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The rank of apparition of powers of Lucas sequence

Prasanta Kumar RAY[1,*], **Nurettin IRMAK**[2], **Bijan Kumar PATEL**[3]
[1]Department of Mathematics, Sambalpur University, Jyoti Vihar, Burla, India
[2]Department of Mathematics, Niğde Ömer Halisdemir University, Niğde, Turkey
[3]Department of Mathematics, International Institute of Information Technology, Bhubaneswar, India

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Abstract: This note is devoted to studying the divisibility relation $u_n^{k+1}|u_m$ for a least positive integer m, where $\{u_n\}_{n\geq 0}$ is a nondegenerate Lucas sequence with characteristic polynomial x^2-ax-b, for some relatively prime integers a and b.

Key words: Lucas sequence, periodicity, rank of apparition, p-adic order

1. Introduction

Let $\{u_n\}_{n\geq 0}$ be a Lucas sequence of integers, so that $u_0=0$, $u_1=1$, and

$$u_n = au_{n-1} + bu_{n-2}; \quad n \geq 2,$$

where a and b are relatively prime integers. Here $\{u_n\}_{n\geq 0}$ is assumed to be a nondegenerate Lucas sequence. That is, $b \neq 0$ and for the roots α, β of the characteristic equation $x^2-ax-b=0$, α/β is not a root of unity, which excludes the pairs

$$(a,b) \in \{(\pm 2,-1),(\pm 1,-1),(0,\pm 1),(\pm 1,0)\}$$

([7], pp. 5–6).

Many properties of the linear recurring sequences have been investigated by several authors from different point of views. Renault [6] studied the period, rank, and order of the (a,b)-Fibonacci sequence modulo any positive integer m, where $\gcd(m,b)=1$. For each positive integer m relatively prime with b,

$$\alpha(m) = min\{n \geq 1 : m|u_n\}$$

is well defined and is called the rank of apparition (order of appearance) of m. Clearly, $\alpha(u_n) \leq n$ for all positive integers n, while from the primitive divisor theorem [1] it follows that $\alpha(u_n)=n$ for all integers $n>30$ (and all the exceptionals $n \leq 30$ such that $\alpha(u_n)<n$ can be computed). The following result about the rank of apparition is found in [6].

Lemma 1.1 *For each integer $m \geq 1$, we have $m|u_n$ for some positive integer n if and only if $\gcd(m,b)=1$ and $\alpha(m)|n$.*

*Correspondence: rayprasanta2008@gmail.com
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For a prime number p and a nonzero integer m, the p-adic valuation of m denoted by $\nu_p(m)$ is the exponent of p in the factorization of m. Recently, Sanna [8] derived a formula for the p-adic valuation of nondegenerate Lucas sequences as follows.

Lemma 1.2 *If p is a prime number such that $p \nmid b$, then*

$$\nu_p(u_n) = \begin{cases} \nu_p(n) + \nu_p(u_p) - 1, & \text{if } p|\Delta, p|n; \\ 0, & \text{if } p|\Delta, p \nmid n; \\ \nu_p(n) + \nu_p(u_{p\alpha(p)}) - 1, & \text{if } p \nmid \Delta, \alpha(p)|n, p|n; \\ \nu_p(u_{p\alpha(p)}), & \text{if } p \nmid \Delta, \alpha(p)|n, p \nmid n; \\ 0, & \text{if } p \nmid \Delta, \alpha(p) \nmid n; \end{cases}$$

for each positive integer n, where $\Delta = a^2 + 4b$.

In a subsequent paper, using the p-adic valuation of $\{u_n\}$, Sanna derived some formulas for the rank of apparition of the power of a prime number ([9], Lemma 2.5).

Indeed, the result in Lemma 1.2 is a reflection of the formula derived by Bilu et al. ([2], Proposition 2.1) as follows.

Lemma 1.3 *For all prime $p \nmid b$,*

$$\nu_p(u_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{\alpha(p)}; \\ \nu_p(u_{\alpha(p)}) + \nu_p(n/\alpha(p)), & \text{if } n \equiv 0 \pmod{\alpha(p)},\ p = \text{odd}; \\ \nu_2(u_2) + \nu_2(n/2), & \text{if } n \equiv 0 \pmod{2},\ p = 2,\ a = \text{even}; \\ \nu_2(u_3), & \text{if } n \equiv 3 \pmod{6},\ p = 2,\ a = \text{odd}; \\ \nu_2(u_6) + \nu_2(n/2), & \text{if } n \equiv 0 \pmod{6},\ p = 2,\ a = \text{odd}. \end{cases}$$

In this paper, we study the rank of apparition of powers of Lucas sequences $\{u_n\}$; that is, we obtain a divisibility relation $u_n^{k+1}|u_m$ for a least positive integer m with $k \geq 0$. The results we prove subsequently are indeed the generalization of some of the previous results of Marques [4]. He derived $\alpha(u_n^{k+1})$ for $a = b = 1$, i.e. for the sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$, as follows.

Lemma 1.4 *If F_n denotes the nth Fibonacci number, then*

$$\alpha(F_n^{k+1}) = \begin{cases} \frac{n}{2}F_n^k, & \text{if } n \equiv 3 \pmod{6} \text{ and } k \geq 2; \\ nF_n^k & \textit{otherwise.} \end{cases}$$

In [4], Marques also established the formula $\alpha(L_n^k)$ in some cases of n and k, where $\{L_n\}_{n\geq 0}$ denote the sequence of Lucas numbers. Subsequently, Pongsriiam [5] derived the same formula for all $n, k \geq 1$.

Our main results are the following.

Theorem 1.5 *For even a and for $b \equiv 1 \pmod{4}$, $\alpha(u_n^{k+1}) = nu_n^k$ with $k \geq 0$.*

Theorem 1.6 *For odd a and for $b=1$, we have*

$$\alpha(u_n^{k+1}) = \begin{cases} \frac{n}{2}u_n^k, & \text{if } n \equiv 3 \pmod 6 \text{ and } k \geq 2; \\ nu_n^k & \textit{otherwise.} \end{cases}$$

Theorem 1.7 *For any a with $b \equiv -1 \pmod 4$ and $k \geq 0$, $\alpha(u_n^{k+1}) = nu_n^k$.*

2. The proofs

Proof [Proof of Theorem 1.5] For a prime $p \nmid b$ with a even, we need only to consider the case $\alpha(p)|n$. By virtue of Lemma 1.3, we obtain

$$\begin{aligned} \nu_p(u_{nu_n^k}) &= \nu_p(nu_n^k) + \nu_p(u_{\alpha(p)}) - \nu_p(\alpha(p)) \\ &= \nu_p(n) + \nu_p(u_{\alpha(p)}) - \nu_p(\alpha(p)) + \nu_p(u_n^k) \\ &= \nu_p(u_n) + \nu_p(u_n^k) \\ &= \nu_p(u_n^{k+1}). \end{aligned}$$

Now consider the case for $p=2$ with a even. For $n \equiv 0 \pmod 2$, $nu_n^k \equiv 0 \pmod 2$ for $k \geq 0$. Using Lemma 1.3 again, we have

$$\begin{aligned} \nu_2(u_{nu_n^k}) &= \nu_2(u_2) + \nu_2(n) + k\nu_2(u_n) - 1 \\ &= (k+1)(\nu_2(u_2) + \nu_2(n) - 1) \\ &= \nu_2(u_n^{k+1}). \end{aligned}$$

Furthermore, as $n \equiv 0 \pmod 2$, $\frac{nu_n^k}{2} \equiv 0 \pmod 2$. Therefore,

$$\nu_2(u_{nu_n^k/2}) = (k+1)(\nu_2(u_2) + \nu_2(n) - 1) - 1 < \nu_2(u_n^{k+1}),$$

which completes the proof. □

The following lemma is useful while proving the subsequent theorem.

Lemma 2.1 *For odd a and $b \equiv 1 \pmod 4$, $\nu_2(u_6) - \nu_2(u_3) = 2$.*

Proof We have $u_6/u_3 = a(a^2+3b)$ and $a^2+3b \equiv 4 \pmod 8$, since a is odd and $b=1$, so the claim follows. □

Proof [Proof of Theorem 1.6] For an odd prime $p \nmid b$ with a odd, we need only to consider the case $\alpha(p)|n$. Using Lemma 1.3, we obtain

$$\begin{aligned} \nu_p(u_{nu_n^k}) &= \nu_p(nu_n^k) + \nu_p(u_{\alpha(p)}) - \nu_p(\alpha(p)) \\ &= \nu_p(n) + \nu_p(u_{\alpha(p)}) - \nu_p(\alpha(p)) + \nu_p(u_n^k) \\ &= \nu_p(u_n) + \nu_p(u_n^k) \\ &= \nu_p(u_n^{k+1}). \end{aligned}$$

Since a is odd and $\alpha(p)|n$, for $p=2$ then $\alpha(2)=3|n$, so that $n\equiv 0 \pmod 6$ or $n\equiv 3 \pmod 6$. Consider the case $n\equiv 0 \pmod 6$; then $nu_n^k\equiv 0 \pmod 6$. Therefore, the use of Lemma 1.3 again gives

$$\begin{aligned}\nu_2(u_{nu_n^k}) &= \nu_2(nu_n^k)+\nu_2(u_6)-1\\ &= \nu_2(n)+k\nu_2(u_n)+\nu_2(u_6)-1\\ &= (k+1)\left(\nu_2(n)+\nu_2(u_6)-1\right)\\ &= \nu_2(u_n^{k+1}).\end{aligned}$$

It follows that $u_n^{k+1}|u_{nu_n^k}$ for $n\equiv 0 \pmod 6$ and hence $\alpha(u_n^{k+1})|nu_n^k$. In order to get a conclusion, it suffices to show $u_n^{k+1}\nmid u_{nu_n^k/2}$. Since $\frac{nu_n^k}{2}\equiv 0 \pmod 6$, we have

$$\begin{aligned}\nu_2(u_{nu_n^k/2}) &= \nu_2(nu_n^k)+\nu_2(u_6)-2\\ &= (k+1)\left[\nu_2(u_6)+\nu_2(n)-1\right]-1\\ &< (k+1)\left[\nu_2(u_6)+\nu_2(n)-1\right]\\ &= \nu_2(u_n^{k+1}),\end{aligned}$$

and the case follows. In order to prove the case $n\equiv 3 \pmod 6$, we proceed as follows. Since $n\equiv 3 \pmod 6$, $\frac{nu_n^k}{2}\equiv 0 \pmod 6$ for $k\geq 2$. Using the fact $\nu_2(u_6)\geq \nu_2(u_3)+2$, we have

$$\begin{aligned}\nu_2(u_{nu_n^k}) &> \nu_2\left(u_{\frac{nu_n^k}{2}}\right)\\ &\geq \nu_2(u_n^{k+1}).\end{aligned}$$

Now it is enough to show that $\nu_2(u_n^{k+1})>\nu_2(u_{nu_n^k/4})$ for all $k\geq 2$. Here

$$\begin{aligned}\nu_2\left(u_{\frac{nu_n^k}{4}}\right) &= \nu_2(u_6)+k\nu_2(u_3)-3\\ &= \nu_2(u_3)+k\nu_2(u_3)-1\\ &< \nu_2(u_n^{k+1}),\end{aligned}$$

and the case follows. Finally we show for the case $k=1$. Since $n\equiv 3 \pmod 6$, $nu_n\equiv 0 \pmod 6$. Therefore, by Lemma 2.1,

$$\begin{aligned}\nu_2(u_{nu_n}) &= \nu_2(u_6)+\nu_2(nu_n)-1\\ &= \nu_2(u_3)+2+\nu_2(u_3)-1\\ &= 3>2=2\nu_2(u_3)\\ &= \nu_2(u_n^2).\end{aligned}$$

Also, for $n\equiv 3 \pmod 6$, we have $nu_n/2\equiv 3 \pmod 6$ and we get

$$\nu_2(u_{nu_n/2})=\nu_2(u_3)<2=\nu_2(u_n^2).$$

This ends the proof. □

Proof [Proof of Theorem 1.7] For even a with $b \equiv -1 \pmod 4$ the proof of the result $\alpha(u_n^{k+1}) = nu_n^k$ when $k \geq 0$ is analogous to Theorem 1.5. Consider the case for odd a. For $n \equiv 3 \pmod 6$, $nu_n^k \equiv 0 \pmod 6$ for $k \geq 2$. By virtue of Lemma 1.3, we have

$$\begin{aligned}\nu_2(u_{nu_n^k}) &= \nu_2(u_6) + \nu_2(nu_n^k) - 1\\ &= \nu_2(u_6) + k\nu_2(u_3) - 1\\ &= \nu_2(u_3) + k\nu_2(u_3)\\ &= \nu_2(u_n^{k+1}).\end{aligned}$$

Now it suffies to prove $\nu_2(u_n^{k+1}) > \nu_2(u_{nu_n^k/2})$, for all $k \geq 2$. Using the identity $u_6/u_3 = a(a^2+3b)$, we get

$$\begin{aligned}\nu_2\left(u_{\frac{nu_n^k}{2}}\right) &= \nu_2(u_6) + k\nu_2(u_3) - 2\\ &= \nu_2(u_3) + k\nu_2(u_3) - 1\\ &< \nu_2(u_n^{k+1}),\end{aligned}$$

and hence the result. For the remaining case, the proof is analogous to Theorem 1.6. □

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