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Pythagorean triples containing generalized Lucas numbers

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Abstract: Let P and Q be nonzero integers. Generalized Fibonacci and Lucas sequences are defined as follows: $U_0(P,Q)=0, U_1(P,Q)=1$, and $U_{n+1}(P,Q)=PU_n(P,Q)+QU_{n-1}(P,Q)$ for $n\geq 1$ and $V_0(P,Q)=2, V_1(P,Q)=P$, and $V_{n+1}(P,Q)=PV_n(P,Q)+QV_{n-1}(P,Q)$ for $n\geq 1$, respectively. In this paper, we assume that P and Q are relatively prime odd positive integers and $P^2+4Q>0$. We determine all indices n such that $U_n=(P^2+4Q)x^2$. Moreover, we determine all indices n such that $(P^2+4Q)U_n=x^2$. As a result, we show that the equation $V_n^2(P,1)+V_{n+1}^2(P,1)=x^2$ has solution only for $n=2$, $P=1$, $x=5$ and $V_{n+1}^2(P,-1)=V_n^2(P,-1)+x^2$ has no solutions. Moreover, we solve some Diophantine equations.

Key words: Generalized Fibonacci and Lucas numbers, Diophantine equations

1. Introduction

Let P and Q be nonzero integers. Generalized Fibonacci and Lucas sequences are defined by

$$U_0(P,Q)=0, U_1(P,Q)=1 \text{ and } U_{n+1}(P,Q)=PU_n(P,Q)+QU_{n-1}(P,Q) \text{ for } n\geq 1$$

and

$$V_0(P,Q)=2, V_1(P,Q)=P \text{ and } V_{n+1}(P,Q)=PV_n(P,Q)+QV_{n-1}(P,Q) \text{ for } n\geq 1$$

respectively. $U_n(P,Q)$ and $V_n(P,Q)$ are called n-th generalized Fibonacci numbers and n-th generalized Lucas numbers, respectively. Sometimes, instead of $U_n(P,Q)$ and $V_n(P,Q)$, we will use U_n and V_n, respectively. Generalized Fibonacci and Lucas numbers for negative subscripts are defined as $U_{-n}(P,Q)=-(-Q)^{-n}U_n(P,Q)$ and $V_{-n}(P,Q)=(-Q)^{-n}V_n(P,Q)$ for $n\geq 1$. If $P=Q=1$, we have classical Fibonacci and Lucas sequences (F_n) and (L_n).

In this paper, we assume that P and Q are relatively prime odd positive integers , and $P^2+4Q>0$. We determine all indices n such that $U_n=(P^2+4Q)x^2$. Moreover, we determine all indices n such that $(P^2+4Q)U_n=x^2$. In [2], Bicknell shows that if $(n,k)=1$, there are no solutions in positive integers to the equation $U_n^2(P,1)+U_k^2(P,1)=x^2$, $n>k>0$, when P is odd and $P\geq 3$. Furthermore, when $P=1$, he gives some results on the equation $F_n^2+F_k^2=x^2$ in [1], and he shows that $L_n^2+L_k^2=x^2$, $n>k>0$, has the unique solution $n=3$, $k=2$, or the triple $3-4-5$. He also considers the equations $F_n^2-F_k^2=x^2$ and $L_n^2-L_k^2=x^2$.

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Here we show that the equation $V_n^2(P,1)+V_{n+1}^2(P,1)=x^2$ has a solution only for $n=2,\ P=1,\ x=5$, forming the triple $3-4-5$. Furthermore, we show that the equation $V_{n+1}^2(P,-1)=V_n^2(P,-1)+x^2$ has no solutions.

2. Preliminaries

We begin by listing the properties concerning generalized Fibonacci and Lucas numbers, which will be needed later.

$$(P^2+4Q)U_n=V_{n+1}+QV_{n-1}, \tag{1}$$

$$U_{2n}=U_nV_n, \tag{2}$$

$$V_{2n}=V_n^2-2(-Q)^n \tag{3}$$

for all natural number n.

$$\text{If } n\geq 1, \text{ then } (U_n,V_n)=1 \text{ or } 2. \tag{4}$$

$$\text{If } n \text{ is odd, then } V_n\equiv(-Q)^{\frac{n-1}{2}}P\left(\text{mod}P^2+4Q\right). \tag{5}$$

$$\text{If } n \text{ is even, then } V_n\equiv 2(-Q)^{\frac{n}{2}}\left(\text{mod}P^2+4Q\right). \tag{6}$$

All the above properties are well known (see [7, 8, 10–12]).

Now we give some theorems and lemmas, which will be used in the proofs of the main theorems. The proofs of the following two theorems are given in [12].

Theorem 1 *Let $n,m\in\mathbb{N}\cup\{0\}$ and $r\in\mathbb{Z}$. Then*

$$U_{2mn+r}\equiv\left(-\left(-Q\right)^m\right)^nU_r\left(\text{mod}V_m\right) \tag{7}$$

and

$$V_{2mn+r}\equiv\left(-\left(-Q\right)^m\right)^nV_r\left(\text{mod}V_m\right) \tag{8}$$

such that $mn+r\geq 0$ if $Q\neq\pm1$.

Theorem 2 *Let m be a positive integer, $n\in\mathbb{N}\cup\{0\}$, and $r\in\mathbb{Z}$. Then*

$$U_{2mn+r}\equiv\left(-Q\right)^{mn}U_r\left(\text{mod}U_m\right) \tag{9}$$

and

$$V_{2mn+r}\equiv\left(-Q\right)^{mn}V_r\left(\text{mod}U_m\right) \tag{10}$$

such that $mn+r\geq 0$ if $Q\neq\pm1$.

The proofs of the following theorem and two lemmas can be found in [9].

Theorem 3 *If $V_n=x^2$, then $n=1,3$, or 5, and if $V_5=x^2$, then $Q\equiv 3(mod 8)$. If $V_n=2x^2$, then $n=0,3$, or 6.*

Lemma 4 *Let m be an odd positive integer and $r \geq 1$. Then*

$a)$ If $3|m$, then $V_{2^r m} \equiv 2(\text{mod}8)$.

$b)$ If $3 \nmid m$, then $V_{2^r m} \equiv \begin{cases} 3(\text{mod}8) & \text{if } r=1, \text{ and } Q \equiv 1(\text{mod}8), \\ 7(\text{mod}8) & \text{otherwise.} \end{cases}$

By the above lemma, it is seen that

$$\left(\frac{2}{V_{2^r}}\right) = \begin{cases} -\left(\frac{-1}{Q}\right) & \text{if } r=1, \\ 1 & \text{if } r \geq 2, \end{cases} \tag{11}$$

and if $3 \nmid m$, then

$$\left(\frac{-1}{V_{2^r m}}\right) = -1 \tag{12}$$

for $r \geq 1$.

Lemma 5 *Let r be a positive integer. Then*

(i) $\left(\frac{Q}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right)$,

(ii) if $r \geq 3$, then $\left(\frac{V_2}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right)$,

(iii) $\left(\frac{U_3}{V_{2^r}}\right) = \begin{cases} -\left(\frac{-1}{Q}\right) & \text{if } r=1, \\ 1 & \text{if } r \geq 2, \end{cases}$

(iv) $\left(\frac{P^2+3Q}{V_{2^r}}\right) = \begin{cases} \left(\frac{-1}{Q}\right) & \text{if } r=1, \\ 1 & \text{if } r \geq 2. \end{cases}$

The following lemma can be proved by induction.

Lemma 6 *Let $r \geq 2$. If r is even, then $V_{2^r} \equiv Q^{2^{r-1}-1}U_3(modU_5)$, and if r is odd, then $V_{2^r} \equiv -Q^{2^{r-1}-1}V_2(modU_5)$.*

Lemma 7 *If n is an even positive integer, then $V_n \equiv 2Q^{\frac{n}{2}}(modP^2)$ and if n is an odd positive integer, then $V_n \equiv nPQ^{\frac{n-1}{2}}(modP^2)$.*

The following lemma can be proved by induction on r.

Lemma 8 *$V_{2^r} \equiv 2Q^{2^{r-1}}(modP^2+4Q)$ for all integer $r \geq 2$.*

Using the above lemma, it can be shown that

$$\left(\frac{P^2+4Q}{V_{2^r}}\right) = -1 \tag{13}$$

for $r \geq 1$.

Lastly, we can obtain the following lemma from [4].

Lemma 9 *If n is an even positive integer, then $U_n \equiv (n/2)P(-Q)^{\frac{n-2}{2}}(modD^2)$ and if n is an odd positive integer, then $U_n \equiv n(-Q)^{\frac{n-1}{2}}(modD^2)$, where $D = P^2+4Q$.*

3. Main theorems

Lemma 10 *Let $r \geq 2$. Then* $\left(\frac{U_5}{V_{2^r}}\right) = \begin{cases} +1 & \text{if } r \text{ is even and } Q \equiv 3,7(\bmod 8), \\ -1 & \text{otherwise.} \end{cases}$

Proof Firstly, assume that $Q \equiv 1,5(\text{mod}8)$. Then $U_5 = P^4 + 3P^2Q + Q^2 \equiv 2 + 3Q \equiv 5,1(\text{mod}8)$. If r is even, it follows that

$$\left(\frac{U_5}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{U_5}\right) = \left(\frac{Q}{U_5}\right)\left(\frac{U_3}{U_5}\right) = \left(\frac{U_3}{U_5}\right)$$

by Lemma 6. Moreover, since $U_5 \equiv -Q^2(\text{mod}U_3)$ and $U_3 = P^2 + Q \equiv 2,6(\text{mod}8)$, it is seen that

$$\left(\frac{U_3}{U_5}\right) = \left(\frac{2}{U_5}\right)\left(\frac{U_3/2}{U_5}\right) = \begin{cases} -\left(\frac{U_5}{U_3/2}\right) = -\left(\frac{-1}{U_3/2}\right) = -1 & \text{if } Q \equiv 1(\text{mod}8), \\ \left(\frac{U_5}{U_3/2}\right) = \left(\frac{-1}{U_3/2}\right) = -1 & \text{if } Q \equiv 5(\text{mod}8). \end{cases}$$

If r is odd, then

$$\left(\frac{U_5}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{U_5}\right) = \left(\frac{-1}{U_5}\right)\left(\frac{Q}{U_5}\right)\left(\frac{V_2}{U_5}\right) = \left(\frac{U_5}{V_2}\right) = \left(\frac{-1}{V_2}\right) = -1$$

by Lemma 6 and (12). Now assume that $Q \equiv 3,7(\text{mod}8)$. Then $U_5 = P^4 + 3P^2Q + Q^2 \equiv 2 + 3Q \equiv 3,7(\text{mod}8)$. If r is even, it follows that

$$\left(\frac{U_5}{V_{2^r}}\right) = -\left(\frac{V_{2^r}}{U_5}\right) = -\left(\frac{Q}{U_5}\right)\left(\frac{U_3}{U_5}\right) = \left(\frac{U_3}{U_5}\right)$$

by Lemma 6. Furthermore, since $U_3 = P^2 + Q \equiv 0,4(\text{mod}8)$, we write $U_3 = 2^k t$ for some odd positive integer t with $k \geq 2$. Let $k = 2$. Then $Q \equiv 3(\text{mod}8)$ and thus

$$\left(\frac{U_5}{V_{2^r}}\right) = \left(\frac{U_3}{U_5}\right) = \left(\frac{t}{U_5}\right) = (-1)^{\frac{t-1}{2}}\left(\frac{U_5}{t}\right) = (-1)^{\frac{t-1}{2}}\left(\frac{-1}{t}\right) = 1$$

since $U_5 \equiv -Q^2(\text{mod}U_3)$. Let $k > 2$. Then it can be seen that $Q \equiv 7(\text{mod}8)$ and therefore $U_5 \equiv 7(\text{mod}8)$. This implies that $\left(\frac{2}{U_5}\right) = 1$. Hence, whether k is odd or not, we get

$$\left(\frac{U_5}{V_{2^r}}\right) = \left(\frac{U_3}{U_5}\right) = \left(\frac{t}{U_5}\right) = (-1)^{\frac{t-1}{2}}\left(\frac{U_5}{t}\right) = (-1)^{\frac{t-1}{2}}\left(\frac{-1}{t}\right) = 1.$$

If r is odd, then

$$\left(\frac{U_5}{V_{2^r}}\right) = -\left(\frac{V_{2^r}}{U_5}\right) = -\left(\frac{-1}{U_5}\right)\left(\frac{Q}{U_5}\right)\left(\frac{V_2}{U_5}\right) = -\left(\frac{V_2}{U_5}\right)$$

by Lemma 6. Since $U_5 \equiv -Q^2(\text{mod}V_2)$, it follows that

$$\left(\frac{U_5}{V_2}\right) = -\left(\frac{V_2}{U_5}\right) = \left(\frac{U_5}{V_2}\right) = \left(\frac{-1}{V_2}\right) = -1.$$

This completes the proof. □

Theorem 11 *If* $U_n = (P^2+4Q)x^2$ *for some integer* x, *then* $(n,P,Q,x) = (5,1,1,1)$.

Proof Assume that $U_n = (P^2+4Q)x^2$. Since $\left(P^2+4Q\right)|U_n$, it follows that $\left(P^2+4Q\right)|n$ by Lemma 9. If $Q \geq 1$, it is obvious that $P^2+4Q \geq 5$, i.e. $n \geq 5$. If $Q=-1$, then, since $P^2-4>0$ and P is odd, it follows that $n \geq P^2-4 \geq 5$. If $Q \leq -3$, then it is seen that $P^2+4Q \geq 5$. If $n=5$, then $P^2+4Q=5$ since $\left(P^2+4Q\right)|5$. Thus

$$\begin{aligned} U_5 &= P^4+3P^2Q+Q^2 = \left(P^2+4Q\right)\left(P^2-Q\right)+5Q^2 = 5x^2 \\ &= 5\left(P^2-Q\right)+5Q^2 = 5x^2 \end{aligned}$$

and from here, using the equality $P^2+4Q=5$, we get

$$P^2-Q+Q^2 = x^2,$$

or

$$5-5Q+Q^2 = x^2. \tag{14}$$

Completing the square gives $(2Q-5)^2-(2x)^2=5$. Hence we get $x=1$ and $Q=1$ or $Q=4$. However, since Q is odd, it follows that $Q=1$ and therefore $P=1$. Let $n>5$. Now assume that n is odd. Then we can write $n=8q\mp1$ or $n=8q\mp3$ for some positive integer q. Assume that $n=8q\mp1$. Thus

$$(P^2+4Q)x^2 = U_n = U_{8q\mp1} \equiv \left(Q^{4q}\right) \text{ or } \left(Q^{4q-1}\right)(\text{mod} V_2)$$

by (7). Using Lemma 5 and (13), we get either

$$1 = \left(\frac{P^2+4Q}{V_2}\right) = -1,$$

which is impossible, or

$$1 = \left(\frac{P^2+4Q}{V_2}\right)\left(\frac{Q}{V_2}\right) = -\left(\frac{-1}{Q}\right),$$

which is impossible for $Q \equiv 1,5(\text{mod}8)$. Therefore, $Q \equiv 3,7(\text{mod}8)$ and $n=8q-1$. If we write $n=2(2^r z)-1$ for some odd positive integer z with $r \geq 2$, then it follows that

$$(P^2+4Q)x^2 = U_n = U_{2(2^r z)-1} \equiv -Q^{2^r z-1}(\text{mod} V_{2^r})$$

by (7). Using (12), Lemma 5, and (13), we get

$$1 = J = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2+4Q}{V_{2^r}}\right)\left(\frac{Q}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right) = -1,$$

which is impossible. Assume that $n=8q\mp3$. If $n=8q+3$, then

$$(P^2+4Q)x^2 = U_n = U_{8q+3} \equiv Q^{4q}U_3(\text{mod} V_2),$$

and if $n=8q-3=2(2^r z)-3$ for some odd positive integer z with $r \geq 2$, then

$$(P^2+4Q)x^2 = U_n = U_{2(2^r z)-3} \equiv -Q^{2^r z-3}U_3(\text{mod} V_{2^r})$$

by (7). Using (12), Lemma 5, and (13), from the above two congruences, we get either

$$1 = J_1 = \left(\frac{P^2+4Q}{V_2}\right)\left(\frac{U_3}{V_2}\right) = \left(\frac{-1}{Q}\right)$$

or

$$1 = J_2 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{Q}{V_{2^r}}\right)\left(\frac{P^2+4Q}{V_{2^r}}\right)\left(\frac{U_3}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right),$$

which is impossible for $Q \equiv 3,7(\text{mod}8)$. Now let $Q \equiv 1,5(\text{mod}8)$. If we write $8q \mp 3 = 8a \mp 5 = 2(2^r z) \mp 5$ for some odd positive integer z with $r \geq 2$, then

$$(P^2+4Q)x^2 = U_n = U_{2(2^r z)\mp 5} \equiv \left(-Q^{2^r z}U_5\right) \text{ or } \left(-Q^{2^r z-5}U_5\right)(\text{mod}V_{2^r})$$

by (7). Using (12), Lemma 5, Lemma 10, and (13), it follows that

$$1 = J_1 = \left(\frac{P^2+4Q}{V_{2^r}}\right)\left(\frac{-1}{V_{2^r}}\right)\left(\frac{U_5}{V_{2^r}}\right) = -1$$

or

$$1 = J_2 = \left(\frac{P^2+4Q}{V_{2^r}}\right)\left(\frac{-1}{V_{2^r}}\right)\left(\frac{Q}{V_{2^r}}\right)\left(\frac{U_5}{V_{2^r}}\right) = -\left(\frac{-1}{Q}\right) = -1,$$

which are impossible.

Now let n be even. Then $n = 2m$ for some positive integer m, and thus $(P^2+4Q)x^2 = U_n = U_m V_m$ by (2). By (5) and (6), it is seen that $\left(P^2+4Q, V_m\right) = 1$, and thus $x^2 = \left(U_m/(P^2+4Q)\right)V_m$. From here, we obtain $V_m = u^2$ and $U_m = (P^2+4Q)v^2$ or $V_m = 2u^2$ and $U_m = 2(P^2+4Q)v^2$ for some integers u and v since $(U_m, V_m) = 1$ or 2 by (4). If $V_m = u^2$ and $U_m = (P^2+4Q)v^2$, then $m = 1, 3$, or 5 by Theorem 3. Since $\left(P^2+4Q\right)|U_m$, it follows that $P^2+4Q|m$ by Lemma 9 and therefore $m \geq P^2+4Q \geq 5$. If $m = 5$, then $(n,P,Q) = (5,1,1)$ and so $u^2 = V_5 = 11$, which is impossible. If $V_m = 2u^2$ and $U_m = 2(P^2+4Q)v^2$, then $m = 3$ or 6 by Theorem 3. Similarly, it can be seen that $m = 6$ and so $n = 12$. Thus $(P^2+4Q)x^2 = U_{12} = U_3V_3V_6$ by (2). Since $\left(P^2+4Q, V_3\right) = \left(P^2+4Q, V_6\right) = 1$ by (5) and (6), it follows that $P^2+4Q|U_3$ and so $\left(P^2+4Q\right)|3$ by Lemma 9. This is impossible since $P^2+4Q \geq 5$. This completes the proof. □

Theorem 12 *If $(P^2+4Q)U_n = x^2$ for some integer x, then $n = 5$, $P^2+4Q = 5u^2$, $P^4+3P^2Q+Q^2 = 5v^2$ for some integers u and v, and $Q \equiv 1(\text{mod}8)$.*

Proof Assume that $(P^2+4Q)U_n = x^2$. It can be easily shown that this is impossible for $n \leq 4$. If $n = 5$, then $(P^2+4Q)(P^4+3P^2Q+Q^2) = x^2$. It can be seen that $(P^2+4Q, P^4+3P^2Q+Q^2) = 1$ or 5, and thus

$$P^2+4Q = u^2 \text{ and } P^4+3P^2Q+Q^2 = v^2 \tag{15}$$

or

$$P^2+4Q = 5u^2 \text{ and } P^4+3P^2Q+Q^2 = 5v^2 \tag{16}$$

for some integers u and v. If (15) is satisfied, then $u^2 = P^2+4Q \equiv 5(\text{mod}8)$, which is impossible. Therefore, $P^2+4Q = 5u^2$, $P^4+3P^2Q+Q^2 = 5v^2$, and it is obvious that $Q \equiv 1(\text{mod}8)$. Now assume that $n > 5$. A similar argument to the proof of the previous theorem shows that n cannot be odd. Let n be even. Then $n = 2m$ for some positive integer m and thus $x^2 = (P^2+4Q)U_{2m} = (P^2+4Q)U_m V_m$. Since $\left(P^2+4Q, V_m\right) = 1$ by (5) and (6), it follows that $\left(\left(P^2+4Q\right)U_m, V_m\right) = 1$ or 2 by (4). This implies that either

$$V_m = a^2 \text{ and } \left(P^2+4Q\right)U_m = b^2 \tag{17}$$

or

$$V_m = 2a^2 \text{ and } \left(P^2+4Q\right)U_m = 2b^2 \tag{18}$$

for some integers a and b. If (17) is satisfied, then $m = 1, 3,$ or 5 by Theorem 3. It can be seen that (17) is possible for only $m = 5$. This implies that $P^2+4Q = 5u^2$ and $P^4+3P^2Q+Q^2 = 5v^2$. Also $Q \equiv 3(\text{mod}8)$ by Theorem 3 and therefore $5v^2 = P^4+3P^2Q+Q^2 \equiv 3(\text{mod}8)$. This shows that $v^2 \equiv 7(\text{mod}8)$, which is impossible. If (18) is satisfied, then $m = 3$ or 6 by Theorem 3. If $m = 3$ or 6, then $\left(P^2+4Q\right)U_3 = 2b^2$ or $\left(P^2+4Q\right)U_3V_3 = 2b^2$, and thus $P^2+4Q = r^2$ or $P^2+4Q = 3r^2$ since P^2+4Q is odd, $(P^2+4Q, V_3) = 1$ and $(P^2+4Q, U_3) = 1$ or 3. This shows that $r^2 \equiv 5(\text{mod}8)$ or $r^2 \equiv 7(\text{mod}8)$, which is impossible. This completes the proof. □

Now we can give the following corollary, which shows that the consecutive two generalized Lucas numbers except $V_2(1,1)$ and $V_3(1,1)$ cannot be sides of a right-angled triangle.

Corollary 13 *If $Q = 1$, then the equation $V_n^2 + V_{n+1}^2 = x^2$ has a solution only for $n = 2$, $P = 1$, $x = 5$, forming the triple $3-4-5$.*

Proof Assume that $Q = 1$ and $V_n^2 + V_{n+1}^2 = x^2$. Then, using (1) and (3), it is seen that $(P^2+4)U_{2n+1} = x^2$. Thus $n = 2$, $P^2+4 = 5u^2$ and $P^4+3P^2+1 = 5v^2$ for some integers u and v by Theorem 12. With MAGMA [3], we get $n = 2$, $P = 1$, $x = 5$, forming the triple $3-4-5$. This completes the proof. □

Corollary 14 *If $Q = -1$, then the equation $V_{n+1}^2 - V_n^2 = x^2$ has no solutions.*

Proof Assume that $Q = -1$ and $V_{n+1}^2 - V_n^2 = x^2$. Then, using (1) and (3), it is seen that $(P^2-4)U_{2n+1} = x^2$. Thus $n = 2$, $P^2-4 = 5u^2$ and $P^4-3P^2+1 = 5v^2$ for some integers u and v by Theorem 12. With MAGMA [3], we get $P = 2$, which is impossible since P is odd. This completes the proof. □

3.1. Solutions of some Diophantine equations

In [6], the authors have given all integer solutions of the equations $x^2-Pxy-y^2 = \pm(P^2+4)$ and $x^2-Pxy+y^2 = -(P^2-4)$ when $(P^2 \pm 4)$ is square free. In this section, when r is odd, we give all positive integer solutions of the equations $x^2-Pxy^2-y^4 = \pm(P^2+4)^{2r}$, $x^4-Px^2y-y^2 = \pm(P^2+4)^{2r}$, $x^2-Pxy^2+y^4 = (P^2-4)^{2r}$, and $x^4-Px^2y+y^2 = (P^2-4)^{2r}$. Before it, we will give several theorems from [6](see also [5]).

Theorem 15 *All nonnegative integer solutions of the equations $x^2-(P^2+4)y^2 = 4$ and $x^2-(P^2+4)y^2 = -4$ are given by $(x,y) = (V_{2n}(P,1), U_{2n}(P,1))$ with $n \geq 0$ and $(x,y) = (V_{2n+1}(P,1), U_{2n+1}(P,1))$ with $n \geq 0$, respectively.*

Theorem 16 *Let $P > 3$. Then all nonnegative integer solutions of the equation $x^2 - (P^2-4)y^2 = 4$ are given by $(x, y) = (V_n(P,-1), U_n(P,-1))$ with $n \geq 1$.*

Using the above two theorems, we obtain the following lemmas. Since the proofs of the following two lemmas can be easily done by induction, we omit their proofs.

Lemma 17 *Let $r \geq 0$ and P^2+4 be a square-free integer. All nonnegative integer solutions of the equations $x^2 - Pxy - y^2 = (P^2+4)^{2r}$ and $x^2 - Pxy - y^2 = -(P^2+4)^{2r}$ are given by $(x, y) = ((P^2+4)^r U_{2n+1}(P,1), (P^2+4)^r U_{2n}(P,1))$ with $n \geq 0$ and $(x, y) = ((P^2+4)^r U_{2n+2}(P,1), (P^2+4)^r U_{2n+1}(P,1))$ with $n \geq 0$, respectively.*

Lemma 18 *Let $r \geq 0$, $P > 3$, and P^2-4 be a square-free integer. All nonnegative integer solutions of the equation $x^2 - Pxy + y^2 = (P^2-4)^{2r}$ are given by $(x, y) = ((P^2-4)^r U_{n+1}(P,-1), (P^2-4)^r U_n(P,-1))$ with $n \geq 0$.*

Now, as an application of Theorem 11, we can give the following corollaries.

Corollary 19 *Let $r \geq 1$ be an odd integer and P^2+4 be a square-free integer. All nonnegative integer solutions of the equation $x^2 - Pxy^2 - y^4 = -(P^2+4)^{2r}$ are given by $(x, y, P) = (8 \cdot 5^r, 5^{(r+1)/2}, 1)$.*

Proof Assume that $r \geq 1$ is an odd integer, P^2+4 is a square-free integer, and $x^2 - Pxy^2 - y^4 = -(P^2+4)^{2r}$. Then $y^2 = (P^2+4)^r U_{2n+1}(P,1)$ with $n \geq 0$ by Lemma 17. This shows that $U_{2n+1}(P,1) = (P^2+4)\left(y/(P^2+4)^{(r+1)/2}\right)^2$ since r is odd. By Theorem 11, it follows that $n = 2$, $P = 1$, and $y^2 = 5^{r+1}$, and so we get $(x, y, P) = (8 \cdot 5^r, 5^{(r+1)/2}, 1)$. This completes the proof. □

We can give the following corollary easily by using Lemma 17.

Corollary 20 *Let $r \geq 1$ be an odd integer and P^2+4 be a square-free integer. All nonnegative integer solutions of the equation $x^4 - Px^2y - y^2 = (P^2+4)^{2r}$ are given by $(x, y, P) = (5^{(r+1)/2}, 3 \cdot 5^r, 1)$.*

Corollary 21 *Let $r \geq 1$ be an odd integer and P^2+4 be a square-free integer. Then the equation $x^2 - Pxy^2 - y^4 = (P^2+4)^{2r}$ has no integer solutions.*

Corollary 22 *Let $r \geq 1$ be an odd integer and P^2+4 be a square-free integer. Then the equation $x^4 - Px^2y - y^2 = -(P^2+4)^{2r}$ has no solutions.*

Corollary 23 *Let $r \geq 1$ be an odd integer, $P > 3$, and P^2-4 be a square-free integer. Then the equation $x^4 - Px^2y + y^2 = (P^2-4)^{2r}$ has no integer solutions.*

Proof Assume that $r \geq 1$ is an odd integer, $P > 3$, P^2-4 is a square-free integer and $x^4 - Px^2y + y^2 = (P^2-4)^{2r}$. Then $x^2 = (P^2-4)^r U_{n+1}(P,-1)$ with $n \geq 0$ or $x^2 = (P^2-4)^r U_n(P,-1)$ with $n \geq 0$ by Lemma 18. This shows that $U_{n+1}(P,-1) = (P^2-4)\left(x/(P^2-4)^{(r+1)/2}\right)^2$ since r is odd, which is impossible by Theorem 11.

Similarly, one can see the following corollary from Lemma 18.

Corollary 24 *Let $r \geq 1$ be an odd integer, $P > 3$, and P^2-4 be a square-free integer. Then the equation $x^2 - Pxy^2 + y^4 = (P^2-4)^{2r}$ has no integer solutions.*

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