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Digital Lusternik–Schnirelmann category

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Abstract: In this paper, we define the digital Lusternik–Schnirelmann category cat_κ, introduce some of its properties, and discuss how the adjacency relation affects the digital Lusternik–Schnirelmann category.

Key words: Lusternik–Schnirelmann category, digital topology

1. Introduction

The Lusternik–Schnirelmann category of a topological space X (denoted by $\mathrm{cat}(X)$) is the least integer ℓ such that there exists an open covering of X with cardinality $\ell+1$, where each subset in this covering is contractible to a point in X [8]. For abbreviation, it is usually called *LS cat*. If no such a covering exists, then we write $\mathrm{cat}(X) = \infty$. In this paper, we construct the Lusternik–Schnirelmann category from the digital viewpoint.

A digital image X is a finite subset of $\mathbb{Z}^n$. In order to work on X we impose a relation, called adjacency relation, on $\mathbb{Z}^n$ as follows: For $n = 1$, we say that two points p and q in $\mathbb{Z}$ are 2-*adjacent* if $q = p \pm 1$. For $n = 2$, we have two possible adjacency relations. Two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{Z}^2$ are 4-*adjacent* if at most only one corresponding coordinate differs by 1 and are 8-*adjacent* if their corresponding coordinates either differ by one or are equal. The generalization of the possible adjacency relations on $\mathbb{Z}^n$ is as follows: Two points $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ in $\mathbb{Z}^n$ are said to be c_ℓ-*adjacent* [4] for $1 \le \ell \le n$ whenever

- there are at most ℓ indices i such that $|p_i - q_i| = 1$ and
- $p_j = q_j$ for all other indices j satisfying $|p_i - q_i| \neq 1$.

Here c_ℓ is a positive number that is the number of possible legal moves for the motion of point p under the certain adjacency relation. Then it is easy to observe that $c_1 = 2$ in $\mathbb{Z}$, $c_1 = 4$ and $c_2 = 8$ in $\mathbb{Z}^2$, and $c_1 = 6$, $c_2 = 18$, $c_3 = 26$ in $\mathbb{Z}^3$ are the only possible adjacency relations. Usually an adjacency relation on a digital image is denoted by Greek letters such as κ and λ. The adjacency relation plays a crucial role for digital topology. Any digital concept that is adapted from topology is considered by the given adjacency relation. For more details, see [2, 6, 16–19].

One of our main results is that any two digital images of the same digital homotopy type have the same Lusternik–Schnirelmann category. On the other hand, the digital homotopy type of a digital image depends not

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only on the digital image itself but also on the adjacency relation on that image. Hence the digital LS-category of a digital image depends on the adjacency relation as well.

We give examples of the same digital images with different adjacency relations, and show that they may have different LS-categories.

A digital image usually admits more than one adjacency relation. Another result of this paper gives the relation between LS-categories if the adjaceny relations on the same digital image are different.

2. Digital Lusternik–Schnirelmann Category

We will start this section by recalling some definitions from digital topology.

A *digital interval* [1] is a subset of $\mathbb{Z}$ of the form

$$[a,b]_{\mathbb{Z}} = \{n \in \mathbb{Z} \mid a \leq n \leq b\},$$

where 2-adjacency relation is assumed.

Definition 2.1 *([2, 20]) Let $X \subset \mathbb{Z}^k$ and $Y \subset \mathbb{Z}^m$ be digital images on which the adjacency relations κ and λ are given respectively. A function $f : X \to Y$ is said to be* (κ,λ)-continuous *if the image of every κ-connected subset of X under f is λ-connected.*

An equivalent but more practicable idea for f to be a (κ,λ)-continuous map is to check the λ-adjacency of $f(x)$ and $f(x')$ in Y whenever x and x' are κ-adjacent in X for $x, x' \in X$.

Definition 2.2 *([2, 15]) Let X and Y be digital images and $f, g : X \to Y$ be* (κ,λ)-continuous *functions. If there exist $m \in \mathbb{Z}^+$ and a function*

$$F : X \times [0,m]_{\mathbb{Z}} \to Y$$

with the following conditions, then we say that F is a digital (κ,λ)-homotopy *between f and g, and f and g are digitally* (κ,λ)-homotopic *in Y, which is denoted by $f \simeq_{\kappa,\lambda} g$ or $f \simeq g$ for short.*

(i) For all $x \in X$, $F(x,0) = f(x)$ and $F(x,m) = g(x))$.

(ii) For all $x \in X$, $F_x : [0,m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x,t)$ is $(2,\lambda)$-continuous.

(iii) For all $t \in [0,m]_{\mathbb{Z}}$, $F_t : X \to Y$ defined by $F_t(x) = F(x,t)$ is (κ,λ)-continuous.

We call a (κ,λ)-continuous map $f : X \to Y$ *null-homotopic* if it is (κ,λ)-homotopic to a constant map c_{y_0} for some $y_0 \in Y$.

Definition 2.3 Digital LS-category *of a digital κ-image $X \subset \mathbb{Z}^n$ is the least integer ℓ such that there is a covering $U_1, U_2, \cdots, U_{\ell+1}$ of X, where each inclusion map $i_i : U_i \hookrightarrow X$ for $i = 1, \cdots, \ell+1$ is digitally κ-nullhomotopic in X. This will be denoted by $\mathrm{cat}_\kappa(X) = \ell$.*

Note that $\mathrm{cat}_\kappa(X)$ can be at most the number of lattice points in X.

Recall that (κ,λ)-continuous map $f : X \to Y$ is (κ,λ)-homotopy equivalence [3, 9] if there exists a (κ,λ)-continuous map $g : Y \to X$ such that $g \circ f \simeq_{\kappa,\kappa} 1_X$ and $f \circ g \simeq_{\lambda,\lambda} 1_Y$ where 1_X and 1_Y are the identity maps on X and Y, respectively.

The following theorem states that any two digital images with the same homotopy type in the digital sense have the same digital LS-category.

Theorem 2.4 *Digital LS-category is a homotopy invariant in the digital sense.*

Proof If X and Y are homotopy equivalent, then X dominates Y (i.e. $f : X \to Y$ has a right homotopy inverse) and Y dominates X. Thus it suffices to show that if X dominates Y, then $\mathrm{cat}_\kappa(X) \geq \mathrm{cat}_\lambda(Y)$.

Assume that X dominates Y. Then there exist (κ, λ)-continuous function $f : X \to Y$ and (λ, κ)-continuous function $g : Y \to X$ such that $f \circ g$ is (λ, λ)-homotopic to id_Y. In other words, there exist $m_1 \in \mathbb{Z}^+$ and

$$F : Y \times [0, m_1]_{\mathbb{Z}} \to Y$$

such that

(A1) For all y, $F(y, 0) = (f \circ g)(y)$ and $F(y, m_1) = \mathrm{id}_Y(y)$.

(A2) For all $y \in Y$, $F_y : [0, m_1]_{\mathbb{Z}} \to Y$ defined by $F_y(t) = F(y, t)$ is $(2, \lambda)$-continuous.

(A3) For all $t \in [0, m_1]_{\mathbb{Z}}$, $F_t : Y \to Y$ defined by $F_t(y) = F(y, t)$ is (λ, λ)-continuous.

Now assume that $\mathrm{cat}_\kappa(X) = \ell$. That is, there exist $U_1, U_2, \cdots, U_{\ell+1}$ such that each inclusion map $i_j : U_j \hookrightarrow X$ is digitally κ-nullhomotopic in X. In other words, for each U_j there exist m_2^j and a map

$$H^j : U_j \times [0, m_2^j]_{\mathbb{Z}} \to X$$

such that

(B1) For all $x \in U_j$, $H^j(x, 0) = i_j(x) = x$ and $H^j(x, m_2^j)$ is a constant.

(B2) For all $x \in U_j$, $H_x^j : [0, m_2^j]_{\mathbb{Z}} \to X$ defined by $H_x^j(t) = H^j(x, t)$ is $(2, \kappa)$-continuous.

(B3) For all $t \in [0, m_2^j]_{\mathbb{Z}}$, $H_t^j : U_j \to X$ defined by $H_t^j(x) = H^j(x, t)$ is (κ, κ)-continuous.

Consider the preimages $g^{-1}(U_j) \subset Y$ for $j = 1, \cdots, \ell + 1$. If we show that each inclusion map $\iota_j : g^{-1}(U_j) \hookrightarrow Y$ is digitally λ-nullhomotopic, we will complete the proof.

For each j, let $m_3^j = m_1 + m_2^j$ and define

$$G^j : g^{-1}(U_j) \times [0, m_3^j]_{\mathbb{Z}} \to Y$$

by

$$G^j(y, t) = \begin{cases} F(y, m_1 - t) & , t \in [0, m_1]_{\mathbb{Z}} \\ f(H^j(g(y), t - m_1)) & , t \in [m_1, m_3^j]_{\mathbb{Z}} \end{cases}.$$

Thus G^j satisfies the following statements.

(C1) For all $y \in g^{-1}(U_j)$, $G^j(y, 0) = \iota_j(y) = y$ and $G^j(y, m_3^j)$ is a constant.

(C2) For all $y \in g^{-1}(U_j)$, $G_y^j : [0, m_3^j]_{\mathbb{Z}} \to g^{-1}(U_j)$ defined by $G_y^j(t) = G^j(y,t)$ is $(2,\lambda)$-continuous. For $t \in [0, m_1]_{\mathbb{Z}}$ it follows from (A2) and for $t \in [m_1, m_3^j]_{\mathbb{Z}}$ it follows from that f is (κ,λ)-continuous, $H^j_{g(y)}$ is $(2,\kappa)$-continuous and from the Proposition 2.5 in [2].

(C3) For all $t \in [0, m_3^j]_{\mathbb{Z}}$, $G_t^j : g^{-1}(U_j) \to g^{-1}(U_j)$ defined by $G_t^j(y) = G^j(y,t)$ is (λ,λ)-continuous. For $t \in [0, m_1]_{\mathbb{Z}}$ it follows from (A3). Note that $H^j(g(-), t-m_1) : g^{-1}(U_j) \to X$ defined by $y \to H^j(g(y), t-m_1)$ is (λ,κ)-continuous. From this observation with the (κ,λ)-continuity of f and the Proposition 2.5 in [2], the (λ,λ)-continuity of G_t^j follows for $t \in [m_1, m_3^j]_{\mathbb{Z}}$.

□

Lemma 2.5 *If the inclusion map $i : U \hookrightarrow X$ is nullhomotopic to $x_0 \in X$, there exists a κ-path from each point in U to x_0.*

Proof The (κ,κ) continuous map $H_x : [0,m]_{\mathbb{Z}} \to X$ derived from the homotopy $H : U \times [0,m]_{\mathbb{Z}} \to X$ gives the desired path between each point x in U and x_0. □

A κ-*path* in a digital image $X \subset \mathbb{Z}$ from x_1 and x_2 is a $(2,\kappa)$ continuous function $\gamma : [0,m]_{\mathbb{Z}} \to X$ for some integer m such that $\gamma(0) = x_0$ and $\gamma(m) = x_1$ [5]. Note that if $\gamma(0) = \gamma(m) = x$, then γ is said to be κ-*loop* at x. Then a digital image X is said to be κ-*connected* [14], if there exists a κ-path between any pair of points in it.

A digital image (X,κ) is said to be κ-*contractible* [1, 15] if its identity map is κ-homotopic in X to a constant map. Note that the statement given in Lemma 2.5 is also true for a κ-contractible digital image X: If a digital image (X,κ) is κ-contractible to a point x_0 in X, then there exists a path between each point x in X and x_0.

Moreover, it follows immediately by the definition that if a digital image (X,κ) is κ-*contractible* then $\mathrm{cat}_\kappa(X) = 0$.

Lemma 2.6 *If X is a κ-contractible digital image, then it is κ-connected.*

Proof Let $H : [0,m]_{\mathbb{Z}} \times X \to X$ be the homotopy map between the identity map 1_X and the constant map c_{x_0} where $x_0 \in X$. Note that each H_x is a path between x and x_0. Then the map

$$\gamma : [0,2m]_{\mathbb{Z}} \to X$$

$$t \to \gamma(t) = \begin{cases} H_x(t) & 0 \le t \le m \\ H_y(m-t) & m \le t \le 2m \end{cases}$$

is a path between any two (distinct) points x and y in X. □

We now compute the digital LS-category of the following three digital images (see Figure 1) defined in [11] and will conclude that different adjacency relations on a digital image may give different digital LS-categories.

Example 2.7 *Let MSC_4 be a digital image in $\mathbb{Z}^2$ 4-isomorphic to*

$$\{a = (-1,0), b = (-1,1), c = (0,1), d = (1,1), e = (1,0), f = (1,-1), g = (0,-1), h = (-1,-1)\}.$$

Since MSC_4 *is* 8*-contractible [10, 11],* $\mathrm{cat}_8(\mathrm{MSC}_4) = 0$*. We will show that* $\mathrm{cat}_4(\mathrm{MSC}_4) = 1$*. Define the sets* $U_1 = \{a, b, c, d\}$ *and* $U_2 = \{e, f, g, h\}$*, which cover* MSC_4*. Define the homotopy*

$$H : U_1 \times [0, 3]_{\mathbb{Z}} \to X$$

$$(x, t) \mapsto H(x, t) = \begin{cases} \iota_1(x) & t = 0 \\ a & (t = 1 \text{ and } x = a, b) \text{ or } (t = 2 \text{ and } x = a, b, c) \\ b & (t = 1 \text{ and } x = c) \text{ or } (t = 2 \text{ and } x = d) \\ c & t = 1 \text{ and } x = d \\ a & t = 3 \end{cases}$$

where $\iota_1 : U_1 \hookrightarrow X$ *is an inclusion map. Then* H *is a homotopy between* ι_1 *and the constant map* c_a*. Hence* ι_1 *is nullhomotopic. Next define the homotopy*

$$F : U_2 \times [0, 3]_{\mathbb{Z}} \to X$$

$$(x, t) \mapsto F(x, t) = \begin{cases} \iota_2(x) & t = 0 \\ e & (t = 1 \text{ and } x = e, f) \text{ or } (t = 2 \text{ and } x = e, f, g) \\ f & (t = 1 \text{ and } x = g) \text{ or } (t = 2 \text{ and } x = h) \\ g & t = 1 \text{ and } x = h \\ e & t = 3 \end{cases}$$

where $\iota_2 : U_2 \hookrightarrow X$ *is an inclusion map. Then* F *is a homotopy between* ι_2 *and the constant map* c_e*. Hence* ι_2 *is nullhomotopic.*

Example 2.8 *Let* $\mathrm{MSC}_8^{'}$ *be a digital image in* $\mathbb{Z}^2$ 8*-isomorphic to*

$$\{(-1, 0), (1, 0), (0, -1), (0, 1)\}.$$

Since $\mathrm{MSC}_8^{'}$ *is* 8*-contractible [2],* $\mathrm{cat}_8(\mathrm{MSC}_8^{'}) = 0$*. However, it is not* 4*-contractible: There exists no* 4*-path between any of two points in* $\mathrm{MSC}_8^{'}$*. By Lemma 2.5, the nullhomotopic map* $\iota_j : U_j \to \mathrm{MSC}_8^{'}$ *exists only when* U_j *is a singleton set. Hence* $\mathrm{cat}_4(\mathrm{MSC}_8^{'}) = 3$*.*

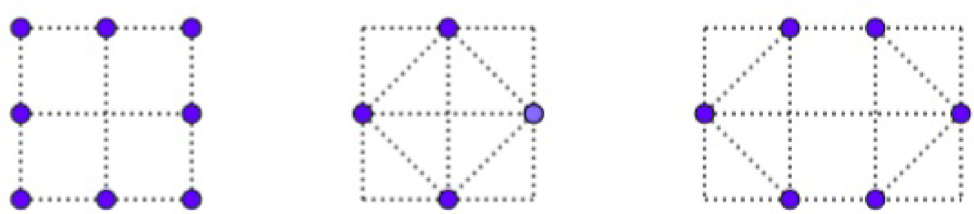

Figure 1. MSC_4, $\mathrm{MSC}_8^{'}$, and MSC_8.

Example 2.9 *Let* MSC_8 *be a digital image in* $\mathbb{Z}^2$ 8*-isomorphic to*

$$\{a = (0, 0), b = (1, 1), c = (2, 1), d = (3, 0), e = (2, -1), f = (1, -1)\}.$$

It is easy to check that it is 8*-connected. However it is not* 8*-contractible [2, 10, 11, 13]. We will show that* $\mathrm{cat}_8(\mathrm{MSC}_8) = 1$*. Let* $U_1 = \{a, b, c\}$ *and* $U_2 = \{d, e, f\}$*. Note that* U_1 *and* U_2 *cover* X*. Define the homotopy*

$$H : U_1 \times [0, 2]_{\mathbb{Z}} \to X$$

$$(x, t) \mapsto H(x, t) = \begin{cases} \iota_1(x) & t = 0 \\ \iota_1(x) & t = 1 \text{ and } x = a, b \\ b & t = 1 \text{ and } x = c \\ a & t = 2 \end{cases}$$

where $\iota_1 : U_1 \hookrightarrow X$ *is an inclusion map. Then* H *is a homotopy between* ι_1 *and the constant map* c_a*. Hence* ι_1 *is nullhomotopic. Next define the homotopy*

$$F : U_2 \times [0, 2]_{\mathbb{Z}} \to X$$

$$(x, t) \mapsto F(x, t) = \begin{cases} \iota_2(x) & t = 0 \\ \iota_2(x) & t = 1 \text{ and } x = e, f \\ e & t = 1 \text{ and } x = d \\ f & t = 2 \end{cases}$$

where $\iota_2 : U_2 \hookrightarrow X$ *is an inclusion map. Then* F *is a homotopy between* ι_2 *and the constant map* c_f*. Hence* ι_2 *is nullhomotopic.*

Note that the digital Lusternik–Schnirelmann category depends on the adjacency relation. For MSC_8*,* $\mathrm{cat}_4(\mathrm{MSC}_8) = 3$*. The idea here is due to Lemma 2.5. The covering* $\{U_1 = \{a\}, U_2 = \{b, c\}, U_3 = \{d\}, U_4 = \{e, f\}\}$ *is the minimal that satisfies the conditions for the digital Lusternik–Schnirelmann category.*

Theorem 2.10 *Let* X *be a digital image in* $\mathbb{Z}^n$ *and let* κ *and* λ *be the two adjacency relations in* $\mathbb{Z}^n$*. Then* $\mathrm{cat}_\kappa(X) \leq \mathrm{cat}_\lambda(X)$ *whenever* $\kappa > \lambda$*.*

Proof Let $\mathrm{cat}_\lambda(X) = \ell$. Then there exists $U_1, \cdots, U_{\ell+1}$ subsets of X covering X such that each inclusion map $i_i : U_i \hookrightarrow X$ for $i = 1, \cdots, \ell+1$ is digitally λ-nullhomotopic in X. On the other hand, two points that are λ-adjacent are also κ-adjacent, since $\kappa > \lambda$. Hence it can easily be verified that all the λ-homotopies between any two maps on U_i are also κ-homotopies. Therefore there are $\ell + 1$ subsets $U_1, \cdots, U_{\ell+1}$ covering X such that each inclusion map $i_i : U_i \hookrightarrow X$ for $i = 1, \cdots, \ell + 1$ is digitally κ-nullhomotopic in X. This completes the proof. □

Note that in some situations the equality $\mathrm{cat}_\kappa(X) = \mathrm{cat}_\lambda(X)$ may occur. The following is an example of such a case.

Example 2.11 *Consider the digital image* $\mathrm{MSS}_{18} = \{c_i\}_{i=0}^{9}$ *[12] in* $\mathbb{Z}^3$ *where*

$$c_0 = (0, 0, 0), c_1 = (1, 1, 0), c_2 = (0, 1, -1), c_3 = (0, 2, -1), c_4 = (1, 2, 0),$$

$$c_5 = (0, 3, 0), c_6 = (-1, 2, 0), c_7 = (0, 2, 1), c_8 = (0, 1, 1), c_9 = (-1, 1, 0).$$

By Lemma 2.5, any inclusion map $\iota : U \to \mathrm{MSS}_{18}$ *is nullhomotopic whenever* U *is a singleton set since no two distinct points in* MSS_{18} *are* 6*-connected* $\mathbb{Z}^3$*. Consider the loop*

$$\gamma : [0, 9]_{\mathbb{Z}} \to \mathrm{MSS}_{18}$$

defined by

$$\gamma(t) := c_{t+1(\mathrm{mod}10)}$$

(see Figure 2). Note that any loop in MSS_{18} *is* 18*-contractible (hence* 26*-contractible) [7] so that by the Theorem 2.10* $\mathrm{cat}_{26}(\mathrm{MSS}_{18}) = \mathrm{cat}_{18}(\mathrm{MSS}_{18}) = 1$.

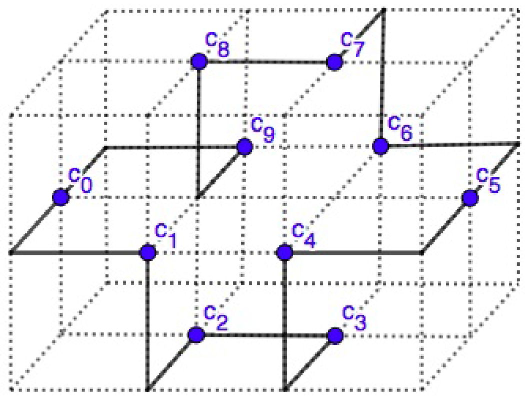

Figure 2. MSS_{18} and the image of the loop γ.

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