Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1913 – 1926

doi:10.3906/mat-1701-22

Regularity of semigroups of transformations with restricted range preserving an alternating orientation order

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Received: 09.01.2017 • **Accepted/Published Online:** 03.05.2018 • **Final Version:** 24.07.2018

Abstract: It is well known that the transformation semigroup on a nonempty set X, which is denoted by $T(X)$, is regular, but its subsemigroups do not need to be. Consider a finite ordered set $X = (X; \leq)$ whose order forms a path with alternating orientation. For a nonempty subset Y of X, two subsemigroups of $T(X)$ are studied. Namely, the semigroup $OT(X,Y) = \{\alpha \in T(X) \mid \alpha \text{ is order-preserving and } X\alpha \subseteq Y\}$ and the semigroup $OS(X,Y) = \{\alpha \in T(X) \mid \alpha \text{ is order-preserving and } Y\alpha \subseteq Y\}$. In this paper, we characterize ordered sets having a coregular semigroup $OT(X,Y)$ and a coregular semigroup $OS(X,Y)$, respectively. Some characterizations of regular semigroups $OT(X,Y)$ and $OS(X,Y)$ are given. We also describe coregular and regular elements of both $OT(X,Y)$ and $OS(X,Y)$.

Key words: Order-preserving, fence, semigroup, regular, coregular

1. Introduction and preliminaries

Regularity is one of the most studied topics in semigroup theory due to its nice algebraic properties and wide applications. An element a in a semigroup S is called *regular* if there is an element $b \in S$ such that $a = aba$. A *regular* semigroup is a semigroup in which every element is regular. There have been many research works studying regularity of semigroups (see [9–11, 13–15, 18, 20, 22]). A special case of a regular element is a coregular element. An element a in a semigroup S is called *coregular* if there is an element $b \in S$ such that $aba = a = bab$ and S is called *coregular* if every element of S is coregular. Clearly, every coregular element is regular. It has been proved that an element a in a semigroup S is coregular if and only if $a^3 = a$ (see [21, Proposition 3]). For a nonempty set X, it is well known that the semigroup $T(X)$ of all transformations of X is regular (see [1, page 33]). However, a subsemigroup of $T(X)$ does not need to be regular. The regularity for various types of subsemigroups of $T(X)$ has been investigated. In 1966, Magill [12] introduced and studied the subsemigroup

$$S(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}$$

of $T(X)$ where Y is a nonempty subset of X. Nenthein et al. [15] described regular elements of $S(X,Y)$ and also determined the number of such elements for a finite set X.

For a nonempty subset Y of X, the subsemigroup

$$T(X,Y) = \{\alpha \in T(X) \mid X\alpha \subseteq Y\}$$

*Correspondence: worawannotai_c@silpakorn.edu
2010 *AMS Mathematics Subject Classification:* 20M20, 20M05, 20M17
This work was supported by the Faculty of Science of Silpakorn University, contract number SRF-PRG-2558-09.

of $T(X)$ was first introduced by Symons [19] in 1975. A characterization of regular elements of $T(X,Y)$ was given [15]. Sanwong and Sommanee [18] obtained the largest regular subsemigroup of $T(X,Y)$ since, in general, $T(X,Y)$ does not need to be regular.

Consider X as the base set of an ordered set $(X;\leq)$. Throughout this paper, we represent an ordered set by its base set. A map $\alpha : X \to X$ is said to be *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X$. The order-preserving counterpart of the semigroup $T(X)$ is denoted by $OT(X)$, the semigroup of all order-preserving transformations of X. Such a semigroup is a subsemigroup of $T(X)$ and plays an important role in the study of algebraic systems. In [5], Gluskin showed that if $OT(X)$ is isomorphic to $OT(Y)$, then the ordered sets X and Y are isomorphic or antiisomorphic. Repnitski and Vernitski [16, Lemma 1.1] proved that every free semigroup can be represented by the semigroup $OT(X)$ of a chain (or a totally ordered set) and every semigroup is a homomorphic image of a free semigroup. Later, Higgins et al. [7] found that the rank of the semigroup $T(X)$ is related to the semigroup $OT(X)$ for some chain X.

There have been many research works focused on the regularity of order-preserving transformation semigroups (see [4, 9–11, 14, 20, 22]). Let X be a chain. Then the semigroup $OT(X)$ is a regular subsemigroup of $T(X)$ if X is finite (see [6, Exercise 6.1.9]). Keprasit and Changphas [9] showed that if X is order-isomorphic to a subchain of $\mathbb{Z}$, then $OT(X)$ is regular. In [4], Fernandes et al. described the largest regular subsemigroup of $OT(X)$.

For a nonempty subset Y of an ordered set X, the semigroups $OS(X,Y)$ and $OT(X,Y)$ are adapted from analogous conditions for $S(X,Y)$ and $T(X,Y)$, respectively. Precisely,

$$OS(X,Y) = \{\alpha \in OT(X) \mid Y\alpha \subseteq Y\}$$

and

$$OT(X,Y) = \{\alpha \in OT(X) \mid X\alpha \subseteq Y\}$$

are subsemigroups of $OT(X)$ and also of $T(X)$. For a chain X, Mora and Kemprasit [14] gave a necessary and sufficient condition for $OT(X,Y)$ to be regular and determined all regular elements. Fernandes et al. [4] characterized the largest regular subsemigroup of $OT(X,Y)$.

The semigroup $OT(X)$ and its subsemigroups have been studied by many mathematicians, but most of this research was done on a chain. Our interest focuses on ordered sets whose simplicity is "next" to that of chains. Such ordered sets are fences.

A *fence* X is an ordered set such that the order forms a path with alternating orientation. Indeed, the only comparability relations in X are either

$$x_1 \leq x_2 \geq x_3, x_3 \leq x_4 \geq x_5, \ldots, x_{2m-1} \leq x_{2m} \geq x_{2m+1}, \ldots$$

or

$$x_1 \geq x_2 \leq x_3, x_3 \geq x_4 \leq x_5, \ldots, x_{2m-1} \geq x_{2m} \leq x_{2m+1}, \ldots$$

where $X = \{x_1, x_2, x_3, \ldots\}$. Every element in X is minimal or maximal. The *cardinality* of a fence X is defined to be the cardinality of X as a set and denoted by $|X|$. Here $|X|$ can be either finite or infinite. A fence X is said to be *trivial* if $|X| = 1$ and *nontrivial* otherwise. A nonempty subset Y of a fence X is called a *subfence* of X if Y is a fence with respect to the order restricted from X.

For $x, y \in X$, the distance $d(x,y)$ from x to y in X is defined by

$$d(x,y) = \inf\{|S| - 1 \;\big|\; S \text{ is a subfence of } X \text{ and } x, y \in S\}.$$

For an element $\alpha \in OT(X)$, let $\operatorname{ran}\alpha = \{x\alpha \mid x \in X\}$. We note that $Y\alpha = \{y\alpha \mid y \in Y\}$ is a subfence of X for every element $\alpha \in OT(X)$ and a subfence Y of X (see [8, Section 2]). In particular, $\operatorname{ran}\alpha = X\alpha$ is subfence of X for $\alpha \in OT(X)$.

Algebraic properties of order-preserving transformations of fences have been long considered (see, for example, [2, 3, 17]). Recently, Jendana and Srithus [8] proved that, for a finite fence X, the semigroup $OT(X)$ is coregular if and only if $|X| \leq 2$, and they characterized coregular elements of $OT(X)$. Later, in 2016, Tanyawong et al. [8] described all regular semigroups of transformations preserving a fence, i.e. $OT(X)$ is regular if and only if $|X| \leq 4$. The regularity of elements in $OT(X)$ was discussed as well.

Throughout this paper, let X be a finite fence and let Y be a nonempty set of X. In general, $OT(X,Y)$ and $OS(X,Y)$ do not need to be regular (see Lemma 2.1). Our main purpose is to investigate the regularity of the semigroups $OS(X,Y)$ and $OT(X,Y)$. In Section 2, we characterize coregular elements in subsemigroups of $OT(X)$. In Section 3, we give necessary and sufficient conditions for $OT(X,Y)$ to be regular. Since an element in $OT(X,Y)$ does not need to be regular, the regular elements of $OT(X,Y)$ are completely determined. Finally, Section 4 is devoted to the study of the regularity of $OS(X,Y)$.

2. Coregular elements in subsemigroups of $OT(X)$

In this section we characterize coregular elements in any subsemigroup of $OT(X)$. Observe that for any element α of $OT(X)$, α is coregular in a subsemigroup of $OT(X)$ if and only if α is coregular in $OT(X)$. Since $OT(X)$ is coregular if and only if $|X| \leq 2$ (see [8, Theorem 2.1]), in general $OT(X)$ does not need to be coregular. We now give an example of a map that is not coregular in $OT(X)$ when $|X| \geq 3$. Moreover, this map is regular in $OT(X)$ but is not regular in either $OT(X,Y)$ or $OS(X,Y)$ for some subset Y of X.

Lemma 2.1 *Let a, b, and c be distinct elements in X satisfying (1) a and b are comparable, and (2) b and c are comparable. Define the map $\alpha : X \to X$ by*

$$x\alpha = \begin{cases} a & \text{if } x = c \\ b & \text{if } x \neq c. \end{cases}$$

Then the following statements hold:

(i) *The map α is an element of $OT(X)$.*

(ii) *The map α is not coregular in $OT(X)$. Consequently, α is not coregular in any subsemigroup of $OT(X)$.*

(iii) *The map α is regular in $OT(X)$.*

(iv) *The map α is not regular in $OT(X,Y)$ for any subset Y of X that contains a and b but does not contain c.*

(v) *The map α is not regular in $OS(X,Y)$ for any subset Y of X that contains a and b but does not contain c.*

Proof

(i) Assumptions (1) and (2) imply that a and c are both minimal or both maximal. Without loss of generality, assume that a and c are both minimal. Then $a < b$ and $c < b$. Let $x, y \in X$ be such that $x \leq y$. Clearly,

$x\alpha \leq y\alpha$ if $x = y$. Assume that $x < y$. Then x is minimal and y is maximal, which implies that $y \neq c$. Hence, $x\alpha = a < b = y\alpha$ if $x = c$, or $x\alpha = b \leq b = y\alpha$ if $x \neq c$. The proof is completed.

(ii) Recall that an element $\gamma \in OT(X)$ is coregular if and only if $\gamma^3 = \gamma$. Since $c\alpha^3 = b \neq a = c\alpha$, α is not regular in $OT(X)$.

(iii) Define the map $\beta : X \to X$ by

$$x\beta = \begin{cases} c & \text{if } x = a \\ b & \text{if } x \neq a. \end{cases}$$

It is easy to check that $\beta \in OT(X)$ and $\alpha\beta\alpha = \alpha$. Therefore, α is regular in $OT(X)$.

(iv) Clearly $\alpha \in OT(X, Y)$. Suppose α is regular in $OT(X, Y)$. Then there exists an element β in $OT(X, Y)$ such that $\alpha\beta\alpha = \alpha$. By the definition of α, we have $c\alpha\beta\alpha = a\beta\alpha$. From $c \notin Y$, we have $a\beta \neq c$, implying $a\beta\alpha = b$. It follows that $c\alpha\beta\alpha = a\beta\alpha = b \neq a = c\alpha$, which is a contradiction. Hence, α is not regular in $OT(X, Y)$.

(v) The proof is similar to the proof of (iv).

□

In 2015, Jendana and Srithus gave a technical lemma that will be a tool for describing coregular elements in $OT(X)$, as stated below.

Lemma 2.2 *([8, Lemma 3.1]) Let S be a subfence of X and let $\alpha \in OT(X)$ with $\operatorname{ran}\alpha = S$ and $\alpha|_S$ is a bijection. Assume that $S = \{x_1, x_2, \ldots, x_n\}$ and $x_k\alpha = x_l$ for some positive integer k and l. Let $w \in \mathbb{N}$ with $w \geq 2$. Then the following statements hold:*

(i) *Assume that $x_{k-1}\alpha = x_{l+1}$. If $x_{k\pm w} \in S$, then $x_{k\pm w}\alpha = x_{l\mp w}$.*

(ii) *Assume that $x_{k-1}\alpha = x_{l-1}$. If $x_{k\pm w} \in S$, then $x_{k\pm w}\alpha = x_{l\pm w}$.*

The following lemma gives useful properties of elements in $OT(X)$.

Lemma 2.3 *Let $\alpha \in OT(X)$, for which $\alpha|_{\operatorname{ran}\alpha}$ is a bijection. Then the following statements hold:*

(i) *If $a, b \in \operatorname{ran}\alpha$ with $a < b$, then $a\alpha < b\alpha$.*

(ii) *If $a \in \operatorname{ran}\alpha$, then a and $a\alpha$ are both minimal or both maximal in X.*

Proof

(i) Let $a, b \in \operatorname{ran}\alpha$ with $a < b$. From α being order-preserving and $a < b$, we have $a\alpha \leq b\alpha$. Since $\alpha|_{\operatorname{ran}\alpha}$ is injective and $a \neq b$, we have $a\alpha \neq b\alpha$, which implies that $a\alpha < b\alpha$.

(ii) Let $a \in \operatorname{ran}\alpha$ with $a\alpha = b$. If $|\operatorname{ran}\alpha| = 1$, then $a = b$. Hence, (ii) is satisfied. Consider $|\operatorname{ran}\alpha| > 1$. We may assume that a and b are minimal and maximal in X, respectively. Since $\operatorname{ran}\alpha$ is a subfence of X, there exists an element $c \in \operatorname{ran}\alpha$ with $a < c$. By (i), $b = a\alpha < c\alpha$. Thus, b is not maximal, a contradiction.

□

In what follows, we restrict our study to the case of a map α in $OT(X,Y)$ for which the restriction to its range is bijective. Theorem 2.4 shows that there are only 2 possibilities for such a map.

Theorem 2.4 *Let $\alpha \in OT(X)$ and let $W = \operatorname{ran}\alpha$. Then $\alpha|_W$ is a bijection if and only if one of the following statements holds:*

(i) *If $|W|$ is even, then $\alpha|_W = id_W$.*

(ii) *If $|W|$ is odd, then either $\alpha|_W = id_W$ or $x_k\alpha = x_{n-(k-1)}$ for all $k \in \{1,2,\dots,n\}$ where $W = \{x_1, x_2, \dots, x_n\}$.*

Proof Assume that $\alpha|_W$ is a bijection. Let $W = \{x_1, x_2, \dots, x_n\}$. First we show that $x_1\alpha \in \{x_1, x_n\}$. Suppose $x_1\alpha = x_j$ for some $j \in \{2,3,\dots,n-1\}$. Then by Lemma 2.2 either $x_{1+s}\alpha = x_{j-s}$ for all $s \in \{0,1,\dots,n-1\}$ or $x_{1+s}\alpha = x_{j+s}$ for all $s \in \{0,1,\dots,n-1\}$. Since $j-s \leq n-1$ and $j+s \geq 2$ for all $s \in \{0,1,\dots,n-1\}$, $x_{1+s}\alpha \neq x_n$ for all $s \in \{0,1,\dots,n-1\}$ or $x_{1+s}\alpha \neq x_1$ for all $s \in \{0,1,\dots,n-1\}$. Hence, $W\alpha \subsetneq W$, which is impossible since $\alpha|_W$ is a bijection.

Therefore, we have $x_1\alpha \in \{x_1, x_n\}$. If $x_1\alpha = x_1$, then $x_n\alpha = x_n$ and $x_{n-1}\alpha = x_{n-1}$ since $\alpha|_W$ is a bijection. By setting $k = n$ and $l = n$ in Lemma 2.2(ii), we have $\alpha|_W = id_W$. In the case where $x_1\alpha = x_n$, we have $x_n\alpha = x_1$ and $x_{n-1}\alpha = x_2$ since $\alpha|_W$ is a bijection. By setting $k = n$ and $l = 1$ in Lemma 2.2(i), it follows that $x_k\alpha = x_{n-k+1}$ for all $k \in \{1,2,\dots,n\}$. If $|W|$ is even, then one of x_1 and x_n is minimal and the other is maximal. Without loss of generality, we assume that x_1 is minimal. By Lemma 2.3(ii) we have that x_1 and $x_1\alpha$ are minimal. Since x_2 is maximal, we get that $x_1\alpha \neq x_n$. Therefore, $x_1\alpha = x_1$ and $\alpha|_W = id_W$.

Conversely, assume that (i) or (ii) holds. Then clearly $\alpha|_W$ is a bijection. □

We shall characterize coregular elements in any subsemigroup of $OT(X)$. To do so we need results concerning coregular elements in $OT(X)$.

Theorem 2.5 *([8, Theorems 3.4 and 3.5]) Let $\alpha \in OT(X)$ and let $W = \operatorname{ran}\alpha$. Then α is coregular if and only if one of the following statements holds:*

(i) *If $|W|$ is even, then $\alpha|_W = id_W$.*

(ii) *If $|W|$ is odd, then either $\alpha|_W = id_W$ or $x_k\alpha = x_{n-(k-1)}$ for all $k \in \{1,2,\dots,n\}$ where $W = \{x_1, x_2, \dots, x_n\}$.*

Summarizing the results, we give a necessary and sufficient condition for an element in a subsemigroup of $OT(X)$ to be coregular.

Theorem 2.6 *Let α be an element in a subsemigroup S of $OT(X)$ and let $W = \operatorname{ran}\alpha$. Then following statements are equivalent:*

(i) *α is coregular.*

(ii) *$\alpha|_W$ is a bijection.*

(iii) *$d(a\alpha, b\alpha) = d(a,b) = |W| - 1$ where a and b are the endpoints of W.*

Proof

(i) $\Leftrightarrow$ (ii): The result follows from Theorems 2.4 and 2.5.

(ii) $\Rightarrow$ (iii): Assume that $\alpha|_W$ is a bijection. Then by Theorem 2.4, $\{a,b\}\alpha = \{a,b\}$. It follows immediately that $d(a\alpha, b\alpha) = d(a,b) = |W| - 1$.

(iii) $\Rightarrow$ (ii): Assume that $d(a\alpha, b\alpha) = d(a,b) = |W| - 1$. Then from the fact that an order-preserving map sends a subfence to a subfence and $\operatorname{ran}\alpha = W$, the image $W\alpha = W$ implies that $\alpha|_W$ is onto. Since W is finite, $\alpha|_W$ is a bijection. □

We close this section with results involving the fixed points of maps in $OT(X)$.

Proposition 2.7 *Let $\alpha \in OT(X)$. Then there exists a positive integer m such that $\operatorname{ran}(\alpha^m) = \operatorname{ran}\left(\alpha^{m+1}\right)$ and the following statements hold.*

(i) *$\alpha^m = \alpha^{m+2}$ and $\alpha^{m+1} = \alpha^{m+3}$.*

(ii) *α^m and α^{m+1} are coregular.*

(iii) *If $\alpha^m = \alpha^{m+1}$, then $\operatorname{ran}(\alpha^m)$ is the set of all fixed points of α; otherwise, α has exactly one fixed point.*

Proof Since $\operatorname{ran}\ \alpha \supseteq \operatorname{ran}\left(\alpha^2\right) \supseteq \operatorname{ran}\left(\alpha^3\right) \supseteq \cdots$ is a chain of finite sets, there exists a positive integer m such that

$$\operatorname{ran}(\alpha^m) = \operatorname{ran}\left(\alpha^{m+1}\right) = (\operatorname{ran}(\alpha^m))\,\alpha. \tag{2.1}$$

Equation (2.1) implies that $\alpha|_{\operatorname{ran}(\alpha^m)}$ is a bijection on $\operatorname{ran}(\alpha^m)$. For simplicity, let $\beta = \alpha|_{\operatorname{ran}(\alpha^m)}$. Then $\operatorname{ran}\beta = \operatorname{ran}(\alpha^m)$ and $\beta|_{\operatorname{ran}\beta} = \left(\alpha|_{\operatorname{ran}(\alpha^m)}\right)|_{\operatorname{ran}(\alpha^m)} = \alpha|_{\operatorname{ran}(\alpha^m)}$ is a bijection onto $\operatorname{ran}(\alpha^m) = \operatorname{ran}\beta$. By Theorem 2.4 the map $\left(\beta|_{\operatorname{ran}\beta}\right)^2 = \left(\alpha|_{\operatorname{ran}(\alpha^m)}\right)^2$ is the identity on $\operatorname{ran}\beta = \operatorname{ran}(\alpha^m)$ and hence $\alpha^m = \alpha^{m+2}$. Similarly, $\alpha^{m+1} = \alpha^{m+3}$. Hence, (i) is proved.

By applying (2.1) recursively, it can be concluded that $\operatorname{ran}(\alpha^m) = (\operatorname{ran}(\alpha^m))\,\alpha^m$. Therefore, $\alpha^m|_{\operatorname{ran}(\alpha^m)}$ is a bijection and hence α^m is coregular by Theorem 2.6. Similarly, α^{m+1} is coregular. The proof of (ii) is completed.

To prove (iii), assume that $\alpha^m = \alpha^{m+1}$. Then $\alpha|_{\operatorname{ran}\alpha^m}$ is the identity on $\operatorname{ran}(\alpha^m)$. Equivalently, $\operatorname{ran}(\alpha^m)$ is the set of all fixed points of α. If $\alpha^m \neq \alpha^{m+1}$, then $\alpha|_{\operatorname{ran}\alpha^m}$ is the involution on $\operatorname{ran}(\alpha^m)$ and α has exactly one fixed point. □

Corollary 2.8 *For $\alpha \in OT(X)$, the fixed points of α form a subfence.*

3. Regularity of $OT(X,Y)$

In this section, we investigate the regularity of $OT(X,Y)$ where Y is a nonempty subset of X. Before doing so, we mention some basic knowledge involving order-preserving maps. It is well known (see [8, Section 2]) that if an ordered set P is connected, i.e. for all $a, b \in P$ there is a subfence of P with endpoints a and b, then every order-preserving map sends an order-connected set to an order-connected set. Because an order-connected subset of a fence is precisely a subfence, an order-preserving map sends a subfence to a subfence.

Observe that for an ordered set P, the identity map id_P and a constant map c_a that maps all elements in P to $a \in P$ are order-preserving. Because $(id_P)^3 = id_P$ and $(c_a)^3 = c_a$, we get that id_P and c_a are coregular and hence regular in $OT(P)$. If X is a trivial fence, then $X = Y$ is a singleton and $OS(X,Y) = OT(X) = OT(X,Y)$ is the set of the identity map. Hence, $OT(X,Y)$ is coregular and also regular. From Lemma 2.1, in general $OT(X,Y)$ does not need to be regular. It is natural to ask when the semigroup $OT(X,Y)$ is regular and coregular, respectively. The answer is shown in the following theorems.

Theorem 3.1 *The semigroup $OT(X,Y)$ is regular if and only if $|X| = |Y| \leq 4$ or Y does not contain nontrivial subfences.*

Proof Assume that $OT(X, Y)$ is regular. If $X = Y$, then $OT(X) = OT(X,Y)$ is regular, and hence $|X| = |Y| \leq 4$ by [20, Theorem 3.9]. Assume that Y is a proper subset of X. To show that Y does not contain non-trivial subfences, we proceed by contradiction. Suppose that Y contains a nontrivial subfence. Then there are two comparable elements a and b in Y and an element $c \in X \setminus Y$ such that b and c are comparable. Then the map α from Lemma 2.1 is an element in $OT(X,Y)$ but it is not regular in $OT(X,Y)$, a contradiction. Therefore, Y does not contain nontrivial subfences.

Conversely, assume that $|X| = |Y| \leq 4$ or Y does not contain nontrivial subfences. If $|X| = |Y| \leq 4$, then $OT(X,Y) = OT(X)$ is regular by [20, Theorem 3.9]. If Y does not contain nontrivial subfences, then $OT(X,Y)$ contains only constant maps. Since constant maps are regular, $OT(X,Y)$ is regular. □

Note that if Y is a subfence of X that does not contain nontrivial subfences, then $|Y| = 1$. Therefore, we have the following corollary.

Corollary 3.2 *If Y is a subfence of X, then $OT(X,Y)$ is regular if and only if $|X| = |Y| \leq 4$ or $|Y| = 1$*

By a similar argument as in the proof of Theorem 3.1 and the fact that $OT(X)$ is coregular if and only if $|X| \leq 2$, the following theorem is obtained.

Theorem 3.3 *The semigroup $OT(X,Y)$ is coregular if and only if $|X| = |Y| \leq 2$ or Y does not contain nontrivial subfences.*

Corollary 3.4 *If Y is a subfence of X, then $OT(X,Y)$ is coregular if and only if $|X| = |Y| \leq 2$ or $|Y| = 1$*

We now characterize regular elements in $OT(X,Y)$.

Theorem 3.5 *Let $\alpha \in OT(X,Y)$. Then the following statements are equivalent:*

(i) *α is regular.*

(ii) *There exists a subfence Z of Y such that $\alpha|_Z$ is a bijection onto $\operatorname{ran}\alpha$.*

(iii) *There exist $x, y \in Y$ such that $x \in a\alpha^{-1}$, $y \in b\alpha^{-1}$, and $d(x,y) = |\operatorname{ran}\alpha| - 1$ where a and b are the endpoints of $\operatorname{ran}\alpha$.*

Proof (i)⇒(ii): Assume that α is regular. Then there exists an element $\beta \in OT(X,Y)$ such that $\alpha\beta\alpha = \alpha$. Define $Z = (\text{ran } \alpha)\beta$. Clearly Z is a subfence of Y and $|Z| \leq |\text{ran } \alpha|$. Now $Z\alpha = (\text{ran } \alpha)\beta\alpha = X\alpha\beta\alpha =$

$X\alpha = \text{ran } \alpha$. In particular, $|Z| \geq |\text{ran } \alpha|$. Therefore, $|Z| = |\text{ran } \alpha| = |Z\alpha|$, and hence $\alpha\big|_Z$ is a bijection onto ran α.

(ii)$\Rightarrow$(i): Let $X = \{x_1, x_2, \ldots, x_n\}$ and let Z be a subfence of Y such that $\alpha\big|_Z$ is a bijection onto ran α. Define $\gamma = \left(\alpha\big|_Z\right)^{-1}$. Let ran $\alpha = \{x_l, x_{l+1}, \ldots, x_m\}$ for some $1 \leq l \leq m \leq n$. Define a map $\beta : X \to Y$ by

$$x_i\beta = \begin{cases} x_l\gamma & \text{if } 1 \leq i < l, \\ x_i\gamma & \text{if } l \leq i \leq m, \\ x_m\gamma & \text{if } m < i \leq n. \end{cases}$$

Observe that $\beta \in OT(X,Y)$ and $\beta\alpha$ is the identity on $\operatorname{ran}\alpha$. Therefore, $x\alpha\beta\alpha = (x\alpha)(\beta\alpha) = x\alpha$ for all $x \in X$. Hence, α is regular.

(ii)$\Rightarrow$(iii): Assume that there exists a subfence Z of Y such that $\alpha|_Z$ is a bijection onto $\operatorname{ran}\alpha$. Then $|Z| = |\operatorname{ran}\alpha|$ and $\operatorname{ran}\alpha = Z\alpha$. Let a and b be the endpoints of $\operatorname{ran}\alpha$ and let x and y be the endpoints of Z. Then $x, y \in Y$ and $d(x,y) = |Z| - 1 = |\operatorname{ran}\alpha| - 1$. Since $\alpha|_Z$ is a bijection onto $\operatorname{ran}\alpha$, either $a = x\alpha$ and $b = y\alpha$ or $b = x\alpha$ and $a = y\alpha$. The desired result follows.

(iii)$\Rightarrow$(ii): Assume that there exist $x, y \in Y$ such that $x \in a\alpha^{-1}$, $y \in b\alpha^{-1}$, and $d(x,y) = |\operatorname{ran}\alpha| - 1$ where a and b are the endpoints of $\operatorname{ran}\alpha$. Let Z be the subfence of Y whose endpoints are x and y. Then $|Z| = d(x,y) + 1 = |\operatorname{ran}\alpha| = d(a,b) + 1 = d(x\alpha, y\alpha) + 1 \leq |Z\alpha| \leq |Z|$. It follows that $|Z| = |\operatorname{ran}\alpha|$ and $\operatorname{ran}\alpha = Z\alpha$. Hence, $\alpha|_Z$ is a bijection onto $\operatorname{ran}\alpha$. □

In general, the product of two regular elements in $OT(X,Y)$ might not be regular. A necessary and sufficient condition for a product to be regular is given below.

Theorem 3.6 *Let α be a regular element of $OT(X,Y)$ and let $\beta \in OT(X,Y)$. Then the following statements are equivalent:*

(i) *$\alpha\beta$ is regular.*

(ii) *There exists a subfence W of $\operatorname{ran}\alpha$ such that $\beta|_W$ is a bijection onto $\operatorname{ran}(\alpha\beta)$.*

(iii) *There exists $x, y \in Y$ such that $x \in a\beta^{-1} \cap \operatorname{ran}\alpha$, $y \in b\beta^{-1} \cap \operatorname{ran}\alpha$, and $d(x,y) = d(a,b) = |\operatorname{ran}(\alpha\beta)| - 1$ where a and b are the endpoints of $\operatorname{ran}\alpha$.*

Proof (i)$\Rightarrow$(ii): Suppose $\alpha\beta$ is regular. By Theorem 3.5, there exists a fence $Z \subseteq Y$ such that $(\alpha\beta)|_Z$ is a bijection onto $\operatorname{ran}(\alpha\beta) = Z\alpha\beta$. Clearly $W = Z\alpha$ is the desired subfence.

(ii)$\Rightarrow$(i): Let W be a subfence of $\operatorname{ran}\alpha$ such that $\beta|_W$ is a bijection onto $\operatorname{ran}(\alpha\beta) = W\beta$. Since α is regular, by Theorem 3.5 there exists a fence $Z' \subseteq Y$ such that $\alpha|_{Z'}$ is a bijection onto $\operatorname{ran}\alpha$. Since W is a subfence of $\operatorname{ran}\alpha = Z'\alpha$, there exists a subfence Z of Z' such that $\alpha|_Z$ is a bijection onto $W = Z\alpha$. Now $Z\alpha\beta = W\beta = \operatorname{ran}(\alpha\beta)$ and $(\alpha\beta)|_Z$ is a bijection onto $\operatorname{ran}(\alpha\beta)$. By Theorem 3.5 the product $\alpha\beta$ is regular.

(ii)$\Leftrightarrow$(iii): By Theorem 3.5 (ii)$\Leftrightarrow$(iii). □

In particular, if α and β are regular elements in $OT(X,Y)$, Theorem 3.6 gives necessary and sufficient conditions for $\alpha\beta$ to be regular as well.

We close this section with some properties of regular elements in $OT(X,Y)$.

Proposition 3.7 *Let α be a regular element in $OT(X,Y)$. The following statements hold:*

(i) $\operatorname{ran}\alpha = Y\alpha$.

(ii) *If* $\operatorname{ran}\alpha = Y$, *then α is coregular.*

Proof (i) Since α is regular, there exists $\beta \in OT(X,Y)$ such that $\alpha\beta\alpha = \alpha$. Let $z \in \operatorname{ran}\alpha$. Then $z\alpha = z\alpha\beta\alpha \in Y\alpha$. Therefore, $\operatorname{ran}\alpha = Y\alpha$.

(ii) Suppose ran $\alpha = Y$. Then (ran $\alpha)\alpha = Y\alpha =$ ran α where the latter equality holds by (i). Therefore, α is a bijection on ran α and thus α is coregular by Theorem 2.6. □

4. Regularity in $OS(X,Y)$

In this section, we focus on the regularity of a semigroup $OS(X,Y)$ and its elements. With the use of the map α defined in Lemma 2.1, we obtain that $OS(X,Y)$ does not need to be regular or coregular. Throughout this section, let Y be a subfence of X. In the following results, necessary and sufficient conditions for the semigroup $OS(X,Y)$ to be regular are completely determined. Some lemmas needed in the characterization are given as follows.

Lemma 4.1 *Let $\alpha \in OT(X)$. If $\alpha|_{\text{ran }\alpha}$ is not a bijection, then* $|\text{ran }\alpha| \le |X| - 2$.

Proof Let $X = \{x_1, x_2, \ldots, x_n\}$. Assume that $|\text{ran }\alpha| \ge |X| - 1$. Clearly, $\alpha|_{\text{ran }\alpha}$ is a bijection if $|\text{ran }\alpha| = |X|$. Assume that $|\text{ran }\alpha| = |X| - 1$. Since $\operatorname{ran}\alpha$ is a subfence of X, it follows that $\operatorname{ran}\alpha = \{x_1, x_2, \ldots, x_{n-1}\}$ or $\operatorname{ran}\alpha = \{x_2, x_3, \ldots, x_n\}$. Without loss of generality, assume that $\operatorname{ran}\alpha = \{x_1, x_2, \ldots, x_{n-1}\}$. Then $x_i\alpha \ne x_j\alpha$ for all $1 \le i < j \le n-1$. Hence, $\alpha|_{\operatorname{ran}\alpha}$ is a bijection. □

Lemma 4.2 *Let Y be a proper subfence of X. If $|Y| \ge 2$, then $OS(X,Y)$ is not regular.*

Proof Assume that $|Y| \ge 2$. Then there are two comparable elements a and b in Y and an element $c \in X \setminus Y$ such that b and c are comparable. Then the map α from Lemma 2.1(iv) is an element in $OS(X,Y)$ but it is not regular in $OS(X,Y)$. Therefore, $OS(X,Y)$ is not a regular semigroup. □

Proposition 4.3 *Let $x \in X$. Then $OS(X,\{x\})$ is regular if and only if $X \setminus \{x\}$ does not contain subfences of size greater than* 2.

Proof Let $X = \{x_1, x_2, \ldots, x_n\}$.

Assume that $X \setminus \{x\}$ contains a subfence of size greater than 2. Without loss of generality, assume that $\{x_k = x, x_{k+1}, x_{k+2}, x_{k+3}\} \subseteq X$ for some $1 \le k \le n-3$. Let $\alpha \in OS(X,\{x\})$ be defined by

$$x_i\alpha = \begin{cases} x_k & \text{if } 1 \le i < k+3, \\ x_{k+1} & \text{if } k+3 \le i \le n. \end{cases}$$

Suppose that there exists an element $\beta \in OS(X,\{x\})$ such that $\alpha = \alpha\beta\alpha$. Then $x_{k+1} = x_{k+3}\alpha = x_{k+3}\alpha\beta\alpha = x_{k+1}\beta\alpha$. Since $x_k\beta = x_k$, we have $x_{k+1}\beta \in \{x_{k-1}, x_k, x_{k+1}\}$. It follows that $x_{k+1} = x_{k+1}\beta\alpha \in \{x_{k-1}, x_k, x_{k+1}\}\alpha = \{x_k\}$, a contradiction. Hence, α is not regular in $OS(X,\{x\})$.

Conversely, assume that $X\setminus\{x\}$ does not contain subfences of size greater than 2. Then $n \leq 5$. Precisely, we have 1) $n \leq 3$; 2) $n = 4$ and $x \in \{x_2, x_3\}$; or 3) $n = 5$ and $x = x_3$. Let $\alpha \in OS(X, \{x\})$.

Case 1 $\alpha|_{\text{ran }\alpha}$ is a bijection. By Theorem 3.5, α is regular in $OT(X, \text{ran }\alpha)$. Then there exists $\beta \in OT(X, \text{ran }\alpha)$ such that $\alpha\beta\alpha = \alpha$. Consequently, $x\alpha = x\alpha\beta\alpha = x\beta\alpha$. Since $x\beta \in \text{ran }\alpha$ and $\alpha|_{\text{ran }\alpha}$ is injective, we have $x = x\beta$, which implies that $\beta \in OS(X, \{x\})$. Hence, α is regular in $OS(X, \{x\})$.

Case 2 $\alpha|_{\text{ran }\alpha}$ is not a bijection. By Lemma 4.1, we have $|\text{ran }\alpha| \leq n-2 \leq 5-2 = 3$. If $|\text{ran }\alpha| = 1$, then α is a constant map that is regular in $OS(X, \{x\})$. We consider the remaining two cases.

Case 2.1 $|\operatorname{ran}\alpha| = 2$. Then $3 \leq n \leq 5$. Suppose $x \in \{x_1, x_n\}$. Without loss of generality assume that $x = x_1$. Then $\operatorname{ran}\alpha = \{x, x_2\}$ and $X \setminus \{x\} = \{x_2, x_3\}$. Since $\alpha|_{\text{ran }\alpha}$ is not a bijection, $x_2\alpha = x\alpha = x$ and so $x_3\alpha = x$, i.e. $\operatorname{ran}\alpha = \{x\}$, a contradiction. Therefore, $x = x_k$ for some $2 \leq k \leq n-1$. We then have $\operatorname{ran}\alpha = \{x_{k-1}, x\}$ or $\operatorname{ran}\alpha = \{x, x_{k+1}\}$. Without loss of generality, assume that $\operatorname{ran}\alpha = \{x_{k-1}, x\}$. Since $\alpha|_{\text{ran }\alpha}$ is not a bijection, $x_{k-1}\alpha = x\alpha = x$. If $x_{k+1}\alpha = x$, then $\operatorname{ran}\alpha = \{\text{x}\}$, a contradiction. Thus, $x_{k+1}\alpha = x_{k-1}$ and $\operatorname{ran}\alpha = \{x_{k-1}, x\} = \{x, x_{k+1}\}\alpha$. By setting $Y = \{x_{k-1}, x, x_{k+1}\}$ and $Z = \{x, x_{k+1}\}$ in Theorem 3.5, α is regular in $OT(X, Y)$. Then there exists $\beta \in OT(X, Y)$ such that $\alpha\beta\alpha = \alpha$. In particular, $x\beta \in Y$. If $x\beta = x_{k+1}$, then $x\alpha\beta\alpha = x\beta\alpha = x_{k+1}\alpha = x_{k-1} \neq x = x\alpha$, a contradiction. If $x\beta = x_{k-1}$, then $x_{k-1}\beta = x_{k-1}$ and $x_{k+1}\alpha\beta\alpha = x_{k-1}\beta\alpha = x_{k-1}\alpha = x \neq x_{k-1} = x_{k+1}\alpha$, a contradiction. Hence, $x\beta = x$, which implies that $\beta \in OS(X, \{x\})$. Therefore, the map α is regular in $OS(X, \{x\})$.

Case 2.2 $|\operatorname{ran}\alpha| = 3$. Since $\alpha|_{\text{ran }\alpha}$ is not a bijection, $5 = |\operatorname{ran}\alpha| + 2 \leq n$ by Lemma 4.1. As $n \leq 5$, it follows that $n = 5$ and $x = x_3$. Since $x\alpha = x$, we have $\{x_1, x_2, x\}\alpha = \operatorname{ran}\alpha = \{x, x_4, x_5\}$ or $\{x, x_4, x_5\}\alpha = \operatorname{ran}\alpha = \{x_1, x_2, x\}$. Without loss of generality, assume that $\{x, x_4, x_5\}\alpha = \operatorname{ran}\alpha = \{x_1, x_2, x\}$. By Theorem 3.5, α is regular in $OT(X, \{x_1, x_2, x\})$. There exists $\beta \in OT(X, \{x_1, x_2, x\})$ such that $\alpha\beta\alpha = \alpha$. Then $x\beta \in \text{ran }\alpha = \{x_1, x_2, x\}$. Suppose that $x\beta \in \{x_1, x_2\}$. Then $\{x_1, x_2, x\} = X\alpha = X\alpha\beta\alpha = \{x_1, x_2, x\}\beta\alpha \subseteq \{x_1, x_2\}\alpha$, a contradiction. Hence, $x\beta = x$, which implies that $\beta \in OS(X, \{x\})$. Therefore, α is regular in $OS(X, \{x\})$.

□

Corollary 4.4 *If $|Y| = 1$ and $|X| \geq 6$, then $OS(X, Y)$ is not regular.*

Proof We note that $X \setminus Y$ contains a subfence of size greater than 2 for all $Y \subsetneq X$ such that $|Y| = 1$. Hence, $OS(X, Y)$ is not regular by Proposition 4.3. □

The regularity of $OS(X, Y)$ is characterized in the following theorem.

Theorem 4.5 *Let $X = \{x_1, x_2, \ldots, x_n\}$ and Y is a subfence of X. Then $OS(X, Y)$ is regular if and only if one of the following statements hold:*

(i) $|X| = |Y| \leq 4$.

(ii) $|X| \leq 3$ *and* $|Y| = 1$.

(iii) $|X| = 4$ *and* $Y \in \{\{x_2\}, \{x_3\}\}$.

(iv) $|X| = 5$ *and* $Y = \{x_3\}$.

Proof Assume that $OS(X,Y)$ is regular. If $X=Y$, then $OT(X)=OS(X,Y)$ is regular, and hence $|X|=|Y|\leq 4$ by [20, Theorem 3.9]. Assume that Y is a proper subfence of X. By Lemma 4.2 and Corollary 4.4, we have that $|X|\leq 5$ and $|Y|=1$. By Proposition 4.3, $X\setminus Y$ does not contain subfences of size greater than 2, or equivalently, 1) $n\leq 3$, 2) $n=4$ and $Y\in\{\{x_2\},\{x_3\}\}$, or 3) $n=5$ and $Y=\{x_3\}$.

Conversely, assume that one of statements (i)–(iv) holds. If $|X|=|Y|\leq 4$, then $OS(X,Y)=OT(X)$ is regular by [20, Theorem 3.9]. If one of statements (ii)–(iv) holds, then $X\setminus Y$ does not contain subfences of size greater than 2. Hence, $OS(X,Y)$ is regular by Proposition 4.3. □

From Theorem 4.5, in many cases, $OS(X,Y)$ is not regular. The characterization of regular elements in $OS(X,Y)$ is given as follows.

Lemma 4.6 *If α is regular in $OS(X,Y)$, then $Y\alpha=Y\cap\operatorname{ran}\alpha$.*

Proof Let $\alpha\in OS(X,Y)$. Assume that α is regular. Then there exists $\beta\in OS(X,Y)$ such that $\alpha\beta\alpha=\alpha$. Clearly, $Y\alpha\subseteq Y\cap X\alpha=Y\cap\operatorname{ran}\alpha$. Let $y\in Y\cap\operatorname{ran}\alpha$. Then $y=x\alpha$ for some $x\in X$. It follows that $y=x\alpha=x\alpha\beta\alpha=y\beta\alpha\in Y\beta\alpha\subseteq Y\alpha$. Hence, $Y\alpha=Y\cap\operatorname{ran}\alpha$. □

Theorem 4.7 *Let $\alpha\in OS(X,Y)$. Then the following statements are equivalent:*

(i) *α is regular.*

(ii) *There exist subfences $Z\subseteq X$ and $W\subseteq Y\cap Z$ such that $\alpha|_Z$ is a bijection onto $\operatorname{ran}\alpha$ and $\alpha|_W$ is a bijection onto $Y\cap\operatorname{ran}\alpha$.*

Proof (i)⇒(ii): Assume that α is regular. Then there exists $\beta\in OS(X,Y)$ such that $\alpha\beta\alpha=\alpha$. Let $Z=(\operatorname{ran}\alpha)\beta$. Then Z is a subfence of X and $|Z|\leq|\operatorname{ran}\alpha|$. Since $Z\alpha=(\operatorname{ran}\alpha)\beta\alpha=X\alpha\beta\alpha=X\alpha=\operatorname{ran}\alpha$, we have $|Z|\geq|\operatorname{ran}\alpha|$. Hence, $|Z|=|\operatorname{ran}\alpha|=|Z\alpha|$. Therefore, $\alpha|_Z$ is a bijection onto $\operatorname{ran}\alpha$.

Define $W=(Y\cap\operatorname{ran}\alpha)\beta$. Then $W=(Y\alpha)\beta\subseteq Y\cap Z$ by Lemma 4.6. Since β is a map, it follows that $|W|\leq|Y\alpha|$. We have $W\alpha=(Y\alpha)\beta\alpha=Y\alpha\beta\alpha=Y\alpha$, which implies that $|W|\geq|Y\alpha|$. Hence, $|W|=|Y\alpha|=|W\alpha|$. Therefore, $\alpha|_W$ is a bijection onto $Y\alpha=Y\cap\operatorname{ran}\alpha$.

(ii)⇒(i): Let $Z\subseteq X$ and $W\subseteq Y\cap Z$ such that $\alpha|_Z$ is a bijection onto $\operatorname{ran}\alpha$ and $\alpha|_W$ is a bijection onto $Y\cap\operatorname{ran}\alpha$. Assume that $X=\{x_1,x_2,\ldots,x_n\}$. Let $\gamma=(\alpha|_Z)^{-1}$ and let $\operatorname{ran}\alpha=\{x_l,x_{l+1},\ldots,x_m\}$ for some $l\leq m$. Define a map $\beta:X\to X$ by

$$x_i\beta=\begin{cases}x_l\gamma & \text{if } 1\leq i<l,\\ x_i\gamma & \text{if } l\leq i\leq m,\\ x_m\gamma & \text{if } m<i\leq n.\end{cases}$$

Since $\beta\alpha$ is the identity on $\operatorname{ran}\alpha$, we have $x\alpha\beta\alpha=(x\alpha)(\beta\alpha)=x\alpha$ for all $x\in X$. It is not difficult to see that $\beta\in OT(X)$. Since $\alpha|_W$ is a bijection onto $Y\cap\operatorname{ran}\alpha$, it follows that $W\alpha=Y\cap\operatorname{ran}\alpha$. We consider the following three cases.

Case 1 $Y\subseteq\operatorname{ran}\alpha$. Then $Y=Y\cap\operatorname{ran}\alpha$. Hence, $Y\beta=(Y\cap\operatorname{ran}\alpha)\beta=(W\alpha)\beta=W\subseteq Y$ since $\alpha\beta$ is the identity on W. It follows that $\beta\in OS(X,Y)$.

Case 2 $\operatorname{ran}\alpha\subseteq Y$. It is not difficult to see that $\beta\in OS(X,Y)$.

Case 3 $Y \not\subseteq \operatorname{ran}\alpha$ and $\operatorname{ran}\alpha \not\subseteq Y$. Suppose that $x_l, x_m \notin Y \cap \operatorname{ran}\alpha$. Since $x_l, x_m \in \operatorname{ran}\alpha$, we have $x_l, x_m \notin Y$. Then $Y \subseteq \operatorname{ran}\alpha$, which is a contradiction. Hence, $x_l \in Y \cap \operatorname{ran}\alpha$ or $x_m \in Y \cap \operatorname{ran}\alpha$. Without loss of generality, assume that $x_l \in Y \cap \operatorname{ran}\alpha$. If $x_m \in Y \cap \operatorname{ran}\alpha$, then $\operatorname{ran}\alpha \subseteq Y$, which is impossible. Hence, $x_m \notin Y \cap \operatorname{ran}\alpha$. Then $Y \setminus \operatorname{ran}\alpha \subseteq \{x_1, x_2, \ldots, x_{l-1}\}$. Since $\alpha|_W$ is a bijection from W onto $Y \cap \operatorname{ran}\alpha$ and $x_l \in Y \cap \operatorname{ran}\alpha$, we have $x_l\gamma = x_l(\alpha|_Z)^{-1} = x_l(\alpha|_W)^{-1} \in W \subseteq Y \cap Z \subseteq Y$. It can be deduced that $(Y \setminus \operatorname{ran}\alpha)\beta \subseteq \{x_l\gamma\} \subseteq Y$. Moreover, $(Y \cap \operatorname{ran}\alpha)\beta = (W\alpha)\beta = W \subseteq Y$ since $\alpha\beta$ is the identity on W. Hence, $\beta \in OS(X,Y)$.

Therefore, α is regular in $OS(X,Y)$ as desired. □

Next, relations between the set $Reg(OT(X,Y))$ of regular elements in $OT(X,Y)$ and the set $Reg(OS(X,Y))$ of regular elements in $OS(X,Y)$ are studied.

Lemma 4.8 *We have*

$$Reg(OS(X,Y)) \subseteq Reg(OT(X,Y)) \cup (OS(X,Y) \setminus OT(X,Y)).$$

Proof Let $\alpha \in Reg(OS(X,Y))$. Assume that $\alpha \in OT(X,Y)$. Then $\operatorname{ran}\alpha \subseteq Y$, and hence $\operatorname{ran}\alpha = Y \cap \operatorname{ran}\alpha$. Since α is regular in $OS(X,Y)$, by Theorem 4.7, there exist subfences $Z \subseteq X$ and $W \subseteq Y \cap Z$ such that $\alpha|_W$ is a bijection onto $Y \cap \operatorname{ran}\alpha = \operatorname{ran}\alpha$. Equivalently, $W \subseteq Y$ and $\alpha|_W$ is a bijection onto $\operatorname{ran}\alpha$. Therefore, $\alpha \in Reg(OT(X,Y))$ by Theorem 3.5. □

In some cases, the equality in Lemma 4.8 holds.

Theorem 4.9 *If* $|X \setminus Y| = 1$ *and* $|Y| \leq 4$*, then*

$$Reg(OS(X,Y)) = Reg(OT(X,Y)) \cup (OS(X,Y) \setminus OT(X,Y)).$$

Proof Assume that $|X \setminus Y| = 1$ and $|Y| \leq 4$. By Lemma 4.8, we have

$$Reg(OS(X,Y)) \subseteq Reg(OT(X,Y)) \cup (OS(X,Y) \setminus OT(X,Y)).$$

For the reverse inclusion, assume that $X = Y \cup \{x\}$. Clearly, $Reg(OT(X,Y)) \subseteq Reg(OS(X,Y))$ since $OT(X,Y) \subseteq OS(X,Y)$. Let $\alpha \in OS(X,Y) \setminus OT(X,Y)$. Then $Y\alpha \subseteq Y$. It follows that, for each $a \in X$, $a\alpha = x$ if and only if $x = a$. Hence, $\alpha|_Y \in OT(Y)$. Since $|Y| \leq 4$, $\alpha|_Y$ is regular in $OT(Y)$ by [20, Theorem 3.9]. Then there exists $\beta \in OT(Y)$ such that $\alpha|_Y\beta\alpha|_Y = \alpha|_Y$. Define $\bar{\beta} : X \to X$ by

$$a\bar{\beta} = \begin{cases} a\beta & \text{if } a \in Y, \\ x & \text{if } a = x. \end{cases}$$

It is not difficult to see that $\bar{\beta} \in OS(X,Y)$ and $\alpha\bar{\beta}\alpha = \alpha$. Hence, $\alpha \in Reg(OS(X,Y))$. Therefore, the result follows. □

In the following part, we focus on coregularity of $OS(X,Y)$. First, we determine a necessary and sufficient condition for $OS(X,\{x\})$ to be regular.

Lemma 4.10 *Let* $x \in X$*. Then* $OS(X,\{x\})$ *is coregular if and only if* $X \setminus \{x\}$ *is a fence of size less than or equal to* 2*.*

Proof Assume that $X \setminus \{x\}$ is not a fence of size less than or equal to 2.

Case 1 $X \setminus \{x\}$ is not a fence. Then $X = \{x_1, x_2, \ldots, x_n\}$ for some $n \geq 3$. It follows that there exists an integer $1 \leq k \leq n-2$ such that $\{x_k, x = x_{k+1}, x_{k+2}\} \subseteq X$. Then it is not difficult to verify that the map

$$x_i\alpha = \begin{cases} x = x_{k+1} & \text{if } 1 \leq i \leq k+1, \\ x_k & \text{if } k+2 \leq i \leq n \end{cases}$$

is an element in $OS(X, \{x\})$ that is not coregular. Therefore, $OS(X, \{x\})$ is not coregular.

Case 2 $X \setminus \{x\}$ is a fence of size greater than 2. Then $|X| \geq 4$ and x is one of the end points of X. By Theorem 4.5, $OS(X, \{x\})$ is not regular, which implies that $OS(X, \{x\})$ is not coregular.

Conversely, assume that $X \setminus \{x\}$ is a fence of size less than or equal to 2. If $|X \setminus \{x\}| = 1$, then the elements in $OS(X, \{x\})$ are id_X and c_x, which are regular. Assume that $|X \setminus \{x\}| = 2$. Then $X = \{x_1, x_2, x_3\}$ is a fence of size 3 and $X \setminus \{x\} = \{x_1, x_2\}$ or $X \setminus \{x\} = \{x_2, x_3\}$. We may assume that $X \setminus \{x\} = \{x_1, x_2\}$. In this case, $x = x_3$. It is not difficult to see that the elements in $OS(X, \{x\})$ are id_X, c_{x_3}, $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_2 & x_3 \end{pmatrix}$, and $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_2 & x_3 \end{pmatrix}$, which are coregular. □

Theorem 4.11 *The semigroup $OS(X, Y)$ is coregular if and only if one of the following statements holds:*

(i) $|X| \leq 2$.

(ii) $X = \{x_1, x_2, x_3\}$ *and* $Y \in \{\{x_1\}, \{x_3\}\}$.

Proof Assume that $OS(X, Y)$ is coregular. If $X = Y$, then $OS(X, Y) = OT(X, Y)$. By Corollary 3.4, it can be concluded that $|X| \leq 2$. Assume that Y is a proper subset of X. If $|Y| > 1$, then $OS(X, Y)$ is not regular by Lemma 4.2, and hence it is not coregular. It follows that $|Y| = 1$. By Lemma 4.10, $X \setminus Y$ is a fence of size less than or equal to 2. Consequently, $|X| = 2$ or $X = \{x_1, x_2, x_3\}$ and $Y \in \{\{x_1\}, \{x_3\}\}$.

Conversely, assume that one of the two conditions holds. If $X = Y$ and $|X| \leq 2$, then $OS(X, Y) = OT(X, Y)$ is coregular by Corollary 3.4. Otherwise, $|Y| = 1$ and $X \setminus Y$ is a fence of size less than or equal to 2. Hence, $OS(X, Y)$ is coregular by Lemma 4.10. □

Acknowledgments

The authors would like to thank the anonymous referees for their helpful comments.

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