Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1941 – 1955

doi:10.3906/mat-1709-74

A new approach to H-supplemented modules via homomorphisms

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Received: 26.09.2017 • **Accepted/Published Online:** 04.05.2018 • **Final Version:** 24.07.2018

Abstract: The class of H-supplemented modules, which is a nice generalization of that of lifting modules, has been studied extensively in the last decade. As the concept of homomorphisms plays an important role in module theory, we are interested in H-supplemented modules relative to homomorphisms. Let R be a ring, M a right R-module, and $S = \mathrm{End}_R(M)$. We say that M is endomorphism H-supplemented (briefly, E-H-supplemented) provided that for every $f \in S$ there exists a direct summand D of M such that $Imf + X = M$ if and only if $D + X = M$ for every submodule X of M. In this paper, we deal with the E-H-supplemented property of modules and also a similar property for a module M by considering $\mathrm{Hom}_R(N, M)$ instead of S where N is any module.

Key words: H-Supplemented module, E-H-supplemented module, dual Rickart module, small submodule

1. Introduction

Throughout this paper R denotes an arbitrary associative ring with identity and all modules are unitary right R-modules. A submodule N of a module M is said to be *small* in M if $N + K \neq M$ for any proper submodule K of M, and we denote it by $N \ll M$. A module M is called *small* if it is a small submodule of some module, and equivalently, M is a small submodule of its injective hull. A submodule A of M is called *coclosed* if A has no proper coessential submodules in M, that is, if whenever A/B is small in M/B, then $A = B$. Following [7], B is called an *s-closure* of A in M if B is a coessential submodule of A and B is coclosed in M. Note that in the literature, the two concepts of s-closure and coclosure are the same.

Recall from [18] that a module M is called *(non)cosingular* in the case of ($\overline{Z}(M) = M$) $\overline{Z}(M) = 0$ where $\overline{Z}(M)$ is defined to be $\cap\{N \leq M \mid M/N \ll E(M/N)\}$. From definitions we conclude that a small noncosingular module is zero. On the other hand, a module M is noncosingular if and only if every nonzero homomorphic image of M is nonsmall.

A module M is called *lifting* if every submodule of M lies above a direct summand of M so that for every $N \leq M$ there exists a direct summand D of M such that $N/D \ll M/D$. A submodule N of M is called a *supplement* in M if there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll N$. A module M is called *supplemented* if every submodule of M has a supplement. Recall that M is called *weakly supplemented*

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2010 *AMS Mathematics Subject Classification:* 16D10, 16D40, 16D80

provided that, for every submodule N of M, there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll M$. A module M is called *amply supplemented* if whenever $M = A+B$ then A contains a supplement of B in M. A lifting module is amply supplemented and hence supplemented.

Recall that a module M is called *H-supplemented* if for every submodule N of M there exists a direct summand D of M such that $M = N + X$ if and only if $M = D + X$ for every submodule X of M. Since H-supplemented modules have a different outward definition and may have different structures than other supplemented modules, the class of H-supplemented modules has been studied in the last two decades. After introducing this new notation, some authors tried to investigate them more. The first serious efforts were done in [12] and [11]. After that, in [8], the authors presented some equivalent conditions for a module to be H-supplemented that shows that this class of modules is closely related to the concept of small submodules. One of the nice papers related to H-supplemented modules and their generalizations is [3]. In [3], the authors introduced a new generalization of H-supplemented modules that is Goldie*-supplemented modules via an equivalence relation, namely β^*. Let X and Y be submodules of M. Then $X\beta^*Y$ in M provided $(X+Y)/X \ll M/X$ and $(X+Y)/Y \ll M/Y$. Here it is convenient to state that M is H-supplemented if and only if for each submodule X of M there exists a direct summand D of M such that $X\beta^*D$ in M. There are some works related to H-supplemented modules and their generalizations (see [16], [17], and [20]).

Recently, after the defining of dual Rickart modules in [13], generalizations of dual Rickart modules seem to be interesting for researchers in ring and module theory. In particular, making a connection between the ring of endomorphisms of a module M and the concepts of lifting, H-supplemented, and others may help us to know their structures more. In [1], the author studied a new generalization of both lifting and dual Rickart modules, namely $\mathcal{I}$-lifting modules. A module M is called *$\mathcal{I}$-lifting* provided that, for every nonzero endomorphism f of M, there is a direct summand D of M such that Imf/D is small in M/D. In [1], some properties of $\mathcal{I}$-lifting modules were investigated.

Now it is natural to define H-supplemented modules using homomorphisms. In this work, we call a module M *endomorphism H-supplemented* if for every nonzero endomorphism f of M there is a direct summand K of M such that $M = Imf + X$ if and only if $M = K + X$ for every submodule X of M (that is equivalent to saying that $Imf\beta^*K$ in M). In Section 2, we investigate some properties of endomorphism H-supplemented modules. We observe that the concept of endomorphism H-supplemented modules generalizes that of dual Rickart modules. This relation makes the endomorphism H-supplemented property more impressive to study. We also present conditions under which these two concepts coincide. In Section 3, we bring another perspective to H-supplemented modules via homomorphisms, namely $\$(N,M)$-$H$-supplemented modules, that is related to the class $\$(N,M)$ defined in [19]. Some characterizations of H-supplemented modules are also obtained.

In what follows, $J(R)$ denotes the Jacobson radical of a ring R and $Rad(M)$ stands for the radical of a module M. S denotes the endomorphism ring $\mathrm{End}_R(M)$ of an R-module M. For unexplained terminologies we refer to [2], [4], [14], and [21].

2. E-H-Supplemented modules

In this section we introduce a new generalization of the class of H-supplemented modules in terms of endomorphisms, namely endomorphism H-supplemented modules. We work on factor modules, in particular direct summands, of endomorphism H-supplemented modules. We also deal with the relations between the class of endomorphism H-supplemented modules and the classes of dual Rickart modules and $\mathcal{I}$-lifting modules.

Definition 2.1 *A module M is called endomorphism H-supplemented (E-H-supplemented, for short) if for every $f \in S$ there exists a direct summand D of M such that $Imf + X = M$ if and only if $D + X = M$ for every submodule X of M. From the definition of the β^* relation [3] we can say that M is called E-H-supplemented if for every $f \in S$ there exists a direct summand D of M such that $Imf \beta^* D$ in M.*

It is obvious that every H-supplemented module is E-H-supplemented. In [8], the authors showed that a module M is H-supplemented if and only if for every submodule N of M there is a direct summand D of M such that $\frac{N+D}{D} \ll \frac{M}{D}$ and $\frac{N+D}{N} \ll \frac{M}{N}$. Now it is not hard to check the following result.

Proposition 2.2 *The following are equivalent for a module M:*

(1) M is E-H-supplemented;

(2) For every $f \in S$, there exists a direct summand D of M with $\frac{Imf+D}{D} \ll \frac{M}{D}$ and $\frac{Imf+D}{Imf} \ll \frac{M}{Imf}$;

(3) For every $f \in S$, there exist a direct summand D and a submodule N of M with $Imf \subseteq N$ and $D \subseteq N$ such that $\frac{N}{D} \ll \frac{M}{D}$ and $\frac{N}{Imf} \ll \frac{M}{Imf}$.

Proposition 2.3 *Let M be an indecomposable module. Then the following are equivalent:*

(1) M is E-H-supplemented;

(2) Every nonzero endomorphism $f \in S$ is an epimorphism or $Imf \ll M$.

Proof (1) $\Rightarrow$ (2) Let M be an E-H-supplemented indecomposable module and $0 \neq f \in S$. Then, by assumption, there exists a direct summand D of M such that $\frac{Imf+D}{D} \ll \frac{M}{D}$ and $\frac{Imf+D}{Imf} \ll \frac{M}{Imf}$. Since M is indecomposable, $D = 0$ or $D = M$. If $D = 0$, $Imf \ll M$, and in the other case f is an epimorphism.
(2) $\Rightarrow$ (1) Obvious. □

Recall that a module M is *co-Hopfian* if every monomorphism $f \in S$ is an isomorphism. In [9], the module M is called *$\mathcal{T}$-noncosingular* if for any $f \in S$, Imf being small in M implies $f = 0$. Note that a noncosingular module is clearly $\mathcal{T}$-noncosingular. Following the last result, if M is indecomposable, E-H-supplemented, and $\mathcal{T}$-noncosingular, then M is co-Hopfian. Also, a module M is called *dual Rickart (or d-Rickart)* if the image in M of any single element of S is generated by an idempotent of S, and equivalently, for any $f \in S$, Imf is a direct summand of M. We now give some relations between the classes of dual Rickart modules and E-H-supplemented modules.

Theorem 2.4 *Let M be a module. Then the following statements are equivalent:*

(1) *M is d-Rickart;*

(2) *M is E-H-supplemented and $\mathcal{T}$-noncosingular.*

In particular, if M is a noncosingular E-H-supplemented module, then it is d-Rickart.

Proof (1) $\Rightarrow$ (2) It is clear by definitions.
(2) $\Rightarrow$ (1) Let M be $\mathcal{T}$-noncosingular and E-H-supplemented. Suppose that $f \in S$. Now there is a direct summand D of M such that $\frac{Imf+D}{D} \ll \frac{M}{D}$ and $\frac{Imf+D}{Imf} \ll \frac{M}{Imf}$. Consider the R-homomorphism $\lambda\colon M \to \frac{M}{D}$ defined by $\lambda(m) = f(m) + D$. Set $M = D \oplus D'$ for a submodule D' of M, so that there is an isomorphism $h\colon \frac{M}{D} \to D'$ induced by the decomposition $M = D \oplus D'$. Consider the homomorphism $joho\lambda\colon M \to M$ where $j\colon D' \to M$ is the inclusion map. Since $Im\lambda = \frac{Imf+D}{D} \ll \frac{M}{D}$, we can get $joho\lambda(M) = h(\frac{Imf+D}{D}) \ll D' \subseteq M$, so $Im(joho\lambda) \ll M$. Being M, $\mathcal{T}$-noncosingular implies that $joho\lambda = 0$. It follows that $\frac{Imf+D}{D} \subseteq Kerh$. Hence, $\frac{Imf+D}{D} = \frac{D}{D}$. Therefore, $Imf \subseteq D$. Since $\frac{D}{Imf} \ll \frac{M}{Imf}$ and $\frac{D}{Imf} + \frac{D'+Imf}{Imf} = \frac{M}{Imf}$, we conclude that $D' + Imf = M$. By modularity, $Imf = D$ is a direct summand of M. $\square$

Recall that a ring R is called *von Nuemann regular* provided that for every $a \in R$, there exists $x \in R$ such that $a = axa$. It is known from [9, Corollary 2.7] that a ring R is right (left) $\mathcal{T}$-noncosingular if and only if $J(R) = 0$. The next result is an immediate consequence of Theorem 2.4.

Corollary 2.5 *A ring R is von Neumann regular if and only if $J(R) = 0$ and R_R is E-H-supplemented.*

There are $(E\text{-})H$-supplemented modules that are not d-Rickart, as the next example shows.

Example 2.6 *Let M be a hollow module with at least an endomorphism f, which is distinct from zero and id_M (for example, the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$, where p is prime and $n > 1$). Then clearly M is $(E\text{-})H$-supplemented, which is not d-Rickart.*

The following indicates that the class of E-H-supplemented modules properly contains the class of H-supplemented modules.

Remark 2.7 *Since a d-Rickart module is E-H-supplemented, every injective module over a right hereditary ring is E-H-supplemented by [13, Theorem 2.29]. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$. Since M is injective, M is E-H-supplemented. Also, it is well known that M is not supplemented and hence it is not H-supplemented. Therefore, every nonsupplemented injective module over a right hereditary ring is E-H-supplemented but not H-supplemented.*

A ring R is called a right *V-ring* if every simple right R-module is injective. It is well known that R is a right V-ring if and only if $Rad(M) = 0$ for every right R-module M (see [21, 23.1]). It follows from [18, Proposition 2.5 and Corollary 2.6] that all modules over a right V-ring R are noncosingular. Thus, we have the following.

Corollary 2.8 *Let R be a right V-ring. Then an R-module M is E-H-supplemented if and only if M is d-Rickart.*

We shall deal with homomorphic images of E-H-supplemented modules.

Proposition 2.9 *Let M be an E-H-supplemented module and N be a direct summand of M. Suppose that for every direct summand K of M, there exists a direct summand T/N of M/N such that $(K+N)/N\beta^*T/N$ in M/N. Then M/N is E-H-supplemented.*

Proof Let $M = N \oplus N'$ for some $N' \leq M$ and $f\colon M/N \to M/N$ be an endomorphism. Consider the natural epimorphism $\pi\colon M \to M/N$ defined by $\pi(x) = x + N$ and the isomorphism $h\colon M/N \to N'$ defined by $h(n' + N) = n'$ induced by the decomposition $M = N \oplus N'$. Therefore, $hof o\pi\colon M \to M$ is an endomorphism. Set $Imf = L/N$. It is easy to check that $Im(hof o\pi) = L \cap N'$. Since M is E-H-supplemented, there exists a direct summand K of M such that $M = (L \cap N') + X$ if and only if $M = K + X$ for all $X \leq M$ (i.e. $(L \cap N')\beta^* K$ in M). By assumption, there is a submodule T of M such that T/N is a direct summand of M/N and $(K + N)/N\beta^* T/N$ in M/N. We shall prove that $L/N\beta^*(K + N)/N$ in M/N is equivalent to proving that $L/N + Y/N = M/N$ if and only if $(K + N)/N + Y/N = M/N$ for every $Y/N \leq M/N$. To verify this assertion, let $L/N + Y/N = M/N$ for $Y/N \leq M/N$. Then $L + Y = M$. From modularity, $L = N \oplus (L \cap N')$, so $(L \cap N') + Y = M$. Now $(L \cap N')\beta^* K$ implies that $K + Y = M$, which leads us to $(K + N)/N + Y/N = M/N$. The inverse implication can be verified in the same way. By transitivity of the β^* relation, we conclude that $Imf\beta^* T/N$ ([3, Lemma 2.2]). Hence, M/N is E-H-supplemented. $\square$

Proposition 2.10 *Let M be an E-H-supplemented module and N a direct summand of M. Suppose that for every direct summand K of M with $M = N + K$, $N \cap K$ is also a direct summand of M. Then N is E-H-supplemented.*

Proof Let N' be a submodule of M such that $M = N \oplus N'$. Let $f\colon N \to N$ be an endomorphism. Consider the homomorphism $g\colon M \to M$ defined by $g(n + n') = f(n) + n'$. Then $Img = Imf + N'$. By assumption, there is a direct summand D of M such that $M = Y + D$ if and only if $M = Imf + N' + Y$ for all $Y \leq M$. Since $M = N + Imf + N'$, we have $M = N + D$, so $D \cap N$ is a direct summand of N. Let $X \leq N$ be a submodule. If $N = X + Imf$, then $M = X + Imf + N'$. Thus, $M = X + D$. Hence, $N = X + (D \cap N)$. On the other hand, if $N = X + (D \cap N)$, then $M = X + (D \cap N) + D = X + D$ since $M = N + D$, so $M = X + Imf + N'$. As $X + Imf \leq N$, by modularity condition, we have $N = Imf + X$. Consequently, N is E-H-supplemented. $\square$

Recall that a submodule N of M is said to be *fully invariant (projection invariant)* if for every endomorphism f of M (every $f^2 = f \in S$), we have $f(N) \subseteq N$. Let M be a module with a submodule N. We call N a *plus invariant submodule* of M whenever for each decomposition $M = M_1 \oplus M_2$ we have $N = (N \cap M_1) \oplus (N \cap M_2)$. Every projection invariant submodule (and so every fully invariant submodule) of a module is plus invariant. Every submodule of a distributive module is plus invariant, so every submodule of a semisimple module for which any two distinct simple submodules are not isomorphic is plus invariant. Direct summands of an abelian module M are plus invariant because they are fully invariant in M.

We may find a similar result and proof in the following related papers. We add it for the sake of completeness.

Proposition 2.11 *Let M be a module and N a plus invariant direct summand of M. If M is E-H-supplemented, then M/N and N are E-H-supplemented.*

Proof Let D and D' be submodules of M such that $M = D \oplus D'$. By assumption, we have $N = (D \cap N) \oplus (D' \cap N)$. Then $(D + N) \cap (D' + N) = [D \oplus (D' \cap N)] \cap [(D \cap N) \oplus D'] = (D \cap N) \oplus (D' \cap N) = N$, so $M/N = [(D + N)/N] \oplus [(D' + N)/N]$. Proposition 2.9 shows that M/N is an E-H-supplemented module.

For the latter, let D and D' be submodules of M such that $M = D \oplus D' = N + D$. Since $N =$

$(D \cap N) \oplus (D' \cap N)$ and N is a direct summand of M, we conclude that $D \cap N$ is a direct summand of M. The result follows from Proposition 2.10. □

In [15], a module is called *duo* (resp. *weak duo*) if every submodule (resp. direct summand) of M is fully invariant in M.

Corollary 2.12 *Every direct summand of an E-H-supplemented duo module is E-H-supplemented.*

A module M has D_3 if for any direct summands M_1 and M_2 of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is a direct summand of M. A module M is said to have the summand intersection property (SIP) if the intersection of two direct summands of M is again a direct summand of M.

Theorem 2.13 *Let M be an E-H-supplemented module with D_3 or having the SIP. Then every direct summand of M is E-H-supplemented.*

Proof It follows immediately from Proposition 2.10. □

The condition D_3 is not necessary in Theorem 2.13, as the following example shows.

Example 2.14 *([17, Example 3.9])* Let I and J be two ideals of a commutative local ring R with maximal ideal m such that $I \subset J \subseteq m$ (e.g., R is a discrete valuation ring with maximal ideal m, $I = m^3$ and $J = m^2$). We consider the module $M = \frac{R}{I} \times \frac{R}{J}$ and its submodules $A = R(\bar{1}, \bar{0})$, $B = R(\bar{1}, \bar{1})$, and $C = R(\bar{0}, \bar{1})$. Note that $M = A + B = A \oplus C = B \oplus C$. On the other hand, we have $A \cap B = J/I \times 0$. Hence $A \cap B \subseteq Rad(M)$ and $A \cap B \ll M$. Therefore $0 \neq A \cap B$ is not a direct summand of M. So M does not satisfy D_3. Moreover, every direct summand of M is H-supplemented by [17, Proposition 2.1]. Hence, every direct summand of M is E-H-supplemented.

In the next results, we observe some connections between the concepts of E-H-supplemented modules and $\mathcal{I}$-lifting modules.

Theorem 2.15 *If a module M is $\mathcal{I}$-lifting, then M is E-H-supplemented. The converse holds if M is weak duo.*

Proof The first assertion is clear by definitions. Let M be a weak duo E-H-supplemented module and $f \in S$. Then there exists a direct summand D of M such that $\frac{Imf+D}{D} \ll \frac{M}{D}$ and $\frac{Imf+D}{Imf} \ll \frac{M}{Imf}$. Set $D \oplus D' = M$. Then $f(M) = f(D) \oplus f(D')$. Now $\frac{Imf+D}{Imf} + \frac{Imf+D'}{Imf} = \frac{M}{Imf}$. Since $\frac{Imf+D}{Imf} \ll \frac{M}{Imf}$, we conclude that $Imf + D' = M$. Thus, $f(D) + f(D') + D' = M$. Since M is a weak duo module, $f(D) \oplus D' = M$. By modularity condition, $f(D) = D$. It follows that $D \subseteq Imf$. Therefore, $\frac{Imf}{D} \ll \frac{M}{D}$. □

Theorem 2.16 *Let M be a projective E-H-supplemented module. Then M is $\mathcal{I}$-lifting.*

Proof Let $f \colon M \to M$ be a homomorphism. Then there is a direct summand D of M such that $Imf + X = M$ if and only if $D + X = M$ for all $X \leq M$. Set $D \oplus D' = M$. It follows that $Imf + D' = M$. Since M is projective, there exists a decomposition $T \oplus D' = M$ by [14, Lemma 4.47], where $T \subseteq Imf$. We show that $Imf \cap D' \ll D'$. To prove the last assertion, let $(Imf \cap D') + K = D'$. Then $(Imf \cap D') + K + T = D' + T = M$. Since $T \oplus (D' \cap Imf) = Imf$, we have that $Imf + K = M$. M being an E-H-supplemented module implies that $D + K = M$. By modularity condition and the fact that $K \subseteq D'$, we conclude that $K = D'$. Hence, M is $\mathcal{I}$-lifting by [1, Proposition 2.1]. □

Corollary 2.17 *Let R be a ring. Then the following are equivalent:*

(1) R_R is E-H-supplemented;

(2) R_R is $\mathcal{I}$-lifting;

(3) Every cyclic right ideal of R lies above a direct summand of R_R.

Proof (1) $\Rightarrow$ (2) It follows from Theorem 2.16.
(2) $\Rightarrow$ (3) This follows from the fact that the image of every endomorphism of R_R is a cyclic right ideal of R.
(3) $\Rightarrow$ (1) Let $g\colon R \to R$ be an endomorphism. Then Img is a cyclic right ideal of R, which lies above a direct summand of R_R by assumption. The rest is clear. □

Proposition 2.18 *Let R be a principal ideal domain. Then R_R is E-H-supplemented if and only if R_R is H-supplemented.*

Proof Clear. □

In [10], a module M is called *retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for every nonzero submodule N of M. When we deal with smallness, we give the following definition.

Definition 2.19 *A module M is said to be s-retractable if for every nonzero submodule N of M, there exists a nonzero $f \in Hom_R(M, N)$ such that $\frac{N}{Imf}$ is small in $\frac{M}{Imf}$.*

Clearly, every s-retractable module is retractable, and every retractable hollow module is s-retractable. It is obvious that being an H-supplemented module implies being an E-H-supplemented module. In the next result, we show that the converse holds for s-retractable modules.

Proposition 2.20 *Let M be an E-H-supplemented and s-retractable module. Then M is H-supplemented.*

Proof Let N be a nonzero submodule of M. Then, by hypothesis, there exists $0 \neq f \in \mathrm{Hom}_R(M, N)$ with $\frac{N}{Imf}$ small in $\frac{M}{Imf}$. This means that $N\beta^* Imf$ in M. Since M is E-H-supplemented, there also exists a direct summand D of M such that $Imf\beta^* D$ in M. It follows from [3, Lemma 2.2] that $N\beta^* D$ in M, which implies that M is H-supplemented. □

3. Relatively H-supplemented modules

In [19], Wang and Ding defined the family

$$\$(N, M) = \{A \leq M \mid \exists X \leq N \text{ and } f \in \mathrm{Hom}_R(X, M) \ni A/f(X) \ll M/f(X)\}$$

for modules N and M. Also, a module M is called *N-lifting* (or N-D_1) if for any $A \in \$(N, M)$ there exists a direct summand K of M such that $K \leq A$ and A/K is small in M/K. Motivated by the works on the family $\$(N, M)$ for modules N and M, in this section we make an approach to the notion of H-supplemented modules regarding the above-mentioned family. In this direction, we also focus on $\mathrm{Hom}_R(N, M)$ instead of S for the E-H-supplemented property of a module M.

Definition 3.1 *Let M and N be modules.*

1. *M is called $\$(N,M)$-$H$-supplemented if for every $A \in \$(N,M)$, there exists a direct summand D of M such that $(A+D)/A$ is small in M/A and $(A+D)/D$ is small in M/D. By the way, $A\beta^*D$ in M.*
2. *Let $f \in Hom_R(N,M)$. Then M is called f-H-supplemented (or H-supplemented relative to f) if there exists a direct summand D of M such that $(Imf+D)/Imf$ is small in M/Imf and $(Imf+D)/D$ is small in M/D. This is equivalent to saying that $Imf\beta^*D$ in M.*
3. *M is called N-H-supplemented (or H-supplemented relative to N) if M is f-H-supplemented for every $f \in Hom_R(N,M)$.*

Note that if a module M is N-lifting, then M is $\$(N,M)$-$H$-supplemented. In view of the above definition, a module M is H-supplemented if and only if M is $\$(M,M)$-$H$-supplemented. Also, M is E-H-supplemented if and only if M is M-H-supplemented.

Example 3.2 *Let M be a semisimple module. Then M is $\$(N,M)$-$H$-supplemented and N-H-supplemented for any R-module N. Let p be a prime number. The simple $\mathbb{Z}$-module $\mathbb{Z}_p$ is $\$(\mathbb{Z}_{p^\infty},\mathbb{Z}_p)$-$H$-supplemented and $\mathbb{Z}_{p^\infty}$ is $\$(\mathbb{Z}_p,\mathbb{Z}_{p^\infty})$-$H$-supplemented. Also, $\mathbb{Z}_4$ is $\$(\mathbb{Z}_3,\mathbb{Z}_4)$-$H$-supplemented and $\mathbb{Z}_3$ is $\$(\mathbb{Z}_4,\mathbb{Z}_3)$-$H$-supplemented. On the other hand, $\mathbb{Z}_4$ is $\mathbb{Z}_3$-H-supplemented and $\mathbb{Z}_3$ is $\mathbb{Z}_4$-H-supplemented.*

Theorem 3.3 *Let M and N be modules. Then the following hold.*

1. *If M is $\$(N,M)$-$H$-supplemented, then for every $A \in \$(N,M)$, there exist a submodule X of N and $f \in Hom_R(X,M)$ such that M is f-H-supplemented.*
2. *If M is X-H-supplemented for every submodule X of N, then M is $\$(N,M)$-$H$-supplemented.*

Proof (1) Let M be $\$(N,M)$-$H$-supplemented and $A \in \$(N,M)$. Then there exists a direct summand D of M with $A\beta^*D$ in M. There also exist a submodule X of N and $f \in \mathrm{Hom}_R(X,M)$ with $A/Imf \ll M/Imf$, which is equivalent to $Imf\beta^*A$ in M. β^* being a transitive relation implies that $Imf\beta^*D$ in M. Therefore, M is f-H-supplemented.
(2) Let $A \in \$(N,M)$. Then there exist a submodule X of N and $f \in \mathrm{Hom}_R(X,M)$ with $\frac{A}{Imf} \ll \frac{M}{Imf}$ so that $Imf\beta^*A$ in M. By assumption, M is X-H-supplemented. Hence, there exists a direct summand D of M such that $Imf\beta^*D$ in M. Now it follows from [3, Lemma 2.2] that $A\beta^*D$. Hence, M is $\$(N,M)$-$H$-supplemented. □

Proposition 3.4 *Let M be an indecomposable module. Then the following are equivalent for any module N.*

(1) M is N-H-supplemented;

(2) Every nonzero $f \in Hom_R(N,M)$ is an epimorphism or $Imf \ll M$.

Proof (1) $\Rightarrow$ (2) Let M be an indecomposable N-H-supplemented module and $0 \neq f \in \mathrm{Hom}_R(N,M)$. Then, by assumption, there exists a direct summand D of M such that $\frac{Imf+D}{D} \ll \frac{M}{D}$ and $\frac{Imf+D}{Imf} \ll \frac{M}{Imf}$. Since M is indecomposable, $D = 0$ or $D = M$. If $D = 0$, $Imf \ll M$ and in the other case f is an epimorphism.

(2) $\Rightarrow$ (1) Let $0 \neq f \in \mathrm{Hom}_R(N, M)$. If f is an epimorphism, then we take M as a direct summand of M satisfying that $\frac{Imf+D}{Imf}$ is small in $\frac{M}{Imf}$ and $\frac{Imf+D}{D}$ is small in $\frac{M}{D}$. For the case $Imf \ll M$, the direct summand 0 satisfies the required conditions. □

Note that if Imf is a direct summand of M for any $f \in \mathrm{Hom}_R(N, M)$, then Imf is an H-supplement of itself. On the other hand, it is clear that every small submodule of a module M belongs to $\$(N, M)$ for every module N.

Proposition 3.5 *Let M be an indecomposable module. Then the following are equivalent for any module N.*

(1) M is $\$(N, M)$-$H$-supplemented;

(2) For every $A \in \$(N, M)$, A is small in M or $A = M$.

Proof (1) $\Rightarrow$ (2) Let M be an indecomposable $\$(N, M)$-$H$-supplemented module and $A \in \$(N, M)$. Then there exists a direct summand D of M with $\frac{A+D}{D} \ll \frac{M}{D}$ and $\frac{A+D}{A} \ll \frac{M}{A}$. The module M being indecomposable implies that $D = 0$ or $D = M$. If $D = 0$, then $A \ll M$, and if $D = M$, then $A = M$.

(2) $\Rightarrow$ (1) Let $A \in \$(N, M)$. By (2), $A \ll M$ or $A = M$. For these cases the direct summands 0 and M satisfy the required conditions, respectively. □

Recall that a module M has the summand sum property (SSP) if the sum of each two direct summands of M is a direct summand of M.

Proposition 3.6 *Let M be a module with SSP and M_1, M_2 be disjoint submodules with $M_1 \oplus M_2$ a direct summand of M. Then M_1 is M_2-H-supplemented.*

Proof Let $f \in \mathrm{Hom}_R(M_2, M_1)$ and $N = \{f(m_2) + m_2 | m_2 \in M_2\}$. Then $N \cap M_1 = 0$, $N \oplus M_1 = M_1 \oplus M_2$ and $N + M_2 = Imf \oplus M_2$. Hence, N is a direct summand of $M_1 \oplus M_2$ and so is that of M. By the SSP, $N + M_2$ is a direct summand of M and therefore Imf is also a direct summand of M and so is that of M_1. This implies that M_1 is M_2-H-supplemented. □

The following result is a direct consequence of Proposition 3.6 and the fact that M is E-H-supplemented if and only if M is M-H-supplemented.

Corollary 3.7 *Let M be a module such that $M \oplus M$ has the SSP. Then M is E-H-supplemented.*

Corollary 3.8 *Let M be a module and N a direct sum of copies of M. Assume that M has Σ-SSP, that is, every direct sum of copies of M has the SSP. Then M is N-H-supplemented.*

Proof By hypothesis, $M \oplus N$ has the SSP. Now the result follows from Proposition 3.6. □

Let M and N be modules. We say that the module M is N-*hollow* if every proper submodule $A \in \$(N, M)$ is small in M. It is obvious that a hollow module M is N-hollow for every module N. Note that a module M is hollow if and only if it is M-hollow.

Theorem 3.9 *Let M and N be modules such that M is indecomposable. Then the following are equivalent.*

(1) For every submodule X of N, M is X-H-supplemented;

(2) M is $\$(N,M)$-$H$-supplemented;

(3) M is N-hollow.

Proof (1) $\Rightarrow$ (2) By Theorem 3.3(2).
(2) $\Rightarrow$ (3) Let $A \in \$(N,M)$ and $A \neq M$. Then by Proposition 3.5, A is small in M. Hence, M is N-hollow.
(3) $\Rightarrow$ (1) Let X be a submodule of N and $f \in \mathrm{Hom}_R(X,M)$. If $Imf = M$, then as a direct summand of M, M satisfies the conditions $\frac{Imf+M}{Imf} \ll \frac{M}{Imf}$ and $\frac{Imf+M}{M} \ll \frac{M}{M}$. Assume now that $Imf \neq M$. $\frac{Imf}{Imf} \ll \frac{M}{Imf}$ implies $Imf \in \$(N,M)$. Since M is N-hollow, Imf is small in M. Therefore, M is X-H-supplemented. □

In [6], a nonzero module M is called *P-hollow*, which stands for principally hollow, if every proper cyclic submodule is small in M.

Proposition 3.10 *Every P-hollow module is R-H-supplemented.*

Proof Let M be a P-hollow module and $f \in \mathrm{Hom}_R(R,M)$. Then $Imf = f(1)R$. If $Imf = M$, then there is nothing to show. If $Imf \neq M$, then by hypothesis, Imf is small in M. Therefore, M is R-H-supplemented. □

In [5], a module M is called a *unique coclosure module* (*UCC module* for short) if every submodule of M has a unique coclosure in M.

Theorem 3.11 *Let M be a UCC module and N a module. Then M is $\$(N,M)$-$H$-supplemented if and only if M is N-lifting.*

Proof Assume that M is $\$(N,M)$-$H$-supplemented. Let $A \in \$(N,M)$. Then there exists a direct summand B of M such that $(A+B)/A$ is small in M/A and $(A+B)/B$ is small in M/B. Hence, the direct summand B of M is the unique coclosure of $A+B$. Also, A is coessential in $A+B$. Since M is a UCC module, by [5, Lemma 3.2], $B \leq A$. Thus, M is N-lifting. The converse statement is clear. □

Proposition 3.12 *Let M be an $\$(N,M)$-$H$-supplemented module where N is a projective module over a right hereditary ring R. Then for a plus invariant small submodule K of M, M/K is $\$(N,M/K)$-$H$-supplemented.*

Proof Let $B/K \in \$(N,M/K)$. Then there exist a submodule X of N and $f \in \mathrm{Hom}_R(X,M/K)$ with $(B/K)/Imf$ small in $(M/K)/Imf$. Since R is right hereditary and N is projective, X is also projective. This implies that there exists $g \in \mathrm{Hom}_R(X,M)$ such that $\pi g = f$ where $\pi\colon M \to M/K$ is the natural projection. Note that $Imf = (Img+K)/K$. $Imf \subseteq B/K$ implies that $Img \subseteq B$. On the other hand, since $(B/K)/Imf$ is small in $(M/K)/Imf$, we conclude that B/Img is a small submodule of M/Img. For if $M/\mathrm{Im}g = B/\mathrm{Im}g + C/\mathrm{Im}g$, then $(M/K)/[(\mathrm{Im}g+K)/K] = (B/K)/[(\mathrm{Im}g+K)/K] + [(C+K)/K]/[(\mathrm{Im}g+K)/K]$. This implies that $(M/K)/[(\mathrm{Im}g+K)/K] = [(C+K)/K]/[(\mathrm{Im}g+K)/K]$. Hence, $M = C+K$, so $M = C$ (note that $K \ll M$). Therefore, $B \in \$(N,M)$. By hypothesis, there exists a direct summand D of M such that $\frac{B+D}{B}$ is small in $\frac{M}{B}$ and $\frac{B+D}{D}$ is small in $\frac{M}{D}$. Let $M = D \oplus H$ for some submodule H of M. Since K is a plus invariant submodule of M, we have $K = (D \cap K) \oplus (H \cap K)$. Thus, $\frac{M}{K} = \frac{D+K}{K} \oplus \frac{H+K}{K}$. Note that $\frac{\frac{B}{K}+\frac{D+K}{K}}{\frac{B}{K}}$ is small in $\frac{\frac{M}{K}}{\frac{B}{K}}$ and $\frac{\frac{B}{K}+\frac{D+K}{K}}{\frac{D+K}{K}}$ is small in $\frac{\frac{M}{K}}{\frac{D+K}{K}}$. Therefore, M/K is $\$(N,M/K)$-$H$-supplemented. □

Proposition 3.13 *Let M be an N-H-supplemented module and N an M-projective module. Then for a plus invariant submodule K of M, M/K is N-H-supplemented.*

Proof Let $f \in \mathrm{Hom}_R(N, M/K)$. Now M-projectivity of N implies that there exists $g \in \mathrm{Hom}_R(N, M)$ such that $\pi g = f$ where $\pi\colon M \to M/K$ is the natural projection. Since M is N-H-supplemented, there exists a direct summand D of M with $\frac{Img+D}{Img} \ll \frac{M}{Img}$ and $\frac{Img+D}{D} \ll \frac{M}{D}$. As in the proof of Proposition 3.12, $\frac{M}{K} = \frac{D+K}{K} \oplus \frac{H+K}{K}$ for some submodule H of M. Also, $\pi g = f$ implies $Imf = \frac{Img+K}{K}$. It follows that $\frac{Imf+\frac{D+K}{K}}{Imf}$ is small in $\frac{\frac{M}{K}}{Imf}$ and $\frac{Imf+\frac{D+K}{K}}{\frac{D+K}{K}}$ is small in $\frac{\frac{M}{K}}{\frac{D+K}{K}}$. Therefore, M/K is N-H-supplemented. □

Proposition 3.14 *Let M be an $\$(N,M)$-$H$-supplemented module with every direct summand plus invariant. Then every direct summand K of M is $\$(N,K)$-$H$-supplemented.*

Proof Let $B \in \$(N,K)$. Then there exist a submodule X of N and $f \in \mathrm{Hom}_R(X,K)$ with $\frac{B}{Imf} \ll \frac{K}{Imf}$. Hence, $\frac{B}{Imf}$ is also small in $\frac{M}{Imf}$, and so $B \in \$(N,M)$. Since M is $\$(N,M)$-$H$-supplemented, there exists a direct summand D of M such that $\frac{B+D}{B}$ is small in $\frac{M}{B}$ and $\frac{B+D}{D}$ is small in $\frac{M}{D}$. Since K is plus invariant, $D \cap K$ is a direct summand of K. In order to see $\frac{B+(D\cap K)}{B} \ll \frac{K}{B}$, let $\frac{B+(D\cap K)}{B} + \frac{A}{B} = \frac{K}{B}$ for some submodule A of K. Then $A + (D \cap K) = K$. Assume that $M = K \oplus L$ where L is a submodule of M. Hence, $M = (A+L)+D$, and so $\frac{M}{B} = \frac{A+L}{B} + \frac{B+D}{B}$. The smallness of $\frac{B+D}{B}$ in $\frac{M}{B}$ implies that $\frac{A+L}{B} = \frac{M}{B}$. Thus, $M = A + L$, so $A = K$ as asserted. We now show that $\frac{B+(D\cap K)}{D\cap K} \ll \frac{K}{D\cap K}$. Let $\frac{B+(D\cap K)}{D\cap K} + \frac{C}{D\cap K} = \frac{K}{D\cap K}$ for some submodule C of K. Then $K = B + C$ and so $M = B + C + L$. The submodule $\frac{B+D}{D}$ being small in $\frac{M}{D}$ implies $M = C + L + D$. By hypothesis on direct summands of M, we have $D = (D\cap K) \oplus (D \cap L)$. Then $M = C + L + (D \cap K)$. By modularity and $D \cap K \subseteq C$, we have $K = C$. This completes the proof. □

Proposition 3.15 *Let M be an N-H-supplemented module with every direct summand plus invariant. Then every direct summand K of M is N-H-supplemented.*

Proof Let $f \in \mathrm{Hom}_R(N,K)$. Then $f \in \mathrm{Hom}_R(N,M)$. Hence, there exists a direct summand D of M such that $\frac{Imf+D}{Imf}$ is small in $\frac{M}{Imf}$ and $\frac{Imf+D}{D}$ is small in $\frac{M}{D}$. The submodule K being plus invariant implies that $D \cap K$ is a direct summand of K. Since a submodule of a small submodule is also small, $\frac{Imf+(D\cap K)}{Imf} \ll \frac{M}{Imf} = \frac{K}{Imf} \oplus \frac{L+Imf}{Imf}$ where $M = K \oplus L$ for some submodule L of M. It follows that $\frac{Imf+(D\cap K)}{Imf} \ll \frac{K}{Imf}$. Also, $\frac{Imf+(D\cap K)}{D\cap K} \ll \frac{K}{D\cap K}$ with a similar discussion as in the proof of Proposition 3.14. □

Proposition 3.16 *Let M be an $\$(N,M)$-$H$-supplemented module and $K \in \$(N,M)$ a plus invariant submodule with $\overline{Z}(K) = K$. Then K is a direct summand of M.*

Proof Since $K \in \$(N,M)$ and M is $\$(N,M)$-$H$-supplemented, there exists a direct summand D of M such that $(K+D)/D$ is small in M/D and $(K+D)/K$ is small in M/K. Let $M = D \oplus D'$ for some submodule D' of M. By hypothesis, $K = (K\cap D) \oplus (K \cap D')$. Then $M/K = (K+D)/K \oplus (K+D')/K$. As $(K+D)/K$ is a small submodule and direct summand, it must be the zero submodule of M/K. Hence, $K = K + D$ and so $D \subseteq K$. On the other hand, $(K+D)/D = K/D$ is a small homomorphic image of K. The hypothesis $\overline{Z}(K) = K$ implies $K \subseteq D$ and so $K = D$. Therefore, K is a direct summand of M. □

In the following, we study amply supplemented modules related to the family $\$(N,M)$.

Definition 3.17 *Let M and N be modules. The module M is called $\$(N,M)$-amply supplemented if whenever $A \in \$(N,M)$ and $M = A + B$, then B contains a supplement $C \in \$(N,M)$ of A.*

Proposition 3.18 *Let M and N be modules and M be $\$(N,M)$-amply supplemented. Then every $A \in \$(N,M)$ of M has an s-closure C in M with $C \in \$(N,M)$.*

Proof Let $A \in \$(N,M)$. Then $M = M + A$ implies that A has a supplement $B \in \$(N,M)$ in M. Similarly, there exists a supplement C of B in M such that $C \in \$(N,M)$ and $C \subseteq A$. Hence, $M = B + C$. Since C is a supplement submodule, it is coclosed. We now prove that A/C is small in M/C. Let $M/C = (A/C) + (D/C)$ for some submodule D of M with $C \subseteq D$. Then $M = A + D$. $A = C + (A \cap B)$ implies $M = C + (A \cap B) + D$. Since $A \cap B$ is small in B, it is also small in M. This yields $M = C + D$. Hence, $M = D$, as asserted. Thus, C is an s-closure of A in M. □

Proposition 3.19 *Let M and N be modules and M be $\$(N,M)$-amply supplemented. If $\$(N,M)$ is closed under finite sums, then the following hold.*

(1) For any $U \in \$(N,M)$, every supplement V of U is $\$(N,V)$-amply supplemented.

(2) For any $U \in \$(N,M)$, there exist submodules X and Y of U such that $U = X + Y$ and X is $\$(N,X)$-amply supplemented and Y is small in M.

(3) For any $U \in \$(N,M)$, there exists a submodule X of U such that $\frac{U}{X} \ll \frac{M}{X}$ and X is $\$(N,X)$-amply supplemented.

Proof (1) Let $U \in \$(N,M)$ and V be a supplement of U in M. Then $M = U + V$. Assume that $V = X + Y$ for some submodules X and Y of V with $X \in \$(N,V)$. Hence, $M = U + X + Y$. Note that $X \in \$(N,V)$ implies $X \in \$(N,M)$. Since $\$(N,M)$ is closed under finite sums, $U + X \in \$(N,M)$. The module M being $\$(N,M)$-amply supplemented implies that Y contains a supplement $Z \in \$(N,M)$ of $U + X$. Thus, $X \cap Z \leq (U + X) \cap Z \ll Z$. On the other hand, $M = U + X + Z$, and so $V = X + Z$ by modularity condition and the smallness of $U \cap V$ in V. Therefore, $Z \in \$(N,M)$ is a supplement of X in V. To complete the proof we need to show $Z \in \$(N,V)$. Since $Z \in \$(N,M)$, there exist a submodule K of N and $f \in \mathrm{Hom}_R(K,M)$ such that $\frac{Z}{f(K)} \ll \frac{M}{f(K)}$. It is enough to show $\frac{Z}{f(K)} \ll \frac{V}{f(K)}$. Let T be a submodule of V with $\frac{Z}{f(K)} + \frac{T}{f(K)} = \frac{V}{f(K)}$. As $M = U + V$, we have $\frac{Z}{f(K)} + \frac{T}{f(K)} + \frac{U+f(K)}{f(K)} = \frac{M}{f(K)}$. The smallness of $\frac{Z}{f(K)}$ in $\frac{M}{f(K)}$ implies $\frac{T}{f(K)} + \frac{U+f(K)}{f(K)} = \frac{M}{f(K)}$, and this yields $T + U = M$, but V is minimal with this property. Hence, $T = V$, as asserted. Therefore, $Z \in \$(N,V)$.

(2) Let $U \in \$(N,M)$ and $V \in \$(N,M)$ be a supplement of U in M and $X \in \$(N,M)$ a supplement of V with $X \subseteq U$. Then $M = U + V = X + V$ and $U \cap V$ is small in V, and it is also small in M. Hence, $U = X + (U \cap V)$ by modularity condition. On the other hand, X is $\$(N,X)$-amply supplemented by (1).

(3) Let $U \in \$(N,M)$. By (2), U is of the form $U = X + Y$ with X an $\$(N,X)$-amply supplemented module

and $Y \ll M$. To show that U/X is a small submodule of M/X, let $U/X + T/X = M/X$ with $X \leq T \leq M$. Then $U + T = M$ and since $U = X + Y$, we get $M = U + T = X + Y + T = Y + T$. $Y \ll M$ implies $T = M$ and therefore $U/X \ll M/X$. □

Corollary 3.20 *Let M and N be modules and M be $\$(N, M)$-amply supplemented. If $\$(N, M)$ is closed under finite sums, then any coclosed submodule $U \in \$(N, M)$ of M is $\$(N, U)$-amply supplemented.*

Proof Let $U \in \$(N, M)$ be a coclosed submodule of M. By Proposition 3.19(3), there exists a submodule X of U such that $\frac{U}{X} \ll \frac{M}{X}$ and X is $\$(N, X)$-amply supplemented. The smallness of U/X in M/X means that X is a coessential submodule of U. Thus, $U = X$. This completes the proof. □

We say that a module M has the coclosed sum property (CCSP for short) if the sum of two coclosed submodules of M is coclosed in M. It is known that every supplement submodule is coclosed. If M is a weakly supplemented module and has the CCSP, then the sum of two supplement submodules is again a supplement.

Theorem 3.21 *Let $0 \to A \to B \to C \to 0$ be an exact sequence of modules. If M is $\$(B, M)$-$H$-supplemented, then it is $\$(A, M)$-$H$-supplemented and $\$(C, M)$-$H$-supplemented. The converse holds if M satisfies the following conditions:*

1. *M is $\$(B, M)$-amply supplemented.*
2. *$\$(B, M)$ is closed under finite sums.*
3. *M has the CCSP.*

Proof Assume that M is $\$(B, M)$-$H$-supplemented. Let $X \in \$(A, M)$. There exist $X' \subseteq A$ and $f \in \mathrm{Hom}_R(X', M)$ such that $X/f(X')$ is small in $M/f(X')$. Thus, $X \in \$(B, M)$. Then there exists a direct summand Y of M such that $M = X + K$ if and only if $M = Y + K$ for each $K \leq M$. Hence, M is $\$(A, M)$-$H$-supplemented. Now let $X \in \$(C, M)$. There exist $X' \subseteq C$ and $f \in \mathrm{Hom}_R(X', M)$ such that $X/f(X')$ is small in $M/f(X')$. Let π denote the natural homomorphism from B onto C with kernel A and $U = \pi^{-1}(X')$ and $\pi_{|U}$ the restriction of π on U. Then $f\pi_{|U} \in \mathrm{Hom}_R(U, M)$ and $X/Im(f\pi_{|U})$ is small in $M/Im(f\pi_{|U})$, so $X \in \$(B, M)$. By assumption, there exists a direct summand Y of M such that $M = X + K$ if and only if $M = Y + K$ for each $K \leq M$. Thus, M is $\$(C, M)$-$H$-supplemented.

Conversely, assume that M has conditions (1), (2), and (3). Let $X \in \$(B, M)$. By Proposition 3.18, X has an s-closure $\overline{X} \in \$(B, M)$. Then there exist $Z \leq B$ and $g \in \mathrm{Hom}_R(Z, M)$ such that $\overline{X}/g(Z)$ is small in $M/g(Z)$. Since $\overline{X}$ is coclosed in M, $\overline{X} = g(Z)$. Hence, $X/g(Z)$ is small in $M/g(Z)$. Let $U = Z \cap A$ and consider the homomorphism $g_{|_U} : U \to M$. Then $g(U) \subseteq g(Z) = \overline{X}$. By Corollary 3.20, $\overline{X}$ is $\$(B, \overline{X})$-amply supplemented, so let $\overline{g(U)}$ be the s-closure of $g(U)$ in $\overline{X}$. Hence, $g(U)/\overline{g(U)}$ is small in $M/\overline{g(U)}$ and $\overline{g(U)} \in \$(A, M)$. The module M being $\$(A, M)$-$H$-supplemented implies that there exists a direct summand D of M such that $(\overline{g(U)} + D)/\overline{g(U)}$ is small in $M/\overline{g(U)}$ and $(\overline{g(U)} + D)/D$ is small in M/D. It is easily seen that $\overline{g(U)}$ is coclosed in M and D is coclosed in M as a direct summand, and by the CCSP, $\overline{g(U)} + D$ is also coclosed in M. Thus, $\overline{g(U)} + D = \overline{g(U)} = D$. Write $M = \overline{g(U)} \oplus D'$. It follows that $\overline{X} = \overline{g(U)} \oplus (\overline{X} \cap D')$ by modularity condition. Note that $\overline{g(U)}$ being the s-closure of $g(U)$ in $\overline{X}$ implies that $g(U)/\overline{g(U)}$ is small in

$\overline{X}/\overline{g(U)}$. Thus, $g(U) = \overline{g(U)}$ since $\overline{X}/\overline{g(U)} \cong (\overline{X} \cap D')$ coclosed in $\overline{X}$. Let $H = (Z + A)/A$ and consider the following mappings to define a homomorphism $h : H \to M$:

$$(Z + A)/A \cong Z/Z \cap A \to g(Z)/g(U) = \overline{X}/g(U) \cong \overline{X} \cap D' \leq M.$$

Let $z + A \in (Z + A)/A$. Then $g(z) = a + b$ where $a \in g(U)$, $b \in \overline{X} \cap D'$. Define $h(z + A) = b$.

We show that h is well defined. Let $z + A = z_1 + A \in (Z + A)/A$. Then $z - z_1 \in Z \cap A$. Write $g(z) = a + b$, $g(z_1) = a_1 + b_1$ with $a, a_1 \in g(U)$, $b, b_1 \in \overline{X} \cap D'$. Then $h(z + A) = b$, $h(z_1 + A) = b_1$. We show $b = b_1$. Since $Z \cap A = U$ and $z - z_1 \in U$, $g(z - z_1) = a - a_1 + b - b_1 \in g(U)$. Since $a - a_1 \in g(U)$ and $b - b_1 \in \overline{X} \cap D'$ and $g(U) \cap (\overline{X} \cap D') = 0$, $b = b_1$. Thus, h is well defined. Then $h(H) = \overline{X} \cap D'$ as $\overline{X} = g(Z)$, so $(\overline{X} \cap D')/h(H)$ is small in $M/h(H)$. Hence, $\overline{X} \cap D' \in \$(C, M)$. Since M is $\$(C, M)$-$H$-supplemented, there exists a direct summand K of M such that $((\overline{X} \cap D') + K)/K$ is small in M/K and $((\overline{X} \cap D') + K)/(\overline{X} \cap D')$ is small in $M/(\overline{X} \cap D')$. Since $(\overline{X} \cap D') + K$ is coclosed in M by the CCSP, $(\overline{X} \cap D') + K = K = \overline{X} \cap D'$. Hence, $\overline{X} \cap D'$ is a direct summand of M and so is that of D'. It follows that $\overline{X}$ is a direct summand of M. Therefore, M is $\$(B, M)$-$H$-supplemented. □

We end this paper by observing some characterizations of H-supplemented modules.

Theorem 3.22 *The following are equivalent for a module M.*

1. *M is $\$(N, M)$-$H$-supplemented for every free R-module N;*
2. *M is $\$(N, M)$-$H$-supplemented for every projective R-module N;*
3. *M is $\$(N, M)$-$H$-supplemented for every flat R-module N;*
4. *M is H-supplemented.*

Proof (4) ⇒ (3) ⇒ (2) ⇒ (1) Obvious.
(1) ⇒ (4) Let K be a submodule of M. Since K is an epimorphic image of a free module F, we have $K \in \$(F, M)$. By (1), M is $\$(F, M)$-$H$-supplemented. It follows from the definition that $\frac{K+D}{K} \ll \frac{M}{K}$ and $\frac{K+D}{D} \ll \frac{M}{D}$ for some direct summand D of M. Therefore, M is H-supplemented. □

Theorem 3.23 *The following are equivalent for a module M.*

1. *M is N-H-supplemented for every free R-module N;*
2. *M is N-H-supplemented for every projective R-module N;*
3. *M is N-H-supplemented for every flat R-module N;*
4. *M is H-supplemented.*

Proof (4) ⇒ (3) ⇒ (2) ⇒ (1) Obvious.
(1) ⇒ (4) Let K be a submodule of M. There exists an epimorphism $f: F \to K$ where F is a free module. Since M is F-H-supplemented, there exists a direct summand D of M with $\frac{Imf+D}{Imf} \ll \frac{M}{Imf}$ and $\frac{Imf+D}{D} \ll \frac{M}{D}$. $Imf = K$ implies that M is H-supplemented. □

4. Some open problems

(1) Determine rings for which every module is E-H-supplemented.

(2) Let R be a ring. Provide some conditions under which every R-module is E-H-supplemented if and only if every R-module is H-supplemented.

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