Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1887 – 1903

doi:10.3906/mat-1605-26

Approximation by integral functions of finite degree in variable exponent Lebesgue spaces on the real axis

Ramazan AKGÜN[1], **Arash GHORBANALIZADEH**[2,*]
[1]Department of Mathematics, Faculty of Arts and Sciences, Balıkesīr University, Çağış Yerleşkesi, Balıkesir, Turkey
[2]Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran

Received: 04.05.2016 • **Accepted/Published Online:** 26.04.2018 • **Final Version:** 24.07.2018

Abstract: We obtain several inequalities of approximation by integral functions of finite degree in generalized Lebesgue spaces with variable exponent defined on the real axis. Among them are direct, inverse, and simultaneous estimates of approximation by integral functions of finite degree in $L^{p(\cdot)}$. An equivalence of modulus of continuity with Peetre's K-functional is established. A constructive characterization of Lipschitz class is also obtained.

Key words: Direct theorem, inverse theorem, modulus of continuity, simultaneous approximation, Lipschitz class

1. Introduction

In recent years, variable exponent function spaces and approximation problems in variable exponent Lebesgue spaces $L^{p(x)}$ have attracted more attention (see Cruz-Uribe and Fiorenza [7], Diening et al. [9], and Sharapudinov [42]). Many authors have obtained analogues of classical results in function space with variable exponents because of their applications in elasticity theory [51], fluid mechanics [35, 36], differential operators [10, 36], nonlinear Dirichlet boundary value problems [25], nonstandard growth [27, 51], and variational calculus. Starting from the work of Orlicz [32], the theory of variable exponents and $L^{p(x)}$ was developed in the late 1900s. In fact, $L^{p(x)}$ is a modular space [14, 28] and under the condition $p^+ := \operatorname{ess\,sup}_{x\in\mathbb{R}} p(x) < \infty$, $L^{p(x)}$ becomes a particular case of Musielak–Orlicz spaces [28]. In subsequent years several problems in $L^{p(x)}$ were investigated in [8, 11, 24, 25, 37, 38, 40].

Variable exponent Lebesgue spaces on $[0, 2\pi]$ (or $[0, 1]$) and many fundamental results corresponding to the approximation of the function were developed by Sharapudinov [39, 41, 43–45]. Nowadays many problems for the approximation of the function are solved in these types of spaces defined on $[0, 2\pi] \subset \mathbb{R}$ (see, e.g., [2–5, 12, 13, 19–22]). In this direction, we aim to obtain direct and inverse theorems for approximation by entire functions of finite degree in variable exponent Lebesgue spaces on the whole real axis $\mathbb{R}$.

Recall that studies dealing with approximation by entire function of finite degree in the real domain date back to Bernstein's works, for example [6]. After his works, Wiener and Paley [33], Ackhiezer [1], Nikolskii [30], and Ibragimov [15–17] developed this subject. Various problems related to approximation of functions on

*Correspondence: ghorbanalizadeh@iasbs.ac.ir
2000 *AMS Mathematics Subject Classification:* Primary 46E30; Secondary 41A10, 41A17
The first author was supported by Balıkesir University Scientific Research Project No. 2018/001.

$\mathbb{R}$ by entire functions of exponential type in the L_p space were studied in the papers of Ackhiezer [1], Timan [48], Timan [49], Nikol'skii [30, 31], Ibragimov [15–18], Taberski [46, 47], Nasibov [29], Popov [34], Ligun [26], Vakarchuk [50], and others. Note that an entire function of finite exponential type is merely an entire function of order 1 and finite type, and in approximation theory these often play an important role similar to trigonometric polynomials in the case of approximation of periodic functions. Thus, for example, there are Bernstein-type inequalities for such functions.

In this work, we generalize the works of Ibragimov and Taberski about approximation of functions in Lebesgue spaces on the whole real axis in variable exponent settings. In what follows, $A \lesssim B$ will mean that there exists a positive constant $C_{u,v,\ldots}$ dependent only on the parameters $u, v, \ldots$ and it can be different in different places, such that the inequality $A \leq CB$ holds.

In Theorem 4.1, we obtain that if $p(\cdot) \in P$ (see Definition 2.1), then there exists a positive constant depending only on $p(\cdot)$, such that the following Jackson–Stechkin type inequality holds:

$$A_\sigma(f)_{p(\cdot)} \lesssim \Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)}, \tag{1.1}$$

where $f \in L^{p(\cdot)}$, $h > 0$, $T_h f(x) := \frac{1}{h}\int_0^h f(x+t)\,dt$, $(x \in \mathbb{R})$, $\Omega(f,\delta)_{p(\cdot)} := \sup\limits_{0<h\leq\delta} \|(I - T_h)f\|_{p(\cdot)}$, $\mathcal{G}_\sigma$ is the subspace of integral function $f(z)$ of exponential type $\leq \sigma$ belonging to $L^{p(\cdot)}$ and $A_\sigma(f)_{p(\cdot)} := \inf\limits_g \{\|f-g\|_{p(\cdot)} : g \in \mathcal{G}_\sigma\}$. Let $W_r^{p(\cdot)}$, $r \in \mathbb{N}$, be the class of functions $f \in L^{p(\cdot)}$ such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L^{p(\cdot)}$. In Theorem 6.1, for any $f \in W_r^{p(\cdot)}$, we show the following simultaneous approximation inequality:

$$A_\sigma(f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r} A_\sigma\left(f^{(r)}\right)_{p(\cdot)}, \quad r \in \mathbb{N}.$$

The weak inverse estimate of Theorem 4.1,

$$\Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)} \lesssim \frac{1}{\sigma}\sum_{\nu=0}^{\lfloor\sigma\rfloor} E_\nu(f)_{p(\cdot)},$$

is obtained in Theorem 5.1, where $\lfloor\sigma\rfloor := \max\{n \in \mathbb{Z} : n \leq \sigma\}$. For $0 < \beta < 1$ we define $Lip_\beta p(\cdot) := \{f \in L^{p(\cdot)} : \Omega(f,\delta)_{p(\cdot)} \lesssim \delta^\beta,\ \delta > 0\}$ and $W_{p(\cdot)}^{r,\beta} := \left\{f \in W_{p(\cdot)}^r : f^{(r)} \in Lip_\beta p(\cdot)\right\}$, and using this notation the following constructive description of the Lipschitz class $Lip_\beta p(\cdot)$ is proved.

Let $0 < \beta < 1$ and $r \in \{0\} \cup \mathbb{N}$, and then

$$f^{(r)} \in Lip_\beta p(\cdot) \text{ iff } A_\sigma(f)_{p(\cdot)} \lesssim \sigma^{-\beta-r}.$$

The rest of the paper is organized as follows. In Section 2 we introduce preliminaries and necessary facts. In Section 3, we give the definition of the modulus of continuity $\Omega(f,\cdot)_{p(\cdot)}$ and obtain an equivalence between $\Omega(f,\delta)_{p(\cdot)}$ and K-functional $K(f,\delta,L^{p(\cdot)},1)_{p(\cdot)}$. Sections 4 and 5 contain the direct and inverse theorems in variable exponent Lebesgue spaces on the real line. In Section 6 we obtain some inequalities on simultaneous approximation of functions in the corresponding Sobolev spaces $W_r^{p(\cdot)}$ and in Section 7 we obtain some constructive characterizations of the Lipschitz class $Lip_\beta p(\cdot)$.

2. Preliminaries

Let $p(x):\mathbb{R}\to[1,\infty)$ be a measurable function. We suppose that

$$1<p_{-}:=\text{ess}\inf_{x\in\mathbb{R}}p(x)\text{ and }p^{+}<\infty. \tag{2.1}$$

We define $L^{p(\cdot)}:=L^{p(\cdot)}(\mathbb{R})$ as the set of all functions $f:\mathbb{R}\to\mathbb{C}$ such that

$$I_{p(\cdot)}\left(\frac{f}{\lambda}\right):=\int_{\mathbb{R}}\left|\frac{f(y)}{\lambda}\right|^{p(y)}dy<\infty \tag{2.2}$$

for some $\lambda>0$. The set of of functions $L^{p(\cdot)}$, with norm

$$\|f\|_{p(\cdot)}:=\inf\left\{\eta>0:I_{p(\cdot)}\left(\frac{f}{\eta}\right)<1\right\},$$

is the Banach space.

Consider now an arbitrary, integral function $f(z)$; put

$$M(r)=\max_{|z|=r}|f(z)|,\qquad z=x+iy.$$

We say that f is of exponential type σ if the relation

$$\limsup_{r\to\infty}\frac{\ln M(r)}{r}\leq\sigma,\qquad\sigma<\infty$$

is valid. Let $\mathcal{G}_{\sigma}$ be the subspace of integral function $f(z)$ of exponential type σ belonging to $L^{p(\cdot)}$. The quantity

$$A_{\sigma}(f)_{p(\cdot)}:=\inf_{g}\{\|f-g\|_{p(\cdot)}:g\in\mathcal{G}_{\sigma}\}$$

where $f\in L^{p(\cdot)}$ is the deviation of the function $f\in L^{p(\cdot)}(\mathbb{R})$ from $\mathcal{G}_{\sigma}$.

For $f\in L^{p(\cdot)}$, we consider Steklov's mean operator:

$$T_{h}(f)=\frac{1}{h}\int_{0}^{h}f(x+t)dt.$$

Definition 2.1 *Let P be the class of measurable functions $p(\cdot)$ satisfying the conditions (2.1), $\exists c,C>0$, $C'\in\mathbb{R}$ such that*

$$|p(x)-p(y)|\leq\frac{c}{\log\left(e+1/|x-y|\right)},\quad\forall x,y\in\mathbb{R}, \tag{2.3}$$

$$|p(x)-C'|\leq\frac{C}{\log\left(e+|x|\right)},\quad\forall x\in\mathbb{R}. \tag{2.4}$$

It was proved in [9, Theorem 4.3.8] that if $p(\cdot)\in P$, then for $h>0$, the family of operators $\{T_{h}\}$ is uniformly bounded in $L^{p(\cdot)}$.

3. Modulus of continuity and K-functional

Let $f \in L^{p(\cdot)}$ and $h > 0$, and then we define the Steklov mean type operator:

$$T_h f(x) := \frac{1}{h}\int_0^h f(x+t)\,dt, \quad x \in \mathbb{R}.$$

The modulus of continuity of $f \in L^{p(\cdot)}$ is defined by

$$\Omega(f,\delta)_{p(\cdot)} := \sup_{0<h\leq\delta} \|(I-T_h)f\|_{p(\cdot)}. \tag{3.1}$$

If $f \in L^{p(\cdot)}$ and $\delta \geq 0$, then

$$\Omega(f,\delta)_{p(\cdot)} \lesssim \|f\|_{p(\cdot)} \tag{3.2}$$

holds for some constant depending only on $p(\cdot)$.

Theorem 3.1 *For $f, g \in L^{p(\cdot)}$ and $\delta \geq 0$, the modulus of continuity $\Omega(f,\delta)_{p(\cdot)}$ has the following properties:*

1. *$\Omega(f,\delta)_{p(\cdot)}$ is a nonnegative, nondecreasing function.*
2. *For $f, g \in L^{p(\cdot)}$ and $\delta > 0$,*
$$\Omega_{p(\cdot)}(f+g,\delta) \leq \Omega_{p(\cdot)}(f,\delta) + \Omega_{p(\cdot)}(g,\delta). \tag{3.3}$$
3. *For $f \in L^{p(\cdot)}$,*
$$\lim_{\delta\downarrow 0} \Omega_{p(\cdot)}(f,\delta) = 0. \tag{3.4}$$

Proof Properties (1) and (2), by definition of $\Omega(f,\delta)_{p(\cdot)}$ and the triangle inequality of $L^{p(\cdot)}$, are clearly valid. For proof of (3.4), using $f \in L^{p(\cdot)}$ and $\Omega(f,\delta)_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$, we can find $N > 1$ such that, for any fixed $\varepsilon > 0$,

$$\|f - T_h(f)\|_{L^{p(\cdot)}[(-\infty,-N)\cup(N,\infty)]} \leq \frac{\varepsilon}{2}. \tag{3.5}$$

We may assume that $\delta < 1$ and we have

$$\Omega_{p(\cdot)}(f,\delta) = \sup_{0<h\leq\delta} \|(I-T_h)f\|_{L^{p(\cdot)}(\mathbb{R})} \leq \sup_{0<h\leq\delta} \|(I-T_h)f\|_{L^{p(\cdot)}([-N,N])} + \frac{\varepsilon}{2}. \tag{3.6}$$

On the other hand, by [7, Corollary 2.73], there exists $\phi \in C_c[-N,N]$ such that

$$\|f-\phi\|_{L^{p(\cdot)}[-N,N]} \leq \varepsilon. \tag{3.7}$$

Let $N > 1$ be the same as the number found above. First we prove that in the case of $\phi \in C_c[-N,N]$, we have $\|\phi - T_h\phi\|_{L^{p(\cdot)}([-N,N])} < C_0\varepsilon$. Set

$$\mathrm{I} = I_{p(\cdot)}\left(\frac{\phi - T_h\phi}{N2^{\frac{1}{p_-}+1}\varepsilon}\right).$$

Then we have

$$\mathrm{I} = \int_{[-N,N]} \left| \frac{1}{N\ 2^{\frac{1}{p_-}+1} \varepsilon} (\phi(y) - T_h\phi(y)) \right|^{p(y)} dy$$
$$= \int_{[-N,N]} \left| \frac{1}{N\ 2^{\frac{1}{p_-}+1} \varepsilon} \frac{1}{\tau} \int_0^{\tau} (\phi(y+h) - \phi(y)) dh \right|^{p(y)} dy.$$

There exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$|\phi(x) - \phi(x+h)| < \varepsilon \tag{3.8}$$

for $0 \le h \le \delta_0$ and $x \in [-N, N]$. Hence, for $0 \le h \le \delta_0$, using (3.8), we have

$$\mathrm{I} \le 1.$$

Then we obtain

$$\|\phi - T_h\phi\|_{L^{p(\cdot)}([-N,N])} < N 2^{\frac{1}{p_-}+1} \varepsilon \tag{3.9}$$

for $0 \le h < \delta_0$.

Also, by uniform boundedness of T_h and (3.7), we have

$$\|T_h(\phi) - T_h(f)\|_{L^{p(\cdot)}([-N,N])} \le c(p)\varepsilon. \tag{3.10}$$

By the triangle inequality, we have

$$\begin{aligned} &\|f - T_h(f)\|_{L^{p(\cdot)}([-N,N])} \\ &\le \|f - \phi\|_{L^{p(\cdot)}([-N,N])} + \|\phi - T_h(\phi)\|_{L^{p(\cdot)}([-N,N])} + \|T_h(\phi) - T_h(f)\|_{L^{p(\cdot)}([-N,N])} \end{aligned} \tag{3.11}$$

for any $f \in L^{p(\cdot)}([-N, N])$. Then, by replacing (3.7), (3.9), and (3.10) in (3.11) we have

$$\|f - T_h(f)\|_{L^{p(\cdot)}([-N,N])} \le c(p)\varepsilon, \qquad 0 \le h \le \delta_0(\varepsilon). \tag{3.12}$$

Consequently, in view of (3.6), we have $\lim_{\delta \to 0^+} \Omega(f, \delta)_{p(\cdot)} = 0$ for every $f \in L^{p(\cdot)}$. □

For proof of Theorem 3.4 we need the following lemma.

Lemma 3.2 *Let $f \in W_1^{p(\cdot)}$ be given. Then*

$$\Omega(f, \delta)_{p(\cdot)} \lesssim \delta \|f'\|_{p(\cdot)}, \quad \delta \ge 0 \tag{3.13}$$

holds with some constant depending only on $p(\cdot)$.

Proof [Proof of Lemma 3.2] It is sufficient to prove the following inequality:

$$\|(I - T_h) f\|_{p(\cdot)} \lesssim h \|f'\|_{p(\cdot)}, \quad h > 0 \tag{3.14}$$

for and $f \in L^{p(\cdot)}$. We have

$$(I - T_h) f(x) = \frac{1}{h} \int_0^h (f(x) - f(x+t)) dt = \frac{-1}{h} \int_0^h \int_x^{x+t} f'(s) ds dt.$$

Therefore, from Minkowski's inequality for integrals, we get

$$\begin{aligned}\|(I-T_h)f\|_{p(\cdot)} &= \left\|\frac{1}{h}\int_0^h\int_x^{x+t} f'(s)\,dsdt\right\|_{p(\cdot)} \\ &= \left\|\frac{1}{h}\int_0^h t\frac{1}{t}\int_0^t f'(x+s)\,dsdt\right\|_{p(\cdot)} = \left\|\frac{1}{h}\int_0^h tT_t(f')\,dt\right\|_{p(\cdot)} \\ &\leq \frac{1}{h}\int_0^h t\|T_t(f')\|_{p(\cdot)}\,dt \lesssim \|f'\|_{p(\cdot)}\frac{1}{h}\int_0^h tdt \lesssim h\|f'\|_{p(\cdot)}\end{aligned}$$

and (3.14) follows. Then

$$\Omega(f,\delta)_{p(\cdot)} \lesssim \delta\|f'\|_{p(\cdot)},\ \delta>0$$

for $f \in W_1^{p(\cdot)}$. □

It is known that for proof of the inverse theorem we need Bernstein's inequality. We present the following theorem corresponding to the well-known Bernstein inequality on the derivative of exponential type entire functions of finite order (integral functions) in variable exponent Lebesgue spaces that was proved by Nanobashvili and Kokilashvili in [23].

Theorem 3.3 *[23, Theorem 2] Let $p \in P$ and g_σ be an exponential type entire function of degree $\leq \sigma$. Assume that $g_\sigma \in L^{p(\cdot)}$. Then the inequality*

$$\|g'_\sigma\|_{p(\cdot)} \lesssim \sigma\|g_\sigma\|_{p(\cdot)}$$

holds with a constant, independent of g_σ.

Let $f \in L^{p(\cdot)}$. The K-functional is defined as follows:

$$K(f,t,L^{p(\cdot)},1)_{p(\cdot)} = \inf_{g\in W_1^{p(\cdot)}}\left\{\|f-g\|_{p(\cdot)} + t\|g'\|_{(\cdot)}\right\}$$

for $t>0$.

In the following theorem we show that K-functional $K(f,\delta,L^{p(\cdot)},1)_{p(\cdot)}$ and $\Omega(f,\delta)_{p(\cdot)}$ are equivalent.

Theorem 3.4 *Let $p(\cdot) \in P$. If $L^{p(\cdot)}$, then the K-functional $K\left(f,t;L^{p(\cdot)},1\right)$ and the modulus $\Omega(f,t)_{p(\cdot)}$ are equivalent; namely,*

$$\Omega(f,t)_{p(\cdot)} \lesssim K\left(f,t;L^{p(\cdot)},1\right)_{p(\cdot)} \lesssim \Omega(f,t)_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}$ with some constants, independent of f.

Proof [Proof of Theorem 3.4] Let $t>0$. Then there exists $\sigma \in \mathbb{N}$ such that $1/\sigma \leq t < 2/\sigma$. We define the operator

$$(U_v f)(x) := \frac{2}{v}\int_{v/2}^{v}\left(\frac{1}{h}\int_0^h f(x+t)\,dt\right)dh, \quad x\in\mathbb{R}, \quad f\in L^{p(\cdot)}, \quad v>0.$$

On the other hand, for $0 < v \le 1$, we obtain by Minkowski's inequality for integrals

$$\begin{aligned} \|U_v f\|_{p(\cdot)} &= \left\| \frac{2}{v} \int_{v/2}^{v} \left(\frac{1}{h} \int_0^h f(x+t)\, dt \right) dh \right\|_{p(\cdot)} \\ &\le \frac{1}{v/2} \int_{v/2}^{v} \left\| \frac{1}{h} \int_0^h f(x+t)\, dt \right\|_{p(\cdot)} dh \\ &= \frac{1}{v/2} \int_{v/2}^{v} \|T_h f\|_{p(\cdot)}\, dh \lesssim \|f\|_{p(\cdot)} \frac{1}{v/2} \int_{v/2}^{v} dh \\ &= \|f\|_{p(\cdot)} \end{aligned}$$

and hence $f - U_v f \in L^{p(\cdot)}$. Also, the function $U_v f(x)$ is absolutely continuous [43] and

$$\left| \frac{d}{dx} U_v f(x) \right| = \frac{2}{v} \left| \int_{v/2}^{v} \frac{1}{h} \left(f(x+h) - f(x) \right) dh \right|.$$

For $0 < v \le 1$ we have by Minkowski's inequality for integrals

$$\begin{aligned} \left\| \frac{d}{dx} U_v f(x) \right\|_{p(\cdot)} &\le \frac{2}{v} \left\| \frac{1}{v} \int_0^v \left(f(x+t) - f(x) \right) dt - \frac{1}{v} \int_0^{v/2} \left(f(x+t) - f(x) \right) dt \right\|_{p(\cdot)} + \\ &\quad + \frac{2}{v} \left\| \int_{v/2}^{v} \frac{dh}{h^2} \left[\int_0^h \left(f(x+t) - f(x) \right) dt - \int_0^{v/2} \left(f(x+t) - f(x) \right) dt \right] \right\|_{p(\cdot)} \\ &\le \frac{2}{v} \left\| T_v f(x) - f(x) - \frac{1}{2} \left(T_{v/2} f(x) - f(x) \right) \right\|_{p(\cdot)} \\ &\quad + \frac{2}{v} \left\| \int_{v/2}^{v} \frac{1}{h} \left(T_h f(x) - f(x) - \frac{v}{2h} \left(T_h f(x) - f(x) \right) \right) dh \right\|_{p(\cdot)} \\ &\lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)} + \frac{1}{v} \Omega(f, v/2)_{p(\cdot)} \\ &\quad + \frac{1}{v} \left\| \int_{v/2}^{v} \frac{1}{h} \left(|T_h f(x) - f(x)| - |T_h f(x) - f(x)| \right) dh \right\|_{p(\cdot)} \\ &\lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)} + \frac{1}{v} \left\| \int_{v/2}^{v} \frac{1}{h} |T_h f(x) - f(x)|\, dh \right\|_{p(\cdot)} \\ &\quad + \frac{1}{v} \|T_h f(x) - f(x)\|_{p(\cdot)} \\ &\lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)} + \frac{1}{v} \int_{v/2}^{v} \frac{1}{h} \|T_h f(x) - f(x)\|_{p(\cdot)}\, dh \\ &\lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)} + \Omega(f, v)_{p(\cdot)} \frac{1}{v} \int_{v/2}^{v} \frac{dh}{h} \lesssim \frac{1}{v} \Omega(f, v)_{p(\cdot)}. \end{aligned} \tag{3.15}$$

Hence, for a given $v \in (0,1]$, $\frac{d}{dx} U_v f(x) \in L^{p(\cdot)}$. Then

$$\begin{aligned} K\left(f, t, L^{p(\cdot)}, W_p^1\right) &\le 2K\left(f, 1/\sigma, L^{p(\cdot)}, W_p^1\right) \\ &\lesssim \left\|f - U_{1/\sigma} f\right\|_{p(\cdot)} + \frac{1}{\sigma}\left\|\frac{d}{dx} U_{1/\sigma} f\right\|_{p(\cdot)} =: I_1 + I_2. \end{aligned}$$

We estimate I_1. Using Minkowski's inequality for integrals we obtain

$$\begin{aligned} \left\|f - U_{1/\sigma} f\right\|_{p(\cdot)} &= \left\|2\sigma \int_{1/2\sigma}^{1/\sigma} \left(\frac{1}{h}\int_0^h \left(f\left(x+t\right) - f\left(x\right)\right) dt\right) dh\right\|_{p(\cdot)} \\ &\le \left\|2\sigma \int_{1/2\sigma}^{1/v} \left|T_h f\left(x\right) - f\left(x\right)\right| dh\right\|_{p(\cdot)} \\ &\le 2\sigma \int_{1/2\sigma}^{1/\sigma} \left\|T_h f - f\right\|_{p(\cdot)} dh \\ &\lesssim \sup_{0\le u\le 1/\sigma} \left\|\left(I - T_u\right) f\right\|_{p(\cdot)} 2\sigma \int_{1/2\sigma}^{1/\sigma} dh = \Omega(f, 1/\sigma)_{p(\cdot)}. \end{aligned} \tag{3.16}$$

For the estimate I_2, we find from (3.15) that

$$\frac{1}{\sigma}\left\|\frac{d}{dx} U_{1/\sigma} f\right\|_{p(\cdot)} \lesssim \Omega\left(f, 1/\sigma\right)_{p(\cdot)}. \tag{3.17}$$

Now (3.16)–(3.17) give

$$K\left(f, t, L^{p(\cdot)}, 1\right) \lesssim \Omega\left(f, 1/\sigma\right)_{p(\cdot)} \le \Omega\left(f, t\right)_{p(\cdot)}.$$

By Lemma 3.2, for $g \in W_1^{p(\cdot)}$,

$$\Omega\left(f, t\right)_{p(\cdot)} \lesssim \left\|f - g\right\|_{p(\cdot)} + t\left\|g'\right\|_{p(\cdot)},$$

and taking infimum on $g \in W_1^{p(\cdot)}$ we get

$$\Omega\left(f, t\right)_{p(\cdot)} \lesssim K\left(f, t; L^{p(\cdot)}, 1\right).$$

Now we obtain

$$\Omega\left(f, t\right)_{p(\cdot)} \approx K\left(f, t; L^{p(\cdot)}, 1\right) \tag{3.18}$$

and this is the desired result. □

As a corollary of Theorem 3.4:

Corollary 3.5 *Let $p(\cdot) \in P$. If $\delta, \lambda \in \mathbb{R}^+$, $f \in L^{p(\cdot)}$ and then*

$$\Omega\left(f, \lambda\delta\right)_{p(\cdot)} \lesssim \left(1 + \lfloor\lambda\rfloor\right) \Omega\left(f, \delta\right)_{p(\cdot)} \tag{3.19}$$

holds with some constant depending only on $p(\cdot)$.

Proof [Proof of Corollary 3.5] Using equivalence (3.18) we have

$$\begin{aligned}\Omega(f,lt)_{p(\cdot)} &\lesssim \inf_{g\in W_1^{p(\cdot)}}\left\{\|f-g\|_{p(\cdot)}+lt\,\|g'\|_{p(\cdot)}\right\}\\ &\lesssim (1+\lfloor l\rfloor)\inf_{g\in W_1^{p(\cdot)}}\left\{\|f-g\|_{p(\cdot)}+t\,\|g'\|_{p(\cdot)}\right\}\\ &\lesssim (1+\lfloor l\rfloor)\,\Omega(f,t)_{p(\cdot)},\end{aligned}$$

which gives (3.19). □

4. Direct theorems

Theorem 4.1 *Let $p(\cdot)\in P$. If $f\in L^{p(\cdot)}$, then*

$$A_\sigma(f)_{p(\cdot)} \lesssim \Omega\left(f,\frac{1}{\sigma}\right)_{p(\cdot)} \tag{4.1}$$

holds with some constant depending only on $p(\cdot)$.

Proof [Proof of Theorem 4.1] Let σ and $f\in L^{p(\cdot)}$ be fixed. We consider the operator $U_{1/\sigma}f$. Using (3.16) and (3.17),

$$\begin{aligned}A_\sigma(f)_{p(\cdot)} &= A_\sigma(f-U_{1/\sigma}f+U_{1/\sigma}f)_{p(\cdot)} \le A_\sigma(f-U_{1/\sigma}f)_{p(\cdot)}+A_\sigma(U_{1/\sigma}f)_{p(\cdot)}\\ &\lesssim \|f-U_{1/\sigma}f\|_{p(\cdot)}+\frac{1}{\sigma}\left\|\frac{d}{dx}U_{1/\sigma}f(x)\right\|_{p(\cdot)} \lesssim \Omega\left(f,\frac{1}{\sigma}\right)_{p(\cdot)},\end{aligned} \tag{4.2}$$

and the result follows. □

We define

$$g(x)=\left(\frac{1}{x}\sin\frac{\sigma x}{2r}\right)^{2r}$$

for $r\ge 3/2$. Then $g(x)\in\mathcal{G}_\sigma$ for $r\ge 3/2$. Set

$$\gamma_r := \int_{\mathbb{R}}\left(\frac{1}{t}\sin\frac{\sigma t}{2r}\right)^{2r}dt.$$

In this case,

$$\gamma_r=\sigma^{2r-1}C,$$

where $C>0$ is dependent only on r.

Let

$$D_\sigma f(x) := \frac{1}{\gamma_r}\int_{\mathbb{R}} f(x+t)g(t)dt,\quad \sigma>0. \tag{4.3}$$

Then $D_\sigma f\in\mathcal{G}_\sigma$ ([17]).

Corollary 4.2 *The subspace of integral function* $f(z)$ *of exponential type* σ *belonging to* $L^{p(\cdot)}$ *is dense in* $L^{p(\cdot)}$.

Lemma 4.3 *Let* $p(\cdot) \in P$. *If* $f \in W_1^{p(\cdot)}$, *then*

$$\|f - D_\sigma f\|_{p(\cdot)} \lesssim \frac{1}{\sigma} \|f'\|_{p(\cdot)} \tag{4.4}$$

holds with some constant depending only on $p(\cdot)$.

Proof [Proof of Lemma 4.3] From (4.3), one can write

$$\begin{aligned}
\|f - D_\sigma f\|_{p(\cdot)} &= \left\| \frac{1}{\gamma_r} \int_{\mathbb{R}} (f(x+t) - f(x)) g(t) dt \right\|_{p(\cdot)} \\
&= \frac{1}{\gamma_r} \left\| \int_{\mathbb{R}} (f(x+t) - f(x)) g(t) dt \right\|_{p(\cdot)} \\
&= \frac{1}{\gamma_r} \left\| \int_{\mathbb{R}} \frac{1}{t} \int_x^{x+t} f'(\tau) d\tau t g(t) dt \right\|_{p(\cdot)} = \frac{1}{\gamma_r} \left\| \int_{\mathbb{R}} T_t f'(x) t g(t) dt \right\|_{p(\cdot)} \\
&\lesssim \frac{1}{\gamma_r} \int_{\mathbb{R}} \|T_t f'\|_{p(\cdot)} |t| |g(t)| dt \lesssim \|f'\|_{p(\cdot)} \frac{2}{\gamma_r} \int_0^\infty |t| |g(t)| dt \\
&\lesssim \|f'\|_{p(\cdot)} \left\{ \frac{1}{\gamma_r} \int_{|t| \le 1/\sigma} |t| |g(t)| dt + \frac{1}{\gamma_r} \int_{|t| \ge 1/\sigma} |t| |g(t)| dt \right\} \lesssim \frac{1}{\sigma} \|f'\|_{p(\cdot)},
\end{aligned}$$

which implies inequality (4.4). □

5. Inverse estimate

Now we present the inverse theorem.

Theorem 5.1 *Let* $p(\cdot) \in P$ *and* $f \in L^{p(\cdot)}$. *Then there exists a positive constant, depending only on* $p(\cdot)$, *such that*

$$\Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)} \lesssim \frac{1}{\sigma} \sum_{\nu=1}^{\lfloor \sigma \rfloor} A_\nu(f)_{p(\cdot)}$$

holds, where $\lfloor \sigma \rfloor$ *is the largest integer less than or equal to* σ.

Proof [Proof of Theorem 5.1] Let g_σ be an exponential type entire function of degree $\le \sigma$, belonging to $L^{p(\cdot)}$, as the best approximation of $f \in L^{p(\cdot)}$. Let $2^j \le \sigma < 2^{j+1}$. Thanks to the definition of $K(f, t, L^{p(\cdot)}, 1)_{p(\cdot)}$ we have

$$\begin{aligned}
K\left(f, \frac{1}{\sigma}, L^{p(\cdot)}, 1\right)_{p(\cdot)} &= \inf_{g \in W^{p(\cdot)}} \left\{ \|f - g\|_{p(\cdot)} + \frac{1}{\sigma} \|g'\|_{p(\cdot)} \right\} \\
&\le \|f - g_{2^{j+1}}\|_{p(\cdot)} + \frac{1}{\sigma} \|g'_{2^{j+1}}\|_{p(\cdot)}.
\end{aligned}$$

Using Theorem 3.3, one can write

$$\|g'_{2^{j+1}}\|_{p(\cdot)} = \|g'_0 - g'_1\|_{p(\cdot)} + \sum_{i=0}^{j} \|g'_{2^{i+1}} - g'_{2^i}\|_{p(\cdot)}$$

$$\lesssim \left\{ \|g_1 - g_0\|_{p(\cdot)} + \sum_{i=0}^{j} 2^{i+1} \|g_{2^{i+1}} - g_{2^i}\|_{p(\cdot)} \right\}$$

and then we have

$$\|g'_{2^{j+1}}\|_{p(\cdot)} \lesssim \left\{ A_0(f)_{p(\cdot)} + A_1(f)_{p(\cdot)} + \sum_{i=0}^{j} 2^{i+1} \left(A_{2^{i+1}}(f)_{p(\cdot)} + A_{2^i}(f)_{p(\cdot)} \right) \right\}$$

$$\lesssim \left\{ A_0(f)_{p(\cdot)} + \sum_{i=0}^{j} 2^{i+1} A_{2^i}(f)_{p(\cdot)} \right\}$$

$$\lesssim \left\{ A_0(f)_{p(\cdot)} + 2A_1(f)_{p(\cdot)} + \sum_{i=1}^{j} 2^{i+1} A_{2^i}(f)_{p(\cdot)} \right\}.$$

Since

$$2^{i+1} A_{2^i}(f)_{p(\cdot)} \leq 4 \sum_{\nu=2^{i-1}+1}^{2^i} A_\nu(f)_{p(\cdot)}, \tag{5.1}$$

we have

$$\|g'_{2^{j+1}}\|_{p(\cdot)} \lesssim \left\{ A_0(f)_{p(\cdot)} + 2A_1(f)_{p(\cdot)} + 4 \sum_{\nu=2}^{2^j} A_\nu(f)_{p(\cdot)} \right\}.$$

Now, using (5.1), we obtain

$$A_{2^{j+1}}(f)_{p(\cdot)} = \frac{2^{j+1} A_{2^{j+1}}(f)_{p(\cdot)}}{2^{j+1}} \leq \frac{2^{j+1} A_{2^{j+1}}(f)_{p(\cdot)}}{\sigma} \leq \frac{4}{\sigma} \sum_{\nu=2^{j-1}+1}^{2^j} A_\nu(f)_{p(\cdot)}.$$

By Theorem 3.4, one can write

$$\Omega\left(f, \frac{1}{\sigma}\right)_{p(\cdot)} \lesssim K\left(f, \frac{1}{\sigma}, L^{p(\cdot)}, 1\right)_{p(\cdot)} \lesssim \left\{ \|f - g_{2^{j+1}}\|_{p(\cdot)} + \frac{1}{\sigma} \|g'_{2^{j+1}}\|_{p(\cdot)} \right\}$$

$$\lesssim \frac{1}{\sigma} \sum_{\nu=2^{j-1}+1}^{2^j} A_\nu(f)_{p(\cdot)} \lesssim \frac{1}{\sigma} \sum_{\nu=1}^{\lfloor \sigma \rfloor} A_\nu(f)_{p(\cdot)}$$

and this completes the proof. □

Theorem 5.2 *Let $p(\cdot) \in P$ and $f \in L^{p(\cdot)}$. If*

$$\sum_{\nu=0}^{\infty} \nu^{r-1} A_\nu(f)_{p(\cdot)} < \infty$$

holds for some $r\in\mathbb{N}$, *then* $f^{(r)}\in L^{p(\cdot)}$ *and*

$$\Omega\left(f^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)}\lesssim\frac{1}{\sigma}\sum_{\nu=0}^{\lfloor\sigma\rfloor}(\nu+1)^{r}A_{\nu}(f)_{p(\cdot)}+\sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty}\nu^{r-1}A_{\nu}(f)_{p(\cdot)}\tag{5.2}$$

with some constant depending only on $p(\cdot)$.

Proof [Proof of Theorem 5.2] Let g_σ be an exponential type entire function of degree $\le\sigma$, belonging to $L^{p(\cdot)}$, as the best approximation of $f\in L^{p(\cdot)}$. For natural numbers $p\le r$, we consider the series

$$g_1^{(p)}+\sum_{\nu=0}^{\infty}\{g_{2^{\nu+1}}^{(p)}-g_{2^{\nu}}^{(p)}\}.\tag{5.3}$$

Using Bernstein's inequality (see Theorem 3.3) we have

$$\begin{aligned}\|g_{2^{(\nu+1)}}^{(p)}-g_{2^{\nu}}^{(p)}\|_{(\cdot)}&\lesssim\sigma^{p}\|g_{2^{(\nu+1)}}-g_{2^{\nu}}\|_{p(\cdot)}\lesssim 2^{(\nu+1)p}\ \|g_{2^{\nu+1}}-g_{2^{\nu}}\|_{p(\cdot)}\\&\lesssim 2^{(\nu+1)p}\ A_{2^{\nu}}(f)_{p(\cdot)}.\end{aligned}$$

Now, by the following estimation,

$$2^{(\nu+1)p}A_{2^{\nu}}(f)_{p(\cdot)}\le 2^{2p}\sum_{\mu=2^{\nu-1}+1}^{2^{\nu}}\mu^{p-1}A_{\mu}(f)_{p(\cdot)},$$

we have

$$\begin{aligned}\|g_1^{(p)}+\sum_{\nu=0}^{\infty}\{g_{2^{\nu+1}}^{(p)}-g_{2^{\nu}}^{(p)}\}\|_{p(\cdot)}&\le\|g_1^{(p)}\|_{p(\cdot)}+\sum_{\nu=0}^{\infty}\|g_{2^{\nu+1}}^{(p)}-g_{2^{\nu}}^{(p)}\|_{p(\cdot)}\\&\lesssim\|g_1^{(p)}\|_{p(\cdot)}+\sum_{\nu=0}^{\infty}2^{(\nu+1)p}\ A_{2^{\nu}}(f)_{p(\cdot)}\\&\lesssim\|g_1^{(p)}\|_{p(\cdot)}+2^{p}A_1(f)_{p(\cdot)}+\sum_{\mu=2^{\nu-1}+1}^{2^{\nu}}\mu^{p-1}A_{\mu}(f)_{p(\cdot)}\\&\lesssim\|g_1^{(p)}\|_{p(\cdot)}+A_1(f)_{p(\cdot)}+\sum_{\mu=2}^{\infty}\mu^{p-1}A_{\mu}(f)_{p(\cdot)}<\infty.\end{aligned}$$

If we denote the partial sum of the above series by $S_n^{(p)}$, for $p=0,1,2,...,r,$ then the sequence of $S_n^{(p)}$ has convergence in the norm of $L^{p(\cdot)}$. For $p=r$, one can write

$$\Omega\left(f^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)}\le\Omega\left(f^{(r)}-S_n^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)}+\Omega\left(S_n^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)}=I_1+I_2.$$

Now for obtaining inequality (5.2), we must estimate I_1 and I_2. First, let us deal with the first item, I_1. We choose $2^n \le \sigma < 2^{n+1}$. By boundedness of the operator T_h and Bernstein's inequality, we obtain

$$\begin{aligned}\Omega\left(f^{(r)}-S_n^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)} &\lesssim \|f^{(r)}-S_n^{(r)}\|_{p(\cdot)}\\ &=\left\|\sum_{\nu=n+1}^{\infty}\{g_{2^{\nu+1}}^{(r)}-g_{2^{\nu}}^{(r)}\}\right\|_{p(\cdot)} \lesssim \sum_{\nu=n+1}^{\infty} 2^{(\nu+1)r}\ A_{2^{\nu}}(f)_{p(\cdot)}\\ &\lesssim \sum_{\nu=n+1}^{\infty}\left\{2^{2r}\sum_{\mu=2^{\nu-1}+1}^{2^{\nu}}\mu^{r-1}A_{\mu}(f)_{p(\cdot)}\right\}\\ &\lesssim \sum_{\mu=2^n+1}^{\infty}\mu^{r-1}A_{\mu}(f)_{p(\cdot)} \lesssim \sum_{\mu=\lfloor\sigma\rfloor+1}^{\infty}\mu^{r-1}A_{\mu}(f)_{p(\cdot)}.\end{aligned}$$

Next, let us estimate I_2:

$$\Omega\left(S_n^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)} \le \Omega\left(g_1^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)}+\sum_{\nu=0}^{n}\Omega\left(g_{2^{\nu+1}}^{(r)}-g_{2^{\nu}}^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)}.$$

Now by inequality (3.13) and Bernstein's inequality (see Theorem 3.3), we have

$$\begin{aligned}\Omega\left(S_n^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)} &\lesssim \frac{1}{\sigma}\|g_1^{(r+1)}-g_0^{(r+1)}\|_{p(\cdot)}+\frac{1}{\sigma}\sum_{\nu=0}^{n}\|g_{2^{\nu+1}}^{(r+1)}-g_{2^{\nu}}^{(r+1)}\|_{p(\cdot)}\\ &\lesssim \frac{1}{\sigma}\|g_1-g_0\|_{p(\cdot)}+\frac{1}{\sigma}\sum_{\nu=0}^{n}2^{(\nu+1)(r+1)}A_{2^{\nu}}(f)_{(\cdot)}\\ &\lesssim \frac{1}{\sigma}\left\{A_0(f)_{p(\cdot)}+A_1(f)_{p(\cdot)}+\sum_{\nu=1}^{n}2^{2(r+1)}\sum_{\mu=2^{\nu-1}+1}^{2^{\nu}}\mu^{r}A_{\mu}(f)_{p(\cdot)}\right\}\\ &\lesssim \frac{1}{\sigma}\left\{\sum_{\mu=0}^{2^n}(\mu+1)^rA_{\mu}(f)_{p(\cdot)}\right\} \lesssim \frac{1}{\sigma}\left\{\sum_{\mu=0}^{\lfloor\sigma\rfloor}(\mu+1)^rA_{\mu}(f)_{p(\cdot)}\right\}.\end{aligned}$$

The last inequality completes the proof. □

6. Simultaneous approximation

Theorem 6.1 *Let $p(\cdot)\in P$, $r\in\mathbb{N}$, and $f\in W_r^{p(\cdot)}$. Then*

$$A_{\sigma}(f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r}A_{\sigma}\left(f^{(r)}\right)_{p(\cdot)} \tag{6.1}$$

holds with some constant depending only on $p(\cdot)$.

Proof [Proof of Theorem 6.1] Let $r=1$. Suppose that $A_{\sigma}(f')_{p(\cdot)}=\|f'-\Theta_n(f')\|_{p(\cdot)}$, $\Theta_n(f')\in\mathcal{G}_{\sigma}$ and

$$F(x):=\int_0^x \Theta_n(f')(t)\,dt$$

for $x>0$. Then $F\in\mathcal{G}_\sigma$ ([17]) and $F'(x)=\Theta_n(f')(x)$. Thus,

$$\begin{aligned}A_\sigma(f)_{p(\cdot)} &= A_\sigma(f-F)_{p(\cdot)}\lesssim\frac{1}{\sigma}\left\|(f-F)'\right\|_{p(\cdot)}\\ &= \frac{1}{\sigma}\left\|f'-F'\right\|_{p(\cdot)}=\frac{1}{\sigma}\left\|f'-\Theta_n(f')\right\|_{p(\cdot)}\\ &\lesssim \frac{1}{\sigma}A_\sigma(f')_{p(\cdot)}\end{aligned}$$

(6.1) follows from the last inequality. □

Corollary 6.2 *Let $p(\cdot)\in P$. Then for every $f\in W_r^{p(\cdot)}$, $r\in\{0\}\cup\mathbb{N}$, the inequalities*

$$A_\sigma(f)_{p(\cdot)}\lesssim\frac{1}{\sigma^r}\Omega\left(f^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)} \tag{6.2}$$

hold with constants depending only on $p(\cdot)$.

7. constructive characterization of Lipschitz classes

Theorem 7.1 *Under the conditions of Theorem 4.1, if the inequality*

$$A_\sigma(f)_{p(\cdot)}\lesssim\sigma^{-\beta}$$

holds for some $\beta>0$, then we have

$$\Omega(f,\delta)_{p(\cdot)}\lesssim\begin{cases}\delta^\beta & ,\quad 1>\beta;\\ \delta^\beta\log\frac{1}{\delta} & ,\quad 1=\beta;\\ \delta & ,\quad 1<\beta.\end{cases}$$

Proof [Proof of Theorem 7.1] Let $f\in L^{p(\cdot)}$ and

$$A_\sigma(f)_{p(\cdot)}\lesssim\sigma^{-\beta}$$

for some $\beta>0$. We suppose that $\delta>0$ and $N:=\lfloor 1/\delta\rfloor$. From Theorem 5.1 we get

$$\begin{aligned}\Omega(f,\delta)_{p(\cdot)} &\le \Omega\left(f,\frac{1}{N}\right)_{p(\cdot)}\lesssim\frac{1}{N}\sum_{\nu=0}^{N}A_\nu(f)_{p(\cdot)}\\ &\lesssim \frac{1}{N}A_0(f)_{p(\cdot)}+\frac{1}{N}\sum_{\nu=1}^{N}A_\nu(f)_{p(\cdot)}\\ &\lesssim \frac{1}{N}\left(\|f\|_{p(\cdot)}+\sum_{\nu=1}^{N}\frac{1}{\nu^\beta}\right).\end{aligned}$$

If $1>\beta$, then by some computations we get

$$\Omega_r(f,\delta)_{p(\cdot)}\lesssim\frac{1}{N}\left(\|f\|_{p(\cdot)}+\sum_{\nu=1}^{N}\frac{1}{\nu^\beta}\right)\lesssim\delta^\beta.$$

If $1=\beta$, then

$$\sum_{\nu=1}^{N}\nu^{-\beta}=\sum_{\nu=1}^{n}\nu^{-1}\leq 1+\log(1/\delta)$$

and hence

$$\Omega\left(f,\delta\right)_{p(\cdot)}\lesssim\delta^{\beta}\log(1/\delta).$$

If $1<\beta$, then the series $\sum_{j=0}^{\infty}j^{-\beta}$ is convergent and

$$\Omega\left(f,\delta\right)_{p(\cdot)}\lesssim\delta\left(A_0(f)_{p(\cdot)}+\sum_{j=1}^{\infty}j^{-\beta}\right)\lesssim\delta$$

holds. □

Using Theorem 5.2 we similarly get the following:

Corollary 7.2 *Let $p(\cdot)\in P$ and $f\in L^{p(\cdot)}$. If*

$$A_{\sigma}\left(f\right)_{p(\cdot)}\lesssim\frac{1}{\sigma^{r+\alpha}},\quad\alpha>0,$$

then $f\in W_{p(\cdot)}^{r}$ and

$$\Omega\left(f^{(r)},\delta\right)_{p(\cdot)}\lesssim\begin{cases}\delta^{\alpha} & ,1>\alpha,\\ \delta^{\alpha}\log\left(1/\delta\right) & ,1=\alpha,\\ \delta & ,1<\alpha.\end{cases}$$

Theorem 7.3 *Let $0<\beta<1$ and $r\in\mathbb{N}$. Under the conditions of Theorem 4.1, we have:*

$$(i)\quad f\ \in\ Lip_{\beta}p\left(\cdot\right)\quad \textit{iff}\quad A_{\sigma}\left(f\right)_{p(\cdot)}\lesssim\sigma^{-\beta}.$$

$$(ii)\quad f\ \in\ W_{p(\cdot)}^{r,\beta}\quad \textit{iff}\quad A_{\sigma}\left(f\right)_{p(\cdot)}\lesssim\sigma^{-\beta-r}.$$

Acknowledgment

The authors are indebted to the referees for valuable comments and suggestions.

References

[1] Ackhiezer NI. Theory of Approximation. English Translation of Second Edition. New York, NY, USA: Frederick Ungar, 1956.

[2] Akgün R, Israfilov D. Approximation in weighted Orlicz spaces. Mathematica Slovaca 2011; 61: 601-618.

[3] Akgün R, Kokilashvili V. The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces. Georgian Math J 2011; 18: 399-423.

[4] Akgün R, Kokilashvili V. On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces. Banach J Math Anal 2011; 5: 70-82.

[5] Akgün R, Kokilashvili V. Refined estimates of trigonometric approximation for functions with generalized derivatives in weighted variable exponent Lebesgue spaces. Georgian Math J 2012; 19: 611-626.

[6] Bernstein SN. Collected works. Vol. I. Moscow, USSR: Izdatel'stvo Akademii Nauk SSSR, 1952.

[7] Cruz-Uribe D, Fiorenza A. Variable Lebesgue Spaces. Foundations and Harmonic Analysis. New York, NY, USA: Birkhauser, 2013.

[8] Diening L. Maximal function on generalized Lebesgue spaces $L^{p(x)}$. Math Inequal Appl 2004; 7: 245-253.

[9] Diening L, Harjulehto P, Hasto P, Ružička M. Lebesgue and Sobolev Spaces with Variable Exponents. Berlin, Germany: Springer, 2011.

[10] Diening L, Ružička M. Calderon–Zymund Operators on Generalized Lebesgue Spaces $L^{p(x)}$ and Problems Related to Fluid Dynamics. Preprint. Freiburg, Germany: Albert-Ludwings-Universität, 2002.

[11] Fan X, Zhao D. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J Math Anal Appl 2001; 263: 424-446.

[12] Guven A, Israfilov D., Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$. J Math Inequal 2010; 4: 285-299.

[13] Guven A, Israfilov D. On approximation in weighted Orlicz spaces. Mathematica Slovaca 2012; 62: 77-86.

[14] Hudzik H. On generalized Orlicz–Sobolev space. Funct Approximatio Comment Math 1976; 4: 37-51.

[15] Ibragimov II. Extremal problems in a class of entire functions of exponential type. Uspehi Mat Nauk (NS) 1957; 12: 323-328 (in Russian).

[16] Ibragimov II. Extremal problems in a class of entire functions of finite degree. Izv Akad Nauk SSSR Ser Mat 1959; 23: 243-256 (in Russian).

[17] Ibragimov II. The Theory of Approximation by Entire Functions. Baku, Azerbaijan: Elm, 1979 (in Russian).

[18] Ibragimov II, Nasibov FG. The estimation of the best approximation of a summable function on the real axis by means of entire functions of finite degree. Dokl Akad Nauk SSSR 1970; 194: 1013-1016 (in Russian).

[19] Israfilov DM, Testici A. Approximation in Smirnov classes with variable exponent. Complex Var Elliptic 2015; 60: 1243-1253.

[20] Israfilov DM, Testici A. Approximation by Faber–Laurent rational functions in Lebesgue spaces with variable exponent. Indagat Math 2016; 27: 914-922.

[21] Jafarov S. Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents. Ukr Math J 2015; 66: 1509-1518.

[22] Jafarov S. Approximation by trigonometric polynomials in subspace of variable exponent grand Lebesgue spaces. Global J Math 2016; 8: 836-843.

[23] Kokilashvili V, Nanobashvili I. Boundedness criteria for the majorants of Fourier integrals summation means in weighted variable exponent Lebesgue spaces and application. Georgian Math J 2013; 20: 721-727.

[24] Kokilashvili V, Samko S. Singular integrals and potentials in some Banach function spaces with variable exponent. J Funct Space Appl 2003; 1: 45-59.

[25] Kováčik ZO, Rákosnik J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czech Math J 1991; 41: 592-618.

[26] Ligun AA, Doronin VG. Exact constants in Jackson-type inequalities for the L_2-approximation on a straight line. Ukr Math J 2009; 61: 112-120.

[27] Marcellini P. Regularity and existence of solutions of elliptic equations with p,q-growth conditions. J Differ Equations 1991; 90: 1-30.

[28] Musielak J. Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics. Berlin, Germany: Springer-Verlag, 1983.

[29] Nasibov FG. Approximation in L_2 by entire functions. Akad Nauk Azerbaidzhan SSR Dokl 1986; 42: 3-6 (in Russian).

[30] Nikolskii SM. Inequalities for entire functions of finite degree and their application to the theory of differentiable functions of several variables. Amer Math Soc Transl Ser 2 1969; 80: 1-38.

[31] Nikolskii SM. Approximation of Functions of Several Variables and Imbedding Theorems. Translated from the Russian by John M. Danskin, Jr. New York, NY, USA: Springer-Verlag, 1975.

[32] Orlicz W. Über konjugierte Exponentenfolgen. Stud Math 1931; 3: 200-212 (in German).

[33] Paley R, Wiener N. Fourier Transforms in the Complex Domain. Providence, RI, USA: American Mathematics Society, 1934.

[34] Popov VY. Best mean square approximations by entire functions of exponential type. Izv Vysš Ucebn Zaved Matematika 1972; 121: 65-73 (in Russian).

[35] Rajagopal KR, Ružička M. On the modeling of electrorheological materials. Mech Res Commun 1996; 23: 401-407.

[36] Ružička M. Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics. Berlin, Germany: Springer-Verlag, 2000.

[37] Samko S. Differentiation and integration of variable order and the spaces $L^{p(x)}$. In: Operator Theory for Complex and Hypercomplex Analysis, Mexico City, 1994. Providence, RI, USA: American Mathematics Society, 1998, pp. 203-219.

[38] Sharapudinov II. The topology of the space $L^{p(t)}([0, 1])$. Mat Zametki 1979; 26: 613-632 (in Russian).

[39] Sharapudinov II. The basis property of the Haar system in the space $L^{p(t)}([0,1])$ and the principle of localization in the mean. Mat Sb 1986; 130: 275-283 (in Russian).

[40] Sharapudinov II. On the uniform boundedness in $L^p(p = p(x))$ of some families of convolution operators. Math Notes 1996; 59: 205–212.

[41] Sharapudinov II. Some problems in approximation theory in the spaces $L^{p(x)}(E)$. Anal Math 2007; 33: 135-153 (in Russian).

[42] Sharapudinov II. Some Questions in the Theory of Approximation in Lebesgue Spaces with Variable Exponent. Itogi Nauki Yug Rossii Mat Monografiya, 5. Vladikavkaz, Russia: Southern Institute of Mathematics of the Vladikavkaz Science Centre of the Russian Academy of Sciences and the Government of the Republic of North Ossetia-Alania, 2012 (in Russian).

[43] Sharapudinov II. Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials. Izv Math 2013; 77: 407-434

[44] Sharapudinov II. On direct and inverse theorems of approximation theory in variable Lebesgue and Sobolev spaces. Azerbaijan Journal of Mathematics 2014; 4: 55-72.

[45] Sharapudinov II. Approximation of functions in Lebesgue and Sobolev spaces with variable exponent by Fourier-Haar sums. Sb Math 2014; 205: 291-306.

[46] Taberski R. Approximation by entire functions of exponential type. Demonstr Math 1981; 14: 151-181.

[47] Taberski R. Contributions to fractional calculus and exponential approximation. Funct Approximatio Comment Math 1986; 15: 81-106.

[48] Timan AF. Theory of Approximation of Functions of a Real Variable. Translated from the Russian by J. Berry. International Series of Monographs in Pure and Applied Mathematics, Vol. 34. New York, NY, USA: Pergamon Press, 1963.

[49] Timan MF. The approximation of functions defined on the whole real axis by entire functions of exponential type. Izv Vyssh Uchebn Zaved Mat 1968; 2: 89-101.

[50] Vakarchuk SB. Best mean-square approximations by entire functions of exponential type and mean ν-widths of classes of functions on the line. Math Notes 2014; 96: 878-896.

[51] Zhikov VV. Averaging of functionals of the calculus of variations and elasticity theory. Izv Akad Nauk SSSR Ser Mat 1986; 50: 675-710 (in Russian).