}_S$ is the quotient ring of R. Let $S \cap Z(R) = \emptyset$, where $Z(R)$ is the set of all zero divisor elements of R. It is also given that if I is an (weakly) n-absorbing δ-primary ideal of R with $I \cap S = \emptyset$, then I_S is an (weakly) n-absorbing δ_S-primary ideal of R_S.

Let $R = R_1 \times ... \times R_n$, where R_i is a commutative ring with nonzero identity and δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for each $i \in \{1, 2, ..., n\}$. Let $\delta_\times$ be a function of $\mathcal{I}(\mathcal{R})$, which is defined by $\delta_\times(I_1 \times I_2 \times ... \times I_n) = \delta_1(I_1) \times \delta_2(I_2) \times ... \times \delta_n(I_n)$ for each ideal I_i of R_i. Then $\delta_\times$ is an expansion function of $\mathcal{I}(\mathcal{R})$. Finally, from Theorem 10 to Theorem 13, we characterize all (weakly) n-absorbing $\delta_\times$-primary ideals of direct product of rings.

2. n-Absorbing δ-primary and weakly n-absorbing δ-primary ideals

Throughout this section, R denotes a commutative ring with nonzero identity, unless otherwise stated.

Definition 1 *Let $\mathcal{I}(\mathcal{R})$ be the set of all ideals of R and $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ be a function of ideals of R. Recall from [8], δ is called an expansion function of $\mathcal{I}(\mathcal{R})$ if it satisfies the following two conditions: (1) $I \subseteq \delta(I)$, (2) If $I \subseteq J$, then $\delta(I) \subseteq \delta(J)$ for any ideals I, J of R.*

Note that there are explanatory examples of expansion functions included in [8, 1.2 Example] and [5, Example 1].

Definition 2 *A proper ideal I of a commutative ring R is called an (weakly) n-absorbing δ-primary ideal if whenever $(0 \neq x_1...x_{n+1} \in I)$ $x_1...x_{n+1} \in I$ for some $x_1, ..., x_{n+1} \in R$, then $x_1...x_n \in I$ or there exists $1 \leq k \leq n$ such that $x_1...\widehat{x_k}...x_{n+1} \in \delta(I)$, where $x_1...\widehat{x_k}...x_{n+1}$ denotes the product of $x_1...x_{k-1}x_{k+1}...x_{n+1}$.*

It is clear that any n-absorbing δ-primary ideal is weakly n-absorbing δ-primary. The following example not only shows that the converse is not true but also gives many illustration of n-absorbing δ-primary ideals.

Example 1 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$.*

(i) If $\delta_i(I) = I$, i.e. δ_i is an identity function, then n-absorbing ideals are equivalent n-absorbing δ_i-primary ideals.

(ii) If $\delta_r(I) = \sqrt{I}$, then I is an n-absorbing δ_r-primary ideal iff I is an n-absorbing primary ideal.

(iii) Every (weakly) 2-absorbing δ-primary ideal is an (weakly) n-absorbing δ-primary ideal.

(iv) Every n-absorbing ideal is an n-absorbing δ-primary ideal, but the converse is not necessarily true. Consider the ring of integers $\mathbb{Z}$ and the expansion function δ_r of $\mathbb{Z}$. Let $I = (p_1^2p_2^2p_3^3...p_n^n)$, where p_i's are distinct prime numbers. Then I is an n-absorbing δ_r-primary ideal of $\mathbb{Z}$ but not an n-absorbing ideal of $\mathbb{Z}$.

(v) Now consider the ring $\mathbb{Z}_m$, where $m = p_1p_2...p_{n+1}$ for some distinct prime numbers $p_1, ..., p_{n+1}$. Then $I = (0)$, the zero ideal, is clearly a weakly n-absorbing δ_r-primary ideal of $\mathbb{Z}_m$. Since $p_1p_2...p_{n+1} \in I$, $p_1p_2...p_n \notin I$ and for each $1 \leq k \leq n$, none of the product of $p_1...\widehat{p_k}...p_{n+1}$ is in $\delta_r(I) = I$. Thus I is not an n-absorbing δ_r-primary ideal of $\mathbb{Z}_m$.

An n-absorbing primary ideal may or may not be an n-absorbing δ-primary ideal as in Example 1 (i). Additionally, an n-absorbing δ-primary ideal is not necessarily an n-absorbing primary ideal. Consider the ring of formal power series $R = F[[X_1, X_2, ..., X_{n+1}]]$, where F is a field. Let us define $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ as $\delta(I) = I + M$ for each ideal I of R, where M is the unique maximal ideal $(X_1, X_2, ..., X_{n+1})$. Then δ is an expansion function of $\mathcal{I}(\mathcal{R})$. Take an ideal $I = (X_1X_2...X_{n+1})$. Then $\sqrt{I} = I$ and I is not an n-absorbing ideal and so it is not an n-absorbing primary ideal. Let $p_1, p_2, ..., p_{n+1} \in R$ such that $p_1p_2...p_{n+1} \in I$. Assume that for some $1 \leq k \leq n$ such that $p_1...\widehat{p_k}...p_{n+1} \notin \delta(I) = M$. Then $p_1...\widehat{p_k}...p_{n+1}$ is a unit of R. Since $p_1p_2...p_{n+1} = (p_1...\widehat{p_k}...p_{n+1})p_k \in I$, we have $p_k \in I$ and so $p_1p_2...p_n \in I$. Thus I is an n-absorbing δ-primary ideal of R.

Theorem 1 *(i) Let δ and γ be expansion functions of $\mathcal{I}(\mathcal{R})$ with $\delta(I) \subseteq \gamma(I)$. If I is an (weakly) n-absorbing δ-primary ideal of R, then I is an (weakly) n-absorbing γ-primary ideal of R.*

(ii) Let γ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I be an n-absorbing primary ideal of R. If $\gamma(I)$ is a radical ideal, i.e. $\sqrt{\gamma(I)} = \gamma(I)$, then I is an n-absorbing γ-primary ideal of R.

Proof (i) It is explicit.

(ii) It can be easily seen that $\sqrt{I} \subseteq \sqrt{\gamma(I)} = \gamma(I)$. Then, by (i), I is an n-absorbing γ-primary ideal of R if I is an n-absorbing primary ideal of R. □

Proposition 1 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$. If $\delta(I)$ is an $(n-1)$-absorbing ideal of R, then I is an n-absorbing δ-primary ideal of R.*

Proof Let $x_1...x_{n+1} \in I$ and $x_1...x_n \notin I$ for some $x_1, ..., x_{n+1} \in R$. Now we have two cases. In the first case, assume that $x_1...x_n \notin \delta(I)$. Since $\delta(I)$ is an $(n-1)$-absorbing ideal and $(x_1x_2)x_3...x_{n+1} \in \delta(I)$, we get $(x_1x_2)...\widehat{x_k}...x_{n+1} \in \delta(I)$ for some $1 \leq k \leq n$. In the second case, assume that $x_1...x_n \in \delta(I)$. This implies that $x_1x_2...x_{n-1} \in \delta(I)$ or $x_1...\widehat{x_k}...x_n \in \delta(I)$ for some $1 \leq k \leq n-1$. Thus, we have $x_1...\widehat{x_n}...x_{n+1} \in \delta(I)$ or $x_1...\widehat{x_k}...x_{n+1} \in \delta(I)$ for some $1 \leq k \leq n-1$, which completes the proof. □

Theorem 2 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$. Every (weakly) n-absorbing δ-primary ideal of R is an (weakly) m-absorbing δ-primary ideal of R for positive integers m, n with $m > n$.*

Proof Let I be an n-absorbing δ-primary ideal of R. We will show that I is an $(n+1)$-absorbing δ-primary ideal. Let $x_1x_2...x_{n+2} \in I$ for some $x_{1,}x_2,...,x_{n+2} \in R$. Now take $x_1x_2 = x'$. Then $x'...x_{n+2} \in I$ implies $x'...x_{n+1} \in I$ or $x'...\widehat{x_k}...x_{n+2}$ is in $\delta(I)$ for $x_k = x'$ or some $3 \leq k \leq n+1$. Hence, I is an m-absorbing δ-primary ideal of R for $m > n$. Similarly, it can be verified that a weakly n-absorbing δ-primary ideal is a weakly m-absorbing δ-primary ideal. □

Definition 3 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$. It satisfies the finite intersection property if $\delta(I_1 \cap ... \cap I_n) = \delta(I_1) \cap ... \cap \delta(I_n)$ for some ideals $I_1, ..., I_n$ of R.*

Note that the radical operation on ideals of a commutative ring is an example of an expansion function satisfying the finite intersection property.

Proposition 2 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ satisfying the finite intersection property and $I_1, ..., I_m$ be proper ideals of R. If I_j is an n_j-absorbing δ-primary ideal and $P = \delta(I_j)$ for all $j \in \{1, ..., m\}$, then $I_1 \cap ... \cap I_m$ is an n-absorbing δ-primary with $n_1 + ... + n_m = n$.*

Proof Assume that $x_1...x_{n+1} \in I_1 \cap ... \cap I_m$ and $x_1...x_n \notin I_1 \cap ... \cap I_m$ for some $x_1, ..., x_{n+1} \in R$. Then $x_1...x_n \notin I_k$ for some $1 \leq k \leq m$. Since I_k is an n_k-absorbing δ-primary ideal, then I_k is an n-absorbing δ-primary ideal by Theorem 2 and so $x_1...\widehat{x_t}...x_{n+1} \in \delta(I_k) = P$ for some $1 \leq t \leq n$. Also note that $\delta(I_1 \cap ... \cap I_m) = \delta(I_1) \cap ... \cap \delta(I_m) = P$ since $\delta(I_j) = P$ for all $1 \leq j \leq m$ and δ satisfies the finite intersection property. Thus $I_1 \cap ... \cap I_m$ is n-absorbing δ-primary. □

Theorem 3 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I, J, and K be proper ideals of R with $J \subseteq K \subseteq I$ and $\delta(I) = \delta(J)$. If I is (weakly) an n-absorbing δ-primary ideal, then K is an (weakly) m-absorbing δ-primary ideal for positive integers $m > n$.*

Proof We will show that if I is an n-absorbing δ-primary ideal of R, K is an $(n+1)$-absorbing δ-primary ideal of R. Assume that $n = 1$. Let $x_1x_2x_3 \in K$ and $x_1x_2 \notin K$. In the first case, suppose that $x_1x_2 \in I$. Then $x_1 \in I$ or $x_2 \in \delta(I)$. Thus $x_1x_3 \in I$ or $x_2x_3 \in \delta(K)$ since $\delta(I) = \delta(J) \subseteq \delta(K)$. This implies that $x_1x_3 \in \delta(K)$ or $x_2x_3 \in \delta(K)$. In the second case, let $x_1x_2 \notin I$. Then $x_3 \in \delta(I)$ and hence $x_1x_3 \in \delta(K)$ and $x_2x_3 \in \delta(K)$. Consequently, K is a 2-absorbing δ-primary ideal of R. Assume that if I is a k-absorbing δ-primary ideal, K is a $(k+1)$-absorbing δ-primary ideal for some positive integer k. Now we show that K is a $(k+2)$-absorbing δ-primary ideal when I is a $(k+1)$-absorbing δ-primary ideal for some positive integer k. Let I be a $(k+1)$-absorbing δ-primary ideal of R. Let $x_1...x_{k+3} \in K$ and $x_1...x_{k+2} \notin K$. In the first case, let $x_1...x_{k+2} \in I$. Then $x_1...x_{k+1} \in I$ or there exists $1 \leq t \leq k+1$ such that $x_1...\widehat{x_t}...x_{k+2}$ is in $\delta(I)$. This yields that $x_1...\widehat{x_{k+2}}...x_{k+3}$ is in $\delta(K)$ or $x_1...\widehat{x_t}...x_{k+3}$ for some $1 \leq t \leq k+1$. In the second case, let $x_1...x_{k+2} \notin I$. Since I is a $(k+1)$-absorbing δ-primary ideal, we get $x_1...\widehat{x_t}...x_{k+3}$ is in $\delta(I) = \delta(K)$ for some $1 \leq t \leq k+2$. Consequently, K is a $(k+2)$-absorbing δ-primary ideal. Similarly, it can be verified for a weakly n-absorbing δ-primary ideal. □

Corollary 1 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I, J be proper ideals of R with $J \subseteq I$ and $\delta(I) = \delta(J)$. Then J is an (weakly) m-absorbing δ-primary ideal in the case I is an (weakly) n-absorbing δ-primary ideal for some positive integers $m > n$.*

Definition 4 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$, I be a weakly n-absorbing δ-primary ideal of R, and $x_1, ..., x_{n+1} \in R$. We say that $(x_1, ..., x_{n+1})$ is a δ-$(n+1)$-tuple-zero of I if $x_1...x_{n+1} = 0$, $x_1...x_n \notin I$ and for each $1 \leq k \leq n$, $x_1...\widehat{x_k}...x_{n+1}$ is not in $\delta(I)$.*

Note that if I is a weakly n-absorbing δ-primary ideal of R that is not an n-absorbing δ-primary ideal, then I has a δ-$(n+1)$-tuple-zero $(x_1, ..., x_{n+1})$ for some $x_1, ..., x_{n+1} \in R$.

Theorem 4 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I be a weakly n-absorbing δ-primary ideal of R. Assume that $(x_1, ..., x_{n+1})$ is a δ-$(n+1)$-tuple-zero of I for some $x_1, ..., x_{n+1} \in R$. Then*

$$x_1...\widehat{x_{i_1}}\widehat{x_{i_2}}...\widehat{x_{i_{m-1}}}\widehat{x_{i_m}}...x_{n+1}I^m = (0)$$

for each $1 \leq i_1, ..., i_m \leq n+1$, $1 \leq m \leq n$.

Proof Let $m = 1$. Assume that $x_1...\widehat{x_{i_1}}...x_{n+1}I \neq (0)$. Then $x_1...\widehat{x_{i_1}}...x_{n+1}y \neq 0$ for some $y \in I$. This yields that $x_1...\widehat{x_{i_1}}...x_{n+1}(x_{i_1} + y) \neq 0$. Since $(x_1, ..., x_{n+1})$ is a δ-$(n+1)$-tuple-zero and I is a weakly n-absorbing δ-primary ideal of R, we conclude that $x_1...\widehat{x_{i_1}}...\widehat{x_j}...x_{n+1}(x_{i_1} + y) \in \delta(I)$ for some $1 \leq j \leq n+1$ and $j \neq i_1$. Thus $x_1...\widehat{x_j}...x_{n+1} \in \delta(I)$, yielding a contradiction. Therefore, it must be $x_1...\widehat{x_{i_1}}...x_{n+1}I = (0)$.

Assume that the claim holds for all positive integers less than $m > 1$. Let $x_1...\widehat{x_{i_1}}\widehat{x_{i_2}}...\widehat{x_{i_{m-1}}}\widehat{x_{i_m}}...x_{n+1}I^m \neq (0)$. Then there are elements $y_1, ..., y_m$ of I such that $x_1...\widehat{x_{i_1}}\widehat{x_{i_2}}...\widehat{x_{i_{m-1}}}\widehat{x_{i_m}}...x_{n+1}y_1...y_m \neq 0$. By hypothesis, we have

$$\begin{aligned} &x_1...\widehat{x_{i_1}}\widehat{x_{i_2}}...\widehat{x_{i_{m-1}}}\widehat{x_{i_m}}...x_{n+1}(x_{i_1} + y_1)(x_{i_2} + y_2)...(x_{i_m} + y_m) \\ &= x_1...x_{i-1}...x_{i+m+1}...x_{n+1}y_1...y_m \neq 0. \end{aligned}$$

Since I is a weakly n-absorbing δ-primary ideal, without loss of generality, we may assume that

$$x_1...\widehat{x_{i_1}}\widehat{x_{i_2}}...\widehat{x_{i_{m-1}}}\widehat{x_{i_m}}...x_{n+1}(x_{i_1} + y_1)...(\widehat{x_{i_t} + y_t})...(x_{i_m} + y_m) \in \delta(I)$$

for some $1 \leq t \leq m$. Since $y_1, ..., y_m$ of I, we get $x_1...\widehat{x_{i_t}}...x_{n+1} \in \delta(I)$, which is a contradiction. Consequently, it must be

$$x_1...\widehat{x_{i_1}}\widehat{x_{i_2}}...\widehat{x_{i_{m-1}}}\widehat{x_{i_m}}...x_{n+1}I^m = (0).$$

□

In the following theorem, Nakayama's lemma is considered for weakly n-absorbing δ-primary ideals.

Theorem 5 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I be a weakly n-absorbing δ-primary ideal of R but it is not an n-absorbing δ-primary ideal. Then $I^{n+1} = (0)$.*

Proof By our assumption, I has a δ-$(n+1)$-tuple-zero $(x_1,...,x_{n+1})$ for some $x_1,...,x_{n+1} \in R$. Let $0 \neq y_1...y_{n+1}$ for some $y_1,...,y_{n+1} \in I$. By Theorem 4, we have $(x_1+y_1)...(x_{n+1}+y_{n+1}) = y_1...y_{n+1} \neq 0$ and $(x_1+y_1)...(x_{n+1}+y_{n+1}) \in I$. Thus we conclude that $(x_1+y_1)...(x_n+y_n) \in I$ or $(x_1+y_1)...(\widehat{x_k+y_k})...(x_{n+1}+y_{n+1}) \in \delta(I)$ for some $k \in \{1,...,n\}$. Therefore, we have $x_1...x_n \in I$ or $x_1...\widehat{x_k}...x_{n+1} \in \delta(I)$, a contradiction. Consequently, $I^{n+1} = (0)$. □

We give the next theorem as a result of Theorem 5.

Theorem 6 *Let δ be an expansion function of $\mathcal{I(R)}$ and I be a weakly n-absorbing δ-primary ideal of R but it is not an n-absorbing δ-primary ideal. Thus,*

1. *$Rad(I) = Nil(R)$.*
2. *If M is a finitely generated R-module with $IM = M$, then $M = (0)$.*

Proof The proof is clear from Theorem 5. □

In Theorem 5, the condition $I^{n+1} = (0)$ does not assure that I is a weakly n-absorbing δ-primary ideal. We give an example for this case:

Example 2 *Let $R = \mathbb{Z}_{p^{n+2}}$ for some prime number p and nonnegative integer n. Consider the expansion function δ_i, which is defined in Example 1. Then $I = (p^{n+1})$ is a proper ideal of R and $I^{n+1} = (0)$, but I is not weakly n-absorbing δ-primary since $p^{n+1} \in I$ and $p^n \notin \delta_i(I)$.*

Corollary 2 *Let δ be an expansion function of $\mathcal{I(R)}$.*

(i) If I is a proper ideal of R with $\delta(\delta(I)) = \delta(I)$, then $\delta(I)$ is an n-absorbing ideal of R if and only if $\delta(I)$ is an n-absorbing δ-primary ideal of R.

(ii) Suppose that $\delta(0)$ is an n-absorbing δ-primary ideal of R with $\delta(\delta(0)) = \delta(0)$. Then $\delta(0)$ is an n-absorbing ideal of R.

Proof (i) The necessary part is clear. For the sufficient part, assume that $x_1...x_{n+1} \in \delta(I)$ and $x_1...x_n \notin \delta(I)$ for some $x_1,...,x_{n+1} \in R$. Since $\delta(I)$ is an n-absorbing δ-primary ideal, then we have $x_1...\widehat{x_k}...x_{n+1} \in \delta(\delta(I)) = \delta(I)$ for some $1 \leq k \leq n$. Hence $\delta(I)$ is an n-absorbing ideal.

(ii) Follows similar to (i). □

Definition 5 *Let $f : R \to S$ be a ring homomorphism and δ, γ expansion functions of $\mathcal{I(R)}$ and $\mathcal{I(S)}$, respectively. Then f is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$ for all ideals J of S.*

If we consider that γ_r is a radical operation on S and δ_r is a radical operation on R, then any homomorphsim from R to S is an example of $\delta_r\gamma_r$-homomorphism. Also note that if f is a $\delta\gamma$-epimorphism and I is an ideal of R containing $\ker(f)$, then $\gamma(f(I)) = f(\delta(I))$.

Theorem 7 *Let $f : R \to S$ be a $\delta\gamma$-homomorphism, where δ is an expansion function of $\mathcal{I(R)}$ and γ is an expansion function of $\mathcal{I(S)}$. Then the following are satisfied:*

(i) If J is an n-absorbing γ-primary ideal of S, then $f^{-1}(J)$ is an n-absorbing δ-primary ideal of R.

(ii) Suppose that f is an epimorphism and I is a proper ideal of R with $\ker(f) \subseteq I$. Then I is an n-absorbing δ-primary ideal of R if and only if $f(I)$ is an n-absorbing γ-primary ideal of S.

Proof (i) Let $x_1...x_{n+1} \in f^{-1}(J)$ for some $x_1,...,x_{n+1} \in R$. Then $f(x_1...x_{n+1}) = f(x_1)...f(x_{n+1}) \in J$. By our assumption, we have $f(x_1)...f(x_n) \in J$ or there exists $1 \leq k \leq n$ such that $f(x_1)...\widehat{f(x_k)}...f(x_{n+1})$ is in $\gamma(J)$. Thus $x_1...x_n \in f^{-1}(J)$ or $x_1...\widehat{x_k}...x_{n+1}$ is in $f^{-1}(\gamma(J))$. Since $\delta\left(f^{-1}(J)\right) = f^{-1}(\gamma(J))$, we get either $x_1...x_n \in f^{-1}(J)$ or $x_1...\widehat{x_k}...x_{n+1}$ is in $\delta\left(f^{-1}(J)\right)$. Therefore, $f^{-1}(J)$ is an n-absorbing δ-primary ideal of R.

(ii) Let $f(I)$ be an n-absorbing γ-primary ideal of S. Since $I = f^{-1}(f(I))$, we conclude that I is an n-absorbing δ-primary ideal of R by (i). Assume that I is an n-absorbing δ-primary ideal of R and $y_1y_2...y_{n+1} \in f(I)$ for some $y_1, y_2, ..., y_{n+1} \in S$. Since f is epimorphism, we have $f(x_i) = y_i$ for each $1 \leq i \leq n+1$. This implies that $f(x_1)f(x_2)...f(x_{n+1}) \in f(I)$ and so $x_1...x_{n+1} \in I$ since $\ker(f) \subseteq I$. As I is an n-absorbing δ-primary ideal, we conclude either $x_1...x_n \in I$ or there exists $1 \leq k \leq n$ such that $x_1...\widehat{x_k}...x_{n+1} \in \delta(I)$. Then we have $y_1...y_n \in f(I)$ or $y_1...\widehat{y_k}...y_{n+1} \in \gamma(f(I))$, which completes the proof. □

Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I an ideal of R. Then the function $\delta_q : R/I \to R/I$, defined by $\delta_q(J/I) = \delta(J)/I$ for all ideals $I \subseteq J$, becomes an expansion function of R/I.

Theorem 8 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I, J be proper ideals of R with $I \subseteq J$. Then the following hold:*

(i) J is an n-absorbing δ-primary ideal of R if and only if J/I is an n-absorbing δ_q-primary ideal of R/I.

(ii) If J is a weakly n-absorbing δ-primary ideal of R, then J/I is a weakly n-absorbing δ_q-primary ideal of R/I.

(iii) Let S be a multiplicatively closed subset of R and δ_S an expansion function of $\mathcal{I}(\mathcal{R}_S)$ such that $\delta_S(I_S) = (\delta(I))_S$. If I is an n-absorbing δ-primary ideal of R with $I \cap S = \emptyset$, then I_S is an n-absorbing δ_S-primary ideal of R_S. Moreover, if I is a weakly n-absorbing δ-primary ideal of R, then I_S is a weakly n-absorbing δ_S-primary ideal of R_S.

Proof (i) It is a result of Theorem 7.

(ii) Let $0_{R/I} \neq \overline{x_1}...\overline{x_{n+1}} \in J/I$ for some $\overline{x_1,...,x_{n+1}} \in R/I$. Then $x_1...x_{n+1} \in R - I$ and also $0 \neq x_1...x_{n+1} \in J$. Since J is weakly n-absorbing δ-primary, we conclude either $x_1...x_n \in J$ or there exists $1 \leq k \leq n$ such that $x_1...\widehat{x_k}...x_{n+1}$ is in $\delta(J)$. Hence $\overline{x_1}...\overline{x_n} \in J/I$ or $\overline{x_1}...\widehat{\overline{x_k}}...\overline{x_n}$ is in $\delta(J)/I = \delta_q(J/I)$, that is, J/I is a weakly n-absorbing δ_q-primary ideal of R/I.

(iii) Let $\frac{x_1}{s_1}...\frac{x_{n+1}}{s_{n+1}} \in I_S$ and $\frac{x_1}{s_1}...\frac{x_n}{s_n} \notin I_S$ for some $x_1,...,x_{n+1} \in R$ and $s_1,...,s_{n+1} \in S$. Then there exists $a \in S$ such that $ax_1...x_{n+1} = (ax_1)...x_{n+1} \in I$. Since I is an n-absorbing δ-primary, we obtain either $(ax_1)...x_n \in I$ or $(ax_1)...\widehat{x_k}...x_{n+1} \in \delta(I)$ for some $x_k = ax_1$ or $2 \leq k \leq n$. If $(ax_1)...x_n \in I$, then $\frac{x_1}{s_1}...\frac{x_n}{s_n} = \frac{ax_1...x_n}{as_1...s_n} \in I_S$. Otherwise, we would have $\frac{x_1}{s_1}...\widehat{\frac{x_k}{s_k}}...\frac{x_{n+1}}{s_{n+1}} = \frac{(ax_1)...\widehat{x_k}...x_{n+1}}{(as_1)...\widehat{s_k}..s_{n+1}} \in (\delta(I))_S = \delta_S(I_S)$ for some k. Therefore, I_S is n-absorbing δ_S-primary. In a similar way, it is easily shown that I_S is weakly n-absorbing δ_S-primary. □

In Theorem 8, the converse of (ii) holds if I is a weakly n-absorbing δ-primary ideal of R. The following theorem explains this situation.

Theorem 9 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$, and J be a proper ideal of R containing a weakly n-absorbing δ-primary ideal I of R. Then J/I is a weakly n-absorbing δ_q-primary ideal of R/I if and only if J is a weakly n-absorbing δ-primary ideal of R.*

Proof $\Leftarrow$: It is clear from Theorem 8 (ii).

$\Rightarrow$: It can be easily seen since I is weakly n-absorbing δ-primary. □

3. n-Absorbing δ-primary and weakly n-absorbing δ-primary ideals in direct product of rings

Theorem 10 *Let $R = R_1 \times ... \times R_m$ be a decomposable ring and*

$$I = I_1 \times ... \times I_{\alpha_1 - 1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times ... \times I_{\alpha_k - 1} \times R_{\alpha_k} \times I_{\alpha_k+1} \times ... \times I_m$$

be a proper ideal of R for $1 \leq \alpha_1, \alpha_2, ..., \alpha_k \leq m$. Then the following are equivalent:

(i) I is an n-absorbing $\delta_\times$-primary ideal of R.

(ii) I is a weakly n-absorbing $\delta_\times$-primary ideal of R.

(iii) $I' = I_1 \times ... \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times ... \times I_{\alpha_k-1} \times I_{\alpha_k+1} \times ... \times I_m$ is an n-absorbing $\delta_\times$-primary ideal of $R' = R_1 \times ... \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times ... \times R_{\alpha_k-1} \times R_{\alpha_k+1} \times ... \times R_m$.

Proof $(i) \Leftrightarrow (ii)$: Since $I^{n+1} \neq (0_R)$, then I is an n-absorbing $\delta_\times$-primary of R by Theorem 5.

$(i) \Rightarrow (iii)$: Let I be an n-absorbing $\delta_\times$-primary ideal of R.

Let $(x_1^{(1)}, ..., x_{(\alpha_1-1)}^{(1)}, x_{(\alpha_1+1)}^{(1)}, ..., x_{(\alpha_k-1)}^{(1)}, x_{(\alpha_k+1)}^{(1)}, ..., x_m^{(1)})...$
$(x_1^{(n+1)}, ..., x_{(\alpha_1-1)}^{(n+1)}, x_{(\alpha_1+1)}^{(n+1)}, ..., x_{(\alpha_k-1)}^{(n+1)}, x_{(\alpha_k+1)}^{(n+1)}, ..., x_m^{(n+1)}) \in I'$ for every $x_i^{(j)} \in R_i$ for $1 \leq i \leq m,\ 1 \leq j \leq n{+}1$.

Note that
$(x_1^{(1)}, ..., x_{(\alpha_1-1)}^{(1)}, 1_{R_{\alpha_1}}, x_{(\alpha_1+1)}^{(1)}, ..., x_{(\alpha_k-1)}^{(1)}, 1_{R_{\alpha_k}}, x_{(\alpha_k+1)}^{(1)}, ..., x_m^{(1)})...$
$(x_1^{(n+1)}, ..., x_{(\alpha_1-1)}^{(n+1)}, 1_{R_{\alpha_1}}, x_{(\alpha_1+1)}^{(n+1)}, ..., x_{(\alpha_k-1)}^{(n+1)}, 1_{R_{\alpha_k}}, x_{(\alpha_k+1)}^{(n+1)}, ..., x_m^{(n+1)}) \in I.$

Then $(x_1^{(1)}, ..., x_{(\alpha_1-1)}^{(1)}, 1_{R_{\alpha_1}}, x_{(\alpha_1+1)}^{(1)}, ..., x_{(\alpha_k-1)}^{(1)}, 1_{R_{\alpha_k}}, x_{(\alpha_k+1)}^{(1)}, ..., x_m^{(1)})...$
$(x_1^{(n)}, ..., x_{(\alpha_1-1)}^{(n)}, 1_{R_{\alpha_1}}, x_{(\alpha_1+1)}^{(n)}, ..., x_{(\alpha_k-1)}^{(n)}, 1_{R_{\alpha_k}}, x_{(\alpha_k+1)}^{(n)}, ..., x_m^{(n)}) \in I$
or there exists $1 \leq k \leq n$ such that

$(x_1^{(1)}, ..., x_{(\alpha_1-1)}^{(1)}, 1_{R\alpha_1}, x_{(\alpha_1+1)}^{(1)}, ..., x_{(\alpha_k-1)}^{(1)}, 1_{R_{\alpha_k}}, x_{(\alpha_k+1)}^{(1)}, ..., x_m^{(1)})...$

$\widehat{(x_1^{(k)}, ..., x_{(\alpha_1-1)}^{(k)}, 1_{R\alpha_1}, x_{(\alpha_1+1)}^{(k)}, ..., x_{(\alpha_k-1)}^{(k)}, 1_{R_{\alpha_k}}, x_{(\alpha_k+1)}^{(k)}, ..., x_m^{(k)})}...$

$(x_1^{(n+1)}, ..., x_{(\alpha_1-1)}^{(n+1)}, 1_{R_{\alpha_1}}, x_{(\alpha_1+1)}^{(n+1)}, ..., x_{(\alpha_k-1)}^{(n+1)}, 1_{R_{\alpha_k}}, x_{(\alpha_k+1)}^{(n+1)}, ..., x_m^{(n+1)}) \in\ \delta_\times(I).$

Thus $(x_1^{(1)}, ..., x_{(\alpha_1-1)}^{(1)}, x_{(\alpha_1+1)}^{(1)}, ..., x_{(\alpha_k-1)}^{(1)}, x_{(\alpha_k+1)}^{(1)}, ..., x_m^{(1)})...$
$(x_1^{(n)}, ..., x_{(\alpha_1-1)}^{(n)}, x_{(\alpha_1+1)}^{(n)}, ..., x_{(\alpha_k-1)}^{(n)}, x_{(\alpha_k+1)}^{(n)}, ..., x_m^{(n)}) \in I'$
or for some $1 \leq k \leq n$,

$(x_1^{(1)}, ..., x_{(\alpha_1-1)}^{(1)}, x_{(\alpha_1+1)}^{(1)}, ..., x_{(\alpha_k-1)}^{(1)}, x_{(\alpha_k+1)}^{(1)}, ..., x_m^{(1)})...$

$\widehat{(x_1^{(k)}, ..., x_{(\alpha_1-1)}^{(k)}, x_{(\alpha_1+1)}^{(k)}, ..., x_{(\alpha_k-1)}^{(k)}, x_{(\alpha_k+1)}^{(k)}, ..., x_m^{(k)})}...$

$(x_1^{(n+1)},...,x_{(\alpha_1-1)}^{(n+1)},x_{(\alpha_1+1)}^{(n+1)},...,x_{(\alpha_k-1)}^{(n+1)},x_{(\alpha_k+1)}^{(n+1)},...,x_m^{(n+1)})$ is in $\delta_\times(I')$.

$(iii) \Rightarrow (i)$: Assume that I' is an n-absorbing $\delta_\times$-primary ideal of R'. In a similar way, it can be seen that I is n-absorbing $\delta_\times$-primary. □

Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$. Then we say that δ satisfies (*) property if $\delta(I) = R$ implies $I = R$, i.e. $\delta(I) \neq R$ for all proper ideals I of R. Note that δ_r and δ_i, defined in Example 1, are examples of expansion functions satisfying (*) property. Moreover, if $R = R_1 \times ... \times R_n$ is a decomposable ring and δ_i's are expansion functions of $\mathcal{I}(\mathcal{R}_i)$ with (*) property, then $\delta_\times$ is an expansion function of $\mathcal{I}(\mathcal{R})$ satisfying (*) property.

Theorem 11 *Let $R = R_1 \times ... \times R_n$ be a decomposable ring and $I = I_1 \times ... \times I_n$ be an ideal of R such that $I_1 \neq 0$ and $\delta_i(I_i) \neq R_i$ for each $1 \leq i \leq n-1$. Suppose that for some $2 \leq k \leq n$, I_k is a nonzero ideal of R_k and δ_i's are expansion functions of $\mathcal{I}(\mathcal{R}_i)$ satisfying (*) property for each $i \in \{1,...,n\}$. Then the following are equivalent:*

(i) I is a weakly n-absorbing $\delta_\times$-primary ideal of R.

(ii) $I_n = R_n$ and $I' = I_1 \times ... \times I_{n-1}$ is an n-absorbing $\delta_\times$-primary ideal of $R' = R_1 \times ... \times R_{n-1}$ or I_i is a δ_i-primary ideal of R_i for each $i \in \{1,...n\}$.

(iii) $I = I_1 \times ... \times I_n$ is an n-absorbing $\delta_\times$-primary ideal of R.

Proof $(i) \Rightarrow (ii)$: Let $I_n = R_n$. Then $I' = I_1 \times ... \times I_{n-1}$ is an n-absorbing $\delta_\times$-primary ideal of $R' = R_1 \times ... \times R_{n-1}$ by Theorem 10. Assume that $I_i \neq R_i$ for every $i \in \{1,...n\}$. To prove that I_i is a δ_i-primary ideal of R_i, take $a_ib_i \in I_i$ for some $a_i, b_i \in R_i$. Then

$$0_R \neq (a_1,1_{R_2},...,1_{R_n})(1_{R_1},...,1_{R_{i-1}},a_i,1_{R_{i+1}},...,1_{R_n})$$
$$(1_{R_1},0,1_{R_3},...,1_{R_n})(1_{R_1},1_{R_2},0_{R_3},1_{R_4},...,1_{R_n})...$$
$$(1_{R_1},...,0_{R_{i-1}},1_{R_i},..,1_{R_n})(1_{R_1},...,1_{R_i},0_{R_{i+1}},1_{R_{i+2}},...,1_{R_n})...$$
$$(1_{R_1},...,1_{R_{n-1}},0_{R_n})(1_{R_1},...,1_{R_{i-1}},b_i,1_{R_{i+1}},...,1_{R_n})$$
$$= (a_1,0_{R_2},...,0_{R_{i-1}},a_ib_i,0_{R_{i+1}},...,0_{R_n}) \in I.$$

Since δ_i satisfies (*) property, $\delta_i(I_i) \neq R_i$ for every $i \in \{1,2,...,n\}$ and so we conclude either

$$(a_1,1_{R_2},...,1_{R_n})(1_{R_1},...,1_{R_{i-1}},a_i,1_{R_{i+1}},...,1_{R_n})(1_{R_1},0,1_{R_3},...,1_{R_n})$$
$$(1_{R_1},1_{R_2},0_{R_3},1_{R_4},...,1_{R_n})...(1_{R_1},...,0_{R_{i-1}},1_{R_i},..,1_{R_n})$$
$$(1_{R_1},...1_{R_i},0_{R_{i+1}},1_{R_{i+2}},...,1_{R_n})...(1_{R_1},...,1_{R_{n-1}},0_{R_n}) \in I$$

or

$$(a_1,1_{R_2},...,1_{R_n})(1_{R_1},0,1_{R_3},...,1_{R_n})(1_{R_1},1_{R_2},0_{R_3},1_{R_4},...,1_{R_n})$$
$$...(1_{R_1},...,0_{R_{i-1}},1_{R_i},..,1_{R_n})(1_{R_1},...1_{R_i},0_{R_{i+1}},1_{R_{i+2}},...,1_{R_n})...$$
$$(1_{R_1},...,1_{R_{n-1}},0_{R_n})(1_{R_1},...,1_{R_{i-1}},b_i,1_{R_{i+1}},...,1_{R_n}) \in \delta(I).$$

Hence $a_i \in I_i$ or $b_i \in \delta_i(I_i)$. Therefore, I_i is a δ_i-primary ideal of R_i. Furthermore, it can be similarly shown that I_1 is a δ_1-primary ideal since $I_k \neq 0$ for some $2 \leq k \leq n$.

$(ii) \Rightarrow (iii)$: Let $I_n = R_n$ and $I' = I_1 \times ... \times I_{n-1}$ be an n-absorbing $\delta_\times$-primary ideal of $R' = R_1 \times ... \times R_{n-1}$. Then $I = I_1 \times ... \times I_n$ is an n-absorbing $\delta_\times$-primary ideal of R by Theorem 10. Assume that I_i is a δ_i-primary ideal of R_i for every $i \in \{1,...n\}$. Let $(x_1^{(1)},...,x_n^{(1)})...(x_1^{(n+1)},...,x_n^{(n+1)}) \in I = I_1 \times ... \times I_n$ for every $x_i^{(j)} \in R_i$ for $1 \leq i \leq n$, $1 \leq j \leq n+1$. At least one of the $x_i^{(j)}$ is in I_i or $\delta_i(I_i)$ for any

$i \in \{1,...,n\}$, $j \in \{1,...,n+1\}$. Thus we can see that $I = I_1 \times ... \times I_n$ is an n-absorbing $\delta_\times$-primary ideal of R.

$(iii) \Rightarrow (i)$: is clear. □

Theorem 12 *Let $R = R_1 \times ... \times R_n$ be a decomposable ring and $I = I_1 \times ... \times I_n$ be an ideal of R such that $I_1 \neq 0$ and $\delta_i(I_i) \neq R_i$ for each $2 \leq i \leq n$. Assume that δ_i's are expansion function of $\mathcal{I}(\mathcal{R}_i)$ satisfying (*) property for each $i \in \{1,...,n\}$. Then the following are equivalent:*

(i) $I = I_1 \times ... \times I_n$ is a weakly n-absorbing $\delta_\times$-primary ideal of R that is not an n-absorbing $\delta_\times$-primary ideal of R.

(ii) I_1 is a weakly δ_1-primary ideal of R_1 that is not a δ_1-primary ideal and $I_i = (0)$ is a δ_i-primary ideal of R_i for each $i \in \{2,...,n\}$.

Proof $(i) \Rightarrow (ii)$: Suppose that $I = I_1 \times ... \times I_n$ is a weakly n-absorbing $\delta_\times$-primary ideal of R that is not n-absorbing $\delta_\times$-primary. Let $I_i \neq (0)$ for some $i \in \{2,...,n\}$. Then $I = I_1 \times ... \times I_n$ is an n-absorbing $\delta_\times$-primary ideal of R by Theorem 11, yielding a contradiction. It must be $I_i = (0)$ for every $i \in \{2,...,n\}$. It is clear that $I_i = (0)$ is a δ_i-primary ideal. Now we assume that $0 \neq xy \in I_1$ for some $x, y \in R_1$. Then

$$0_R \neq (x, 1_{R_2}, ..., 1_{R_n})(1_{R_1}, 0_{R_2}, 1_{R_3}, ..., 1_{R_n})$$
$$(1_{R_1}, 1_{R_2}, 0_{R_3}, 1_{R_4}, ..., 1_{R_n})...(1_{R_1}, ..., 1_{R_{n-1}}, 0_{R_n})(y, 1_{R_2}, ..., 1_{R_n})$$
$$= (xy, 0_{R_2}, ..., 0_{R_n}) \in I_1 \times 0 \times ... \times 0.$$

We obtain that $x \in I_1$ or $y \in \delta_1(I_1)$ since $I_1 \times 0 \times ... \times 0$ is a weakly n-absorbing $\delta_\times$-primary ideal of R. Consequently, I_1 is weakly δ_1-primary. If I_1 is a δ_1-primary ideal of R_1, then I_i is a δ_i-primary ideal of R_i for every $i \in \{1,...,n\}$. Hence, it is easily seen that I is an n-absorbing $\delta_\times$-primary ideal of R, a contradiction.

$(ii) \Rightarrow (i)$: Assume that I_1 is a weakly δ_1-primary ideal of R_1 that is not a δ_1-primary ideal and $I_i = (0)$ is a δ_i-primary ideal of R_i for every $i \in \{2,...,n\}$. Let $0_R \neq (x_1^{(1)}, ..., x_n^{(1)})...(x_1^{(n+1)}, ..., x_n^{(n+1)}) \in I_1 \times 0 \times ... \times 0$ for every $x_i^{(j)} \in R_i$ for $1 \leq i \leq n$, $1 \leq j \leq n+1$. Then at least one of the $x_1^{(j)}$ is in I_1 or in $\delta_1(I_1)$ and for any $i \in \{2,...,n\}, j \in \{1,...,n+1\}$, at least one of the $x_i^{(j)} = 0$ or is in $\delta_i(0)$. Thus we have that $I_1 \times 0 \times ... \times 0$ is a weakly n-absorbing $\delta_\times$-primary ideal of R. Since I_1 is not a δ_1-primary ideal, there are $x, y \in R_1$ with $xy = 0$ but $x \notin I_1$ and $y \notin \delta_1(I_1)$. Then we get

$$(x, 1_{R_2}, ..., 1_{R_n})(1_{R_1}, 0_{R_2}, 1_{R_3}, ..., 1_{R_n})$$
$$(1_{R_1}, 1_{R_2}, 0_{R_3}, 1_{R_4}, ..., 1_{R_n})...(1_{R_1}, ..., 1_{R_{n-1}}, 0_{R_n})(y, 1_{R_2}, ..., 1_{R_n})$$

$= (0_{R_1}, 0_{R_2}, ..., 0_{R_n})$. However, products of n elements of $(x, 1_{R_2}, ..., 1_{R_n}), (1_{R_1}, 0_{R_2}, 1_{R_3}, ..., 1_{R_n}), (1_{R_1}, 1_{R_2}, 0_{R_3}, ..., 1_{R_n})$, $(1_{R_1}, 1_{R_2}, 0_{R_3}, 1_{R_4}, ..., 1_{R_n}), ..., (1_{R_1}, ..., 1_{R_{n-1}}, 0_{R_n}), (y, 1_{R_2}, ..., 1_{R_n})$ are neither in $I_1 \times 0 \times ... \times 0$ nor in $\delta_\times(I_1 \times 0 \times ... \times 0)$. Thus $I_1 \times 0 \times ... \times 0$ is not an n-absorbing $\delta_\times$-primary ideal of R. □

Theorem 13 *Let $R = R_1 \times ... \times R_{n+1}$ be a decomposable ring and $I = I_1 \times ... \times I_{n+1}$ be a nonzero proper ideal of R such that $\delta_i(I_i) \neq R_i$ for each $1 \leq i \leq n+1$. Assume that δ_i's are expansion functions of $\mathcal{I}(\mathcal{R}_i)$ satisfying (*) property for each $i \in \{1,...,n+1\}$. Then the following are equivalent:*

(i) I is a weakly n-absorbing $\delta_\times$-primary ideal of R.

(ii) I is an n-absorbing $\delta_\times$-primary ideal of R.

(iii) $I_k = R_k$ *for some* $1 \le k \le n+1$ *and* I_j *is a* δ_j*-primary ideal of* R_j *for each* $j \in \{1,...,n+1\} - \{k\}$ *or* $I = I_1 \times ... \times I_{\alpha_1 - 1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times ... \times I_{\alpha_k - 1} \times R_{\alpha_k} \times I_{\alpha_k+1} \times ... \times I_{n+1}$*, where* $I' = I_1 \times ... \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times ... \times I_{\alpha_k-1} \times I_{\alpha_k+1} \times ... \times I_{n+1}$ *is an* n*-absorbing* $\delta_\times$*-primary ideal of* $R' = R_1 \times ... \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times ... \times R_{\alpha_k-1} \times R_{\alpha_k+1} \times ... \times R_{n+1}$ *for some* $1 \le \alpha_1, \alpha_2, ..., \alpha_k \le n+1$.

Proof $(i) \Rightarrow (ii)$: Take $(0,...,0) \neq (a_1,...,a_{n+1}) \in I$. Then
$(0,...,0) \neq (a_1,...,a_{n+1}) = (a_1, 1_{R_2},...,1_{R_{n+1}})...(1_{R_1},...,1_{R_n}, a_{n+1}) \in I$. Since I is weakly n-absorbing δ-primary, then
$(a_1, 1_{R_2},...,1_{R_{n+1}})...(1_{R_1},...,a_n,1_{R_{n+1}}) \in I$ or there exists $1 \le k \le n$ such that

$(a_1, 1_{R_2},...,1_{R_{n+1}})...(1_{R_1},\widehat{...,a_k,...},1_{R_{n+1}})...(1_{R_1},...,1_{R_n},a_{n+1})$ is in $\delta(I)$. Then $I_i = R_i$ or $\delta_j(I_j) = R_j$ for some $1 \le i,j \le n+1$. Since δ_j satisfies (*) property, we get $I_i = R_i$ for some $1 \le i \le n+1$. Thus $I^{n+1} \neq 0_R$. By Theorem 5, I is n-absorbing $\delta_\times$-primary.

$(ii) \Rightarrow (iii)$: Let I be an n-absorbing $\delta_\times$-primary ideal. Then $I_i = R_i$ for some $1 \le i \le n+1$ by the previous proof. Assume that $I = I_1 \times ... \times R_i \times ... \times I_{n+1}$ for some $i \in \{1,...,n+1\}$ and I_j is a proper ideal of R_j for every $j \in \{1,...,n\} - \{i\}$. Now we show that I_j is a δ_j-primary ideal of R_j. Let $x_jy_j \in I_j$ for $x_j, y_j \in R_j$. Then

$$(1_{R_1},...,1_{R_{j-1}},x_j,1_{R_{j+1}},...,1_{R_{n+1}})(0,1_{R_2},...,1_{R_i},...,1_{R_j},...,1_{R_{n+1}})$$
$$(1_{R_1},0_{R_2},1_{R_3},...,1_{R_i},...,1_{R_j},...,1_{R_{n+1}})...$$
$$(1_{R_1},...,0_{R_{j-1}},1_{R_j},1_{R_{j+1}},...,1_{R_{n+1}})(1_{R_1},...,1_{R_j},0_{R_{j+1}},1_{R_{j+2}},...,1_{R_{n+1}})...$$
$$(1_{R_1},...,1_{R_n},0_{R_{n+1}})(1_{R_1},...,1_{R_{j-1}},y_j,1_{R_{j+1}},...,1_{R_{n+1}})$$

$= (0,...,0,1_{R_i},0,...,0,x_jy_j,0,...,0) \in I$ for some $j \neq i$. Since I is an n-absorbing $\delta_\times$-primary ideal, we have either $x_j \in I_j$ or $y_j \in \delta_j(I_j)$. Therefore, I_j is a δ_j-primary ideal of R_j.

Let $I = I_1 \times ... \times I_{\alpha_1-1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times ... \times I_{\alpha_k-1} \times R_{\alpha_k} \times I_{\alpha_k+1} \times ... \times I_{n+1}$ for some $1 \le \alpha_1, \alpha_2, ..., \alpha_k \le n+1$. Then $I' = I_1 \times ... \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times ... \times I_{\alpha_k-1} \times I_{\alpha_k+1} \times ... \times I_{n+1}$ is an n-absorbing $\delta_\times$-primary ideal of $R' = R_1 \times ... \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times ... \times R_{\alpha_k-1} \times R_{\alpha_k+1} \times ... \times R_{n+1}$ by Theorem 10.

$(iii) \Rightarrow (i)$: It is easily seen that I is a weakly n-absorbing $\delta_\times$-primary ideal of R. □

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