

## Accelerating diffusion by incompressible drift on the two-dimensional torus

Nejib YAAKOUBI\* 

Departement de Mathematique, Faculty des Sciences de Sfax, University of Sfax, Sfax, Tunisia

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**Abstract:** In this paper we construct an explicit sequence of divergence-free vector fields  $b_n$  that pushes the spectral gap of the nonself-adjoint operator  $A_{b_n} = \Delta + b_n \cdot \nabla$  to infinity. The spectral gap is an indicator for the speed at which this diffusion converges toward its equilibrium, which corresponds to the uniform distribution.

**Key words:** Nonself-adjoint operator, spectral gap, divergence-free vector fields, rearrangement, comparison manifold, Faber–Krahn inequality

### 1. Introduction

Let  $M$  be a compact  $d$ -dimensional Riemannian manifold. An interesting subject is to study the speed of the convergence of diffusion generated by the antisymmetric perturbations of the Laplacian of the form  $A_b = \Delta + b \cdot \nabla$ , where  $b$  is a divergence-free vector field, to its equilibrium state. A way for studying this is to analyze the behavior of the spectral gap  $\rho_b$  of  $A_b$  since it is an important indicator for understanding this speed of convergence. Note that all diffusions generated by operators of the above form have the same invariant measure, which is the Riemannian volume on  $M$ . For example, this question is asked when one wants to measure the performance of Markov chain Monte Carlo algorithms. In [10] it was shown that the optimal algorithm is to be found in the class of nonreversible diffusions where the vector field  $b$  is nonzero and the generator  $A_b$  is not self-adjoint. In this context it is then important to understand how a change in the vector field  $b$  affects the convergence to equilibrium. In particular it is interesting to see if one can find vector fields that push the spectral gap of  $A_b$  over a prescribed value of arbitrary size. If the compact manifold has dimension larger than or equal to three then it is known that there exists on this manifold some smooth measure preserving mixing flows (see [9]). The problem on 2-dimensional Riemannian manifolds and 2-dimensional sphere was studied in [6] and [7], where the authors prove that the spectral gap denoted by  $\rho_b$  can be arbitrarily large with a suitable choice of  $b$ . For vector fields  $b$  generating mixing flows, the operator  $b \cdot \nabla$  has no nonconstant eigenfunctions and it then follows that the spectral gap of  $A_{ab}$  goes to infinity as the real parameter  $a$  grows to infinity (see [3] and [5]). It was proved in Hwang et al. [10] that the addition of a divergence-free drift increases the spectral gap if the first eigenspace of the unperturbed operator is not invariant under the drift. This means that  $A_b$  with  $b = 0$  has the smallest spectral gap, and the difference compared with nonzero  $c$  is in general strict. In this paper, we study the case of the torus  $\mathbb{T}^2$ . We will construct explicitly a sequence of vector fields  $b_n$  such that the spectral gap of the operator  $A_{b_n}$  will be pushed to infinity. A similar problem was discussed by Hwang

\*Correspondence: nejibyaakoubi@gmail.com

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and Pai in [11] in a general case for the  $n$ -Torus. However, in their constructions, they prove the existence of such vector fields without explicating the associated flow. However, the technique presented in this manuscript introduces an explicit form of the associated flow in the torus  $\mathbb{T}^2$ .

**1.1. Motivation**

Let  $\mathbb{T}^2$  be the torus in  $\mathbb{R}^3$ . We will identify  $\mathbb{T}^2$  to the set  $[0, 1] \times [0, 1]$  on  $\mathbb{R}^2$ . As a result, we introduce the the Lebesgue measure on  $\mathbb{R}^2$  denoted  $\omega_{\mathbb{R}^2}$  to integrate surfaces on  $\mathbb{T}^2$ . The arc length measure with respect to the measure  $\omega_{\mathbb{R}^2}$  will be noted  $\ell_{\mathbb{R}^2}$ . It is clear that  $\ell_{\mathbb{R}^2}$  is the euclidian arc length measure on  $\mathbb{R}^2$ . For  $f \in C^2(\mathbb{T}^2)$ , we can define the nonself-adjoint operator

$$A_b f = \Delta f + b \cdot \nabla f.$$

This operator generates a diffusion semigroup of operators  $(T_t^{(b)})_{t \geq 0}$  on  $L^2(\mathbb{T}^2)$  through the parabolic evolution equation

$$\partial_t T_t f = A_b T_t f \quad \text{with initial condition } T_0 f = f.$$

This is the well-known Kolmogorov backward equation for a Brownian motion on  $\mathbb{T}^2$ , which is drifted by a stationary incompressible medium with velocity profile  $b$ . Since the formal adjoint of the operator  $A_b$  equals  $A_{-b}$  it follows that the uniform distribution is an invariant distribution for all the diffusions generated by operators of the form  $A_b$ . It is then interesting to know if the choice of the vector-field  $b$  has some influence on the speed at which this equilibrium is reached. The convergence exponent, which measures the speed, is the spectral gap of the operators  $A_b$  defined as

$$\rho_b = - \sup \left\{ \operatorname{Re}(z); z \in \operatorname{sp}(A_b) \setminus \{0\} \right\}.$$

Here  $\operatorname{sp}(A_b)$  denotes the spectrum of the operator  $A_b$ . The spectrum of this operator is located in the complex plane and the eigenfunctions are complex valued. In order to study those objects we introduce the following Hilbert space of mean-zero complex-valued functions:

$$H = \left\{ f = f_1 + if_2; f_1, f_2 \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} f_1 d\omega_{\mathbb{R}^2} = \int_{\mathbb{T}^2} f_2 d\omega_{\mathbb{R}^2} = 0 \right\}$$

with scalar product:

$$\langle f, g \rangle := \int_{\mathbb{T}^2} f \bar{g} d\omega_{\mathbb{R}^2}.$$

We also define the following Sobolev space of mean-zero functions:

$$H^1 = \left\{ f = f_1 + if_2 \in H; \int_{\mathbb{T}^2} |\nabla f_1|^2 d\omega_{\mathbb{R}^2} + \int_{\mathbb{T}^2} |\nabla f_2|^2 d\omega_{\mathbb{R}^2} < \infty \right\}$$

with scalar product:

$$\langle f, g \rangle_1 := \int_{\mathbb{T}^2} \nabla f \cdot \nabla \bar{g} d\omega_{\mathbb{R}^2}.$$

**2. Main result**

**2.1. Main result**

In this article we will study the problem of the spectral gap of the operator  $A_b$  on  $\mathbb{T}^2$ . We will prove the following theorem:

**Theorem 1** *In the 2– dimensional torus  $\mathbb{T}^2$ , there exists a sequence of divergence-free vector fields  $(b_n)_{n \in \mathbb{N}}$  such that one has*

$$\lim_{n \rightarrow \infty} \rho_{b_n} = \infty.$$

The proof of this result is presented at the end of the paper.

**2.2. Construction of the flow lines**

For all  $n \in \mathbb{N}$ , we define the sequence of  $\mathcal{C}^1$  functions by

$$\begin{aligned} \gamma_n : [0, 1] \times [0, 1] &\rightarrow [0, 1]; \\ (z, \theta) &\mapsto \gamma_n(z, \theta). \end{aligned}$$

where

$$\gamma_n(z, \theta) = \begin{cases} \frac{1}{4} \sin(2n\pi\theta) + z & \text{if } 0 \leq \frac{1}{4} \sin(2n\pi\theta) + z \leq 1 \\ \frac{1}{4} \sin(2n\pi\theta) + z - 1 & \text{if } \frac{1}{4} \sin(2n\pi\theta) + z \geq 1 \\ \frac{1}{4} \sin(2n\pi\theta) + z + 1 & \text{if } \frac{1}{4} \sin(2n\pi\theta) + z \leq -1 \end{cases}$$

**Remark 1**

For all  $z \in [0, 1]$ , we define the graph of the function  $\theta \mapsto \gamma_n(z, \theta)$  by

$$\Gamma_z^{(n)} = \left\{ (\theta, \gamma_n(z, \theta)); \theta \in [0, 1] \right\}.$$

**Remark 2** For all  $z \in [0, 1]$ , the curves  $\Gamma_z^{(n)}$  are disjoint since we have

$$\frac{d}{dz} \gamma_n(z, \theta) = 1 > 0$$

**Lemma 1** For all  $z \in [0, 1]$ , one has

$$\lim_{n \rightarrow \infty} \ell_{\mathbb{R}^2}(\Gamma_z^{(n)}) = \infty. \tag{1}$$

**Proof** Let be  $z \in [0, 1]$ . The curve  $\Gamma_z^{(n)}$  oscillates  $n$  times with an amplitude equal to  $\frac{1}{4}$ . It follows so that

$$\ell_{\mathbb{R}^2}(\Gamma_z^{(n)}) \rightarrow \infty.$$

□

**2.3. Construction of the vector fields**

We denote by  $(z, \theta) \in [0, 1] \times [0, 1]$  the coordinates on  $\mathbb{T}^2$ . With those variables we have the following expression for the gradient operator of any  $C^1$  function  $F$ :

$$\nabla F = \left( \frac{\partial F}{\partial z}, \frac{\partial F}{\partial \theta} \right).$$

We define the orthogonal gradient of  $F$  denoted  $\nabla^\perp F$  by

$$\nabla^\perp F = \left( -\frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial z} \right).$$

**Lemma 2** *The orthogonal gradient  $\nabla^\perp F$  of  $F$  satisfies the two following properties:*

1.  $(\nabla F)^t(\nabla^\perp F) = 0$ ;
2.  $\operatorname{div}(\nabla^\perp F) = 0$ .

**Proof** The first assertion is trivial. For the second property, note that

$$\begin{aligned} \operatorname{div}(\nabla^\perp F) &= \operatorname{div}\left(\left(-\frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial z}\right)\right) \\ &= \frac{\partial}{\partial z}\left(-\frac{\partial F}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial F}{\partial z}\right) = 0. \end{aligned}$$

Note that we used here that if we represent the vector field  $u$  in those coordinates as  $u = u_1\partial_z + u_2\partial_\theta$ , then its divergence becomes  $\operatorname{div}(u) = \left(\frac{\partial}{\partial z}u_1 + \frac{\partial}{\partial \theta}u_2\right)$ . □

□

**Definition 1** *Let  $(z, \theta) \in [0, 1] \times [0, 1]$ ; we use the implicit function theorem to define a function  $\alpha_n : [0, 1] \times [0, 1] \rightarrow [0, 1]$ , which verifies that*

$$\gamma_n(\alpha_n(z, \theta), \theta) = z$$

for all  $(z, \theta) \in [0, 1] \times [0, 1]$ .

**Remark 3** *Note that the function  $\alpha_n(z, \theta)$  is constant along any trajectory  $\Gamma_z^{(n)}$  for all  $(z, \theta) \in [0, 1] \times [0, 1]$ . Indeed, for any given point  $A$  of coordinates  $(z, \theta) \in [0, 1] \times [0, 1]$ , the function  $\alpha_n(z, \theta)$  determines for which curve  $\Gamma_z^{(n)}$  belongs to this point.*

From this family of functions, we define a sequence of vector fields  $b_n^0$  on  $\mathbb{T}^2$  by

$$b_n^0 := \nabla^\perp \alpha_n(z, \theta).$$

By construction  $b_n^0$  is a divergence-free vector field that generates a flow following the curves  $\Gamma_z^{(n)}$ .

**Remark 4** Note that for every function  $S \in \ker(\mathfrak{b}_n^0 \cdot \nabla) \cap \mathcal{C}^1(\mathbb{T}^2)$  we have that  $\operatorname{div}(\operatorname{Sb}_n^0) = 0$ . The vector field  $\mathfrak{b}_n = \operatorname{Sb}_n^0$  has the same trajectories as  $\mathfrak{b}_n^0$ . Those trajectories are the level sets of the function  $\gamma_n$  that are closed. Let  $(\phi_t)_{t \in \mathbb{R}}$  be the flow generated by  $\mathfrak{b}_n$ . For  $(z, \theta) \in [0, 1] \times [0, 1]$ , let  $\tau_{(z, \theta)}$  be the return time for the flow  $(\phi_t)_{t \geq 0}$ . In other words, we have

$$\phi_{\tau_{(z, \theta)}}(z, \theta) = (z, \theta).$$

**Lemma 3** For all  $n \in \mathbb{N}$ , there exists a function  $S \in \ker(\mathfrak{b}_n^0 \cdot \nabla) \cap \mathcal{C}^1(\mathbb{T}^2)$  such that the return time of the flow  $(\phi_t)_{t \in \mathbb{R}}$  generated by the vector field  $\mathfrak{b}_n = \operatorname{Sb}_n^0$  satisfies  $\tau_{(z, 0)} = z$  for all  $z \in [0, 1]$ .

**Proof** We introduce the flows  $(\phi_t)_{t \in \mathbb{R}}$  and  $(\phi_t^0)_{t \in \mathbb{R}}$  generated by  $\mathfrak{b}_n$  and  $\mathfrak{b}_n^0$  through the equations  $\dot{\phi}_t^0 = \mathfrak{b}_n^0(\phi_t^0)$  and  $\dot{\phi}_t = \mathfrak{b}_n(\phi_t)$ . This implies the relation  $\phi_t = \phi_{tS}^0$  for all  $t \in \mathbb{R}$ . If we denote by  $\tau_{(z, \theta)}^0$  the return time of  $(\phi_t^0)_{t \in \mathbb{R}}$  then we have that  $\phi_{\tau_{(z, \theta)}^0}^0(z, \theta) = (z, \theta)$ . The statement of our lemma requires that

$$\phi_{\frac{\tau_{(z, 0)}^0}{S(z, 0)}}(z, 0) = \phi_{\tau_{(z, 0)}^0}^0(z, 0) = (z, 0).$$

This implies that  $\tau_{(z, 0)} = \tau_{(z, 0)}^0/S(z, 0)$  is the return time of the flow  $(\phi_t)_{t \in \mathbb{R}}$  to the point  $(z, 0)$ . As a consequence the function  $S(z, \theta)$  has to equal  $\frac{\tau_{(z, 0)}^0}{z}$  when  $(z, \theta) \in \Gamma_z^{(n)}$ . Note also that  $z \mapsto \tau_{(z, 0)}^0$  is continuously differentiable (see [8]) and also in  $\ker(\mathfrak{b}_n^0 \cdot \nabla)$ . □ □

### 3. The comparison argument

#### 3.1. Some results concerning continuity along the trajectory

We define the following space of weak eigen functions of the antisymmetric operator  $\mathfrak{b} \cdot \nabla$ :

$$H_\mu^1 = \left\{ f \in H^1; \int_{\mathbb{T}^2} f(x) \mathfrak{b} \cdot \nabla \psi(x) d\omega_{\mathbb{R}^2} = i\mu \int_{\mathbb{T}^2} f(x) \psi(x) d\omega_{\mathbb{R}^2}, \forall \psi \in C^1(\mathbb{T}^2) \right\}.$$

Let  $\xi_\epsilon : \mathbb{R} \mapsto [0, \infty[$  be a smooth function, with support in  $] -\epsilon, \epsilon[$  and  $\int_{-\epsilon}^\epsilon \xi(s) ds = 1$ . For  $f \in H$  and  $x \in \mathbb{T}^2$ , we consider  $f_\epsilon(x) = \int_{-\epsilon}^\epsilon f \circ \phi_{-s} \xi_\epsilon(s) ds$ .

**Proposition 1** For any  $f \in H_\mu^1$  the function  $f_\epsilon$  satisfies that  $\|f_\epsilon - f\|_1 \rightarrow 0$  as  $\epsilon \downarrow 0$ . Moreover, for all  $x \in \mathbb{T}^2$  we have

$$f_\epsilon \circ \phi_t(x) = e^{i\mu t} f_\epsilon(x), \quad \forall t \in \mathbb{R}.$$

**Proof** The proof from [6] carries over without any changes to our situation. □ □

#### 3.2. The comparison manifold

On the Riemannian manifold  $\mathbb{T}^2$  we define the isoperimetric function

$$L_{\mathfrak{b}_n}(c) = \inf_{A \in I_c} \ell_{\mathbb{R}^2}(\partial A),$$

where

$$I_c = \left\{ A : \omega_{\mathbb{R}^2}(A) = c, A \subset \mathbb{T}^2 \text{ measurable and } \phi - \text{invariant} \right\}$$

Note that  $I_c$  contains all subsets of  $\mathbb{T}^2$  with surface  $c$  enclosed by flow lines of  $(\phi_t^{b_n})_{t \in \mathbb{R}}$ . It is also evident that  $L_{b_n}(c) = L_{b_n}(1 - c)$  because the Riemannian surface  $\mathbb{T}^2$  is considered with an empty boundary.

**Remark 5** *In the following lemma, we will construct a modified isoperimetric function that agrees with properties given in [1] to construct comparison manifolds.*

**Lemma 4** *There exists a sequence of function  $h_n : [0, 1] \rightarrow \mathbb{R}$  with*

1.  $h_n(c) \leq L_{b_n}(c), \quad \forall 0 \leq c \leq 1.$
2.  $h_n(c) \sim 2\sqrt{\pi}\sqrt{c}$  as  $c \rightarrow 0.$
3.  $\inf_{0 < c \leq \frac{1}{2}} \frac{h_n(c)}{c} \xrightarrow{n \rightarrow \infty} \infty.$

**Proof** The proof follows the same arguments as in [7]. The idea is to define the function  $h_n(c)$  that is smaller than  $L_{b_n}(c)$  for all  $0 \leq c \leq 1$  and it verifies the hypothesis given in [1] to construct the comparison manifold associated  $\mathbb{T}^2$ . □

**Proposition 2** *There exists a Riemannian metric  $g^*$  on  $S^2$  such that the resulting Riemannian manifold  $(M^*, g^*)$  satisfies the following four properties:*

1.  $\omega_{M^*}(M^*) = \omega_{\mathbb{R}^2}(\mathbb{T}^2) = 1.$
2. *the elements of  $SO(3)$  fixing the  $x_3$ -axis act as isometries on  $M^*$ .*
3. *Reflection with respect to the  $x_1x_2$ -plane acts as an isometry on  $M^*$ .*
4.  $\ell_{M^*}(\partial\Omega_c^*) = h_n(c)$  for all  $c$ , where  $\ell_{M^*}$  is the arc length measure on  $M^*$  and  $\Omega_c^*$  is the polar cap with volume  $\omega_{M^*}(\Omega_c^*) = c$  around  $n^*$ .

**Proof** See [6] for a proof based on some method from [4]. Note that the point (2) guarantees that the metric  $g^*$  behaves well in the poles  $n^*$  and  $s^*$  of  $S^2$ . As an alternative, one could also use some construction method from [1]. The proof follows the same lines of arguments as in the paper [6]. In other words, we start with the canonical metric  $g_0$  on  $S^2$  and transform it to a new metric  $g^*$ . Let  $\Omega_c$  be a part of  $S^2$ , with volume  $c$ , enclosed into two level profiles volume  $c$ . Its boundary of  $\Omega_c$  is a smooth curve. For some point  $x \in \partial\Omega_c$ , we can decompose any tangential vector  $V$  into a part  $V^\parallel$  tangent to  $\partial\Omega_c$  and a part  $V^\perp$  orthogonal to  $\partial\Omega_c$ . The metric  $g_0$  now satisfies, for any two tangential vectors  $V$  and  $W$ , that

$$g_0(V, W) = g_0(V^\perp, W^\perp) + g_0(V^\parallel, W^\parallel).$$

We define the new metric  $g^*$  through

$$g^*(V, W) = \left( h_n(c)(\ell_M(\partial\Omega_c))^{-1} \right)^2 g_0(V^\parallel, W^\parallel) + \left( h_n(c) \right)^{-1} (\ell_M(\partial\Omega_c))^2 g_0(V^\perp, W^\perp).$$

Finally, with the same arguments as in [4], we can prove that  $(M^*, g^*)$  satisfies the three stated properties.  $\square$

### 3.3. Rearrangement arguments on $M^*$

Let  $A$  be a measurable set of finite volume in  $\mathbb{T}^2$ . Its symmetric rearrangement  $A^*$  is the open ball centered around the zero in  $\mathbb{R}^2$  satisfying

$$\omega_{M^*}(A^*) = \omega_{\mathbb{R}^2}(A).$$

For all nonnegative measurable function  $f$ , the following representation holds almost everywhere with respect to  $\omega_{\mathbb{R}^2}$ :

$$f(x) = \int_0^\infty \chi_{\{y|f(y)>t\}}(x) dt.$$

We define its rearrangement function by

$$f^*(x) = \int_0^\infty \chi_{\{y|f(y)>t\}}^*(x) dt.$$

Here we used the notation  $\chi_A$  for the indicator function over the set  $A$ , which is one for  $x \in A$  and zero for  $x \notin A$ .

Then  $f^*$  is lower semicontinuous (since its level sets are open), and is uniquely determined by the distribution function  $\omega_{\mathbb{R}^2}(\{x; f(x) \geq s\})$  of  $f$ .

**Remark 6** *The function  $f^*$  constructed above verifies, for all  $s \in \mathbb{R}$ ,*

$$\omega_{\mathbb{R}^2}(\{x; f(x) \geq s\}) = \omega_{M^*}(\{x; f^*(x) \geq s\}).$$

*It follows that  $\|f\|_{L^2(\mathbb{T}^2, \omega_{\mathbb{R}^2})} = \|f^*\|_{L^2(M^*, \omega_{M^*})}$ .*

For general bounded function  $f$ , we define  $f^* := (f - \inf f)^* + \inf f$ .

**Remark 7** *Note that  $C^1(\mathbb{T}^2) \cap \ker(\mathbf{b} \cdot \nabla)$  is dense in  $H^1 \cap \ker(\mathbf{b} \cdot \nabla)$  with respect to the  $H^1$ -norm.*

**Proposition 3** *For all  $f \in C^1(\mathbb{R}^2) \cap \ker(\mathbf{b} \cdot \nabla)$  we have that*

$$\int_{\mathbb{R}^2} |\nabla f|^2 d\omega_{\mathbb{T}^2} \geq \int_{M^*} |\nabla f^*|^2 d\omega_{M^*}.$$

**Proof** The proof follows the line of arguments for proving Faber–Krahn-type inequalities as given in [2] and adopted in [6].  $\square$

**4. Proof of the main results**

**4.1. Asymptotics of the spectral gap**

The proof of Theorem 1 relies on the following asymptotic result for the spectral gap  $\rho_{cb}$  as  $|c| \uparrow \infty$ .

**Remark 8** Recall here the main result from [5], which said that

$$\lim_{m \rightarrow \infty} \rho_{mb} = \inf_{\mu \in \mathbb{R}} \inf \left\{ \int_{\mathbb{T}^2} |\nabla f|^2 d\omega_{\mathbb{R}^2}; f \in H_{\mu}^1 : \|f\| = 1 \right\} \tag{2}$$

with

$$H_{\mu}^1 = \left\{ f \in H^1; \int_{\mathbb{T}^2} f(x) b \cdot \nabla \psi(x) dx = i\mu \int_{\mathbb{T}^2} f(x) \psi(x) dx, \forall \psi \in C^1(\mathbb{T}^2) \right\}.$$

The following result simplifies the analysis of the infimum in the previous remark to the space  $H_0^1$  for our vector-fields  $b_n$ .

**Proposition 4** The sequence of divergence-free vector fields  $b_n$  constructed in section 2.3 satisfies that  $H_{\mu}^1 = \{0\}$  for all  $\mu \neq 0$ .

**Proof** Denote by  $\phi_t^{(n)}$  the flow generated by  $b_n$ . Let  $f \in H_{\mu}^1$ . As we saw in Proposition 1, there exists a sequence of function  $f_{\epsilon}$  in  $H_{\mu}^1$  with the property

$$f_{\epsilon} \circ \phi_t^{(n)} = e^{i\mu t} f_{\epsilon}.$$

By the fact that the return time of this flow is equal to  $z$  (see Lemma 3), one has that

$$f_{\epsilon}(x) = f_{\epsilon} \circ \phi_z(x) = e^{i\mu z} f_{\epsilon}(x) \quad \forall x \in \mathbb{T}^2.$$

It follows that, for all  $x \in M$ , we have either

$$\exists k \in \mathbb{Z}, z\mu = 2\pi k$$

or

$$f_{\epsilon}(x) = 0.$$

Since the absolute value of the function  $f_{\epsilon}$  is constant along the trajectories, we obtain that the function  $f_{\epsilon}$  is equal to zero  $\omega_{\mathbb{R}^2}$ -almost everywhere. This implies  $f$  is equal to zero  $\omega_{\mathbb{R}^2}$ -almost everywhere.  $\square \quad \square$

**Remark 9** Note that  $H_0^1$  can be identified to  $\ker(b \cdot \nabla) \oplus i\ker(b \cdot \nabla)$ .

We have to introduce an appropriate Cheeger constant for the space of rotationally invariant functions on  $M^*$ . For this denote by  $\mathcal{S}^*$  the set of all closed subsets in  $M^*$  that are invariant with respect to rotations fixing the  $x_3$ -axis. We define the Cheeger constant by

$$C^* = \inf_{A \in \mathcal{S}^*: \omega_{M^*}(A) < \frac{1}{2}} \frac{\ell_{M^*}(\partial A)}{\omega_{M^*}(A)}$$

For all  $f \in H^1(M^*)$  that are invariant by rotations through the  $x_3$ -axis, we have that: Note that the constant  $C^*$  depends on the vector field  $b_n$  used to construct  $M^*$ . We will emphasize this dependence through the notation  $C_n^*$  in the following Proposition.



**Proposition 5** We have  $\lim_{n \rightarrow \infty} C_n^* = \infty$ .

**Proof**

This result was proved in [6]. It uses as ingredient (3) from Lemma 4 and (4) from Proposition 2. □

The following proposition is based on Cheeger’s method applied to the function from  $M^*$  invariant by rotations fixing the  $x_3$ -axis.

**Proposition 6** For all  $f \in H^1(M^*)$  invariant by rotations through the  $x_3$ -axis, we have that

$$\frac{\int_{M^*} |\nabla f|^2 d\omega_{M^*}}{\int_{M^*} |f|^2 d\omega_{M^*}} \geq \frac{(C^*)^2}{4}.$$

**Proof** For the proof of this proposition, we follow the same arguments as in [6]. Recall that we only take care of functions that are rotationally invariant since their level sets are in  $\mathcal{S}^*$ . □

**Proof of Theorem 1** Let  $K > 0$  be large. By Proposition 6, for the sequence of vector fields  $(b_n)_{n \in \mathbb{N}}$  constructed in the first section satisfies that there exists  $n \in \mathbb{N}$  such that

$$\inf \left\{ \frac{\int_{M_n^*} |\nabla f^*|^2 d\omega_{M_n^*}}{\int_{M_n^*} |f^*|^2 d\omega_{M_n^*}}; f^* \in H^1(M_n^*) \right\} \geq \frac{(C_n^*)^2}{4} \geq 2K,$$

where  $M_n^*$  denotes the comparison manifold, which is associated to  $b_n$ . Further, it follows from Proposition 3 together with Remark 7 that

$$\inf \left\{ \frac{\int_{\mathbb{T}^2} |\nabla f|^2 d\omega_{\mathbb{R}^2}}{\int_{\mathbb{T}^2} |f|^2 d\omega_{\mathbb{R}^2}}; f \in H_0^1 \right\} \geq 2K.$$

Then Remark 8 and Proposition 4 imply for the vector field  $b_n$  chosen above

$$\lim_{a \rightarrow \infty} \rho_{ab_n} \geq 2K.$$

In conclusion, there exists large constant  $a \in \mathbb{R}$  such that  $\rho_{ab_n} > K$ . □

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