Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1970 – 1977
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doi:10.3906/mat-1710-59

On the rank of transformation semigroup $T_{(n,m)}$

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Received: 16.10.2017 • **Accepted/Published Online:** 09.05.2018 • **Final Version:** 24.07.2018

Abstract: Let T_n and S_n be the full transformation semigroup and the symmetric group on $X_n = \{1,\dots,n\}$, respectively. For $n,m \in \mathbb{Z}^+$ with $m \leq n-1$ let

$$T_{(n,m)} = \{\alpha \in T_n : X_m\alpha = X_m\}.$$

In this paper we research generating sets and the rank of $T_{(n,m)}$. In particular, we prove that

$$\operatorname{rank}(T_{(n,m)}) = \begin{cases} 2 & \text{if } (n,m) = (2,1) \text{ or } (3,2) \\ 3 & \text{if } (n,m) = (3,1) \text{ or } 4 \leq n \text{ and } m = n-1 \\ 4 & \text{if } 4 \leq n \text{ and } 1 \leq m \leq n-2. \end{cases}$$

for $1 \leq m \leq n-1$.

Key words: Transformations, permutations, restricted image, generating set, rank

1. Introduction

Let $T(X)$ be the full transformation semigroup on the set X. For a nonempty subset Y of X Symons introduced and studied the subsemigroup $T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$ of $T(X)$ in [8]. In [6] Sanwong and Sommanee proved that the largest regular subsemigroup of $T(X,Y)$ is $F(X,Y) = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$, and, moreover, they researched the rank of $F(X,Y)$ in [7]. For a nonempty subset Y of a finite set X let

$$T_{(X,Y)} = \{\alpha \in T(X) : Y\alpha = Y\}.$$

It is clear that $T_{(X,Y)}$ is a subsemigroup of $T(X)$. In this paper we research the generating sets and the rank of $T_{(X,Y)}$. When X is finite, we take $X = X_n = \{1,\dots,n\}$ and write T_n instead of $T(X_n)$. Let P_n and S_n be the partial transformation semigroup and the symmetric group on X_n, respectively. For $n,m \in \mathbb{Z}^+$ with $m \leq n-1$ let Y be any subset of X_n with $|Y| = m$. If we denote $T_{(X_n,X_m)}$ by $T_{(n,m)}$, that is

$$T_{(n,m)} = \{\alpha \in T_n : X_m\alpha = X_m\},$$

then it is clear that $T_{(X_n,Y)}$ and $T_{(n,m)}$ are isomorphic. Thus, it is enough to consider the subsemigroup $T_{(n,m)}$ of T_n for $1 \leq m \leq n-1$. Observe that if we denote the restriction of any $\alpha \in T_{(n,m)}$ into X_m by $\alpha_{|m} = \alpha_{|X_m}$, then $\alpha_{|m}$ is a permutation of X_m, that is $\alpha_{|m} \in S_m$.

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2010 *AMS Mathematics Subject Classification:* 20M20

Let S be any semigroup, and let A be any nonempty subset of S. Then the subsemigroup generated by A that is the smallest subsemigroup of S containing A is denoted by $\langle A\rangle$. If there exists a finite subset A of a semigroup S with $\langle A\rangle = S$, then S is called a finitely generated semigroup. The *rank* of a finitely generated semigroup S is defined by

$$\text{rank}\,(S) = \min\{\,|A| : \langle A\rangle = S\,\}.$$

The *defect*, *kernel*, *fix*, and *shift* of $\alpha \in T_n$ are defined by

$$\begin{aligned} &\text{defect}\,(\alpha) = n - |\text{im}\,(\alpha)|, && \ker(\alpha) = \{(x,y) \in X_n \times X_n : x\alpha = y\alpha\}, \\ &\text{fix}\,(\alpha) = \{x \in X_n : x\alpha = x\}, \quad \text{and} && \text{shift}\,(\alpha) = \{x \in X_n : x\alpha \neq x\}. \end{aligned}$$

For any $\alpha, \beta \in T_n$ it is well known that $\ker(\alpha) \subseteq \ker(\alpha\beta)$ and $\text{im}\,(\alpha\beta) \subseteq \text{im}\,(\beta)$. Let $\alpha \in T_n$, if for unique $i \in X_n$, $i\alpha = j$ and $k\alpha = k$ for all $k \neq i$, and then we use the notation

$$\alpha = \binom{i}{j}$$

(and so α is an idempotent of defect 1). For $n \geq 3$, it is well known that $\text{rank}\,(S_n) = 2$ and $\text{rank}\,(T_n) = 3$. Moreover,

$$\begin{aligned} S_n &= \langle (1\ 2), (1\ 2\ \cdots\ n)\rangle \text{ and} \\ T_n &= \langle (1\ 2), (1\ 2\ \cdots\ n), \binom{1}{2}\rangle, \end{aligned}$$

where $(1\ 2)$ and $(1\ 2\ \cdots\ n)$ are the transposition and the n-cycle as defined in general, respectively. Let $\text{Sing}_n = T_n \setminus S_n$; it is called singular mappings. Gomes and Howie proved that $\text{rank}\,(\text{Sing}_n) = \frac{n(n-1)}{2}$ [3]. Necessary and sufficient conditions have been found for any set for transformations of defect 1 in Sing_n to be a (minimal) generating set for Sing_n [1]. For $1 \leq r \leq n$, let $K_{n,r} = \{\alpha \in T_n : |\text{im}\,(\alpha)| \leq r\}$. Howie and McFadden proved that $\text{rank}\,(K_{n,r}) = S(n,r)$ for $2 \leq r \leq n-1$ where $S(n,r)$ is the second kind of Stirling number [5]. Let $P_{n,r} = \{\alpha \in P_n : |\text{im}\,(\alpha)| \leq r\}$; Garba proved that $\text{rank}\,(P_{n,r}) = S(n+1, r+1)$ for $2 \leq r \leq n-1$ in [2]. In this paper generating sets and the rank of $T_{(n,m)}$ have been established. We use the same notations as in Howie's book [4].

2. Generating sets

For any $\alpha \in T_{(n,m)}$, if we define $\beta_\alpha, \gamma_\alpha, \lambda_\alpha \in T_n$ as follows:

$$\begin{aligned} i\beta_\alpha &= \begin{cases} i\alpha & i \leq m \\ i & i > m \end{cases}, \\ i\gamma_\alpha &= \begin{cases} i\alpha & i > m \text{ and } i\alpha \in X_m \\ i & \text{other} \end{cases}, \qquad (1) \\ i\lambda_\alpha &= \begin{cases} i\alpha & i\alpha > m \\ i & i\alpha \leq m \end{cases}, \end{aligned}$$

then it is clear that $\beta_\alpha, \gamma_\alpha, \lambda_\alpha \in T_{(n,m)}$. For $1 \leq i \leq m$ it follows from the fact $i\beta_\alpha = i\alpha \in X_m$ that $(i\beta_\alpha)\gamma_\alpha = i\alpha$, and so

$$i(\beta_\alpha\gamma_\alpha\lambda_\alpha) = (i\beta_\alpha)\lambda_\alpha = (i\alpha)\lambda_\alpha = i\alpha,$$

since $(i\alpha)\alpha \leq m$. For $m+1 \leq i \leq n$ we have $i\beta_\alpha = i$. If $i\alpha \in X_m$, then $(i\alpha)\alpha \in X_m$ and $i\gamma_\alpha = i\alpha$, and so

$$i(\beta_\alpha\gamma_\alpha\lambda_\alpha) = (i\gamma_\alpha)\lambda_\alpha = (i\alpha)\lambda_\alpha = i\alpha.$$

If $i\alpha \notin X_m$, then $i\gamma_\alpha = i$ and $i\lambda_\alpha = i\alpha$. Since $i\beta_\alpha = i$, it follows that

$$i(\beta_\alpha\gamma_\alpha\lambda_\alpha) = (i\gamma_\alpha)\lambda_\alpha = i\lambda_\alpha = i\alpha.$$

Therefore, we have the equality $\alpha = \beta_\alpha\gamma_\alpha\lambda_\alpha$.

For an example, if

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 3 & 6 & 1 & 3 & 7 \end{pmatrix} \in T_{(8,4)},$$

then it follows from the definitions that

$$\beta_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 3 & 5 & 6 & 7 & 8 \end{pmatrix}, \quad \gamma_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 1 & 3 & 8 \end{pmatrix},$$

$\lambda_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 6 & 6 & 7 & 7 \end{pmatrix}$, and that $\alpha = \beta_\alpha\gamma_\alpha\lambda_\alpha$.

For any $\alpha \in T_{(n,m)}$, it is clear from the definitions that

$$X_n \setminus X_m \subseteq \text{fix}\,(\beta_\alpha), \quad X_m \subseteq \text{fix}\,(\gamma_\alpha), \quad \text{and} \quad X_m \subseteq \text{fix}\,(\lambda_\alpha).$$

Moreover, for any $i \in X_n \setminus X_m$, we have $i\lambda_\alpha \in X_n \setminus X_m$ and either $i \in \text{fix}\,(\gamma_\alpha)$ or $i\gamma_\alpha \in X_m$. If we define the following sets,

$$U(n,m) = \{\beta_\alpha : \alpha \in T_{(n,m)}\}, \qquad V(n,m) = \{\gamma_\alpha : \alpha \in T_{(n,m)}\}, \quad \text{and}$$
$$W(n,m) = \{\lambda_\alpha : \alpha \in T_{(n,m)}\},$$

then it is clear that $U(n,m)$, $V(n,m)$, and $W(n,m)$ are subsemigroups of $T_{(n,m)}$. Moreover,

$$U(n,m) \cong S_m \text{ and } W(n,m) \cong T_{n-m}.$$

For every $\alpha \in T_{(n,m)}$, since $\gamma_\alpha^2 = \gamma_\alpha$, $V(n,m)$ consists of only idempotents.

Since $T_{(n,1)} \cong P_{n-1}$ and $T_{(n,n)} \cong S_n$, we only consider the case $2 \leq m \leq n-1$. Now we state and prove the following proposition.

Proposition 1 *For $2 \leq m \leq n-2$, if*

$$A_m = \{(1\ 2), (1\ 2\ \cdots\ m), (m+1\ m+2), (m+1\ m+2\ \cdots\ n), \binom{m+1}{m+2}, \binom{m+1}{1}\},$$

then, for all $1 \leq i \leq m$ and for all $m+1 \leq k \leq n$, we have

$$\binom{k}{i} \in \langle A_m \rangle.$$

Proof First of all recall that $(1\ i) \in \langle (1\ 2), (1\ 2\ \cdots\ m) \rangle \cong S_m$ for every $2 \leq i \leq m$, and similarly, $(m+1\ k) \in \langle (m+1\ m+2), (m+1\ m+2\ \cdots\ n) \rangle \cong S_{n-m}$ for every $m+2 \leq k \leq n$. Since

$$\binom{k}{1} = (m+1\ k)\binom{m+1}{1}(m+1\ k)$$

for every $m+2 \leq k \leq n$, it follows that $\binom{k}{1} \in \langle A_m \rangle$ for all $m+1 \leq k \leq n$. Therefore, for all $2 \leq i \leq m$ and for all $m+1 \leq k \leq n$,

$$\binom{k}{i} = (1\ i)\binom{k}{1}(1\ i) \in \langle A_m \rangle,$$

as required. □

Lemma 2 *For* $2 \leq m \leq n-2$,

$$A_m = \{(1\ 2), (1\ 2\ \cdots\ m), (m+1\ m+2), (m+1\ m+2\ \cdots\ n), \binom{m+1}{m+2}, \binom{m+1}{1}\}$$

is a generating set of $T_{(n,m)}$. *Moreover,*

$$A_{n-1} = \{(1\ 2), (1\ 2\ \cdots\ n-1), \binom{n}{1}\}$$

is a generating set of $T_{(n,n-1)}$.

Proof For $2 \leq m \leq n-2$, let $\alpha \in T_{(n,m)}$. Suppose that β_α, γ_α, and λ_α are defined as in Equation (1) so that $\alpha = \beta_\alpha \gamma_\alpha \lambda_\alpha$. First of all it is clear that

$$\beta_\alpha \in \langle (1\ 2), (1\ 2\ \cdots\ m) \rangle \cong S_m$$

and that

$$\lambda_\alpha \in \langle (m+1\ m+2), (m+1\ m+2\ \cdots\ n), \binom{m+1}{m+2} \rangle \cong T_{n-m}.$$

Since $\alpha = \beta_\alpha \gamma_\alpha \lambda_\alpha$, it is enough to show that $\gamma_\alpha \in \langle A_m \rangle$.

Suppose that $S = \text{shift}(\gamma_\alpha) = \{x_1, \ldots, x_r\} \neq \emptyset$ (otherwise $\alpha = \beta_\alpha \lambda_\alpha$, and so there is nothing to show). Since $S \subseteq (X_n \setminus X_m)$ and $S\gamma_\alpha \subseteq X_m$, it follows that, for each $1 \leq i \leq r$, there exists $y_i \in X_m$ such that $x_i \gamma_\alpha = y_i$. Then it is clear that

$$\gamma_\alpha = \binom{x_1}{y_1}\binom{x_2}{y_2}\cdots\binom{x_r}{y_r}.$$

Thus, it follows from Proposition 1 that $\gamma_\alpha \in \langle A_m \rangle$. Therefore, $T_{(n,m)} = \langle A_m \rangle$ for $2 \leq m \leq n-2$.

If $\alpha \in T_{(n,n-1)}$, then it is clear that λ_α is the identity of T_n, and moreover, γ_α is either the identity or an idempotent $\gamma_\alpha = \binom{n}{i}$ for any $1 \leq i \leq n-1$. In other words either $\alpha = \beta_\alpha$ or $\alpha = \beta_\alpha \binom{n}{i}$ for any $1 \leq i \leq n-1$. Now since

$$\binom{n}{i} = (1\ i)\binom{n}{1}(1\ i) \in \langle A_{n-1} \rangle$$

for all $2 \leq i \leq n-1$, it follows that $T_{(n,n-1)} = \langle A_{n-1} \rangle$. □

3. Rank of $T_{(n,m)}$

For $2 \leq m \leq n-1$, let

$$\begin{aligned} G(n,m) &= \{\alpha \in T_{(n,m)} : (X_n \setminus X_m)\alpha = X_n \setminus X_m\} = T_{(n,m)} \cap S_n \quad \text{and} \\ H(n,m) &= \{\alpha \in T_{(n,m)} : (X_n \setminus X_m)\alpha \subseteq X_n \setminus X_m\}. \end{aligned}$$

It is clear that both $G(n,m)$ and $H(n,m)$ are subsemigroups of $T_{(n,m)}$,

$$G(n,m) \leq H(n,m) \text{ and } G(n,m) \cong S_m \times S_{n-m} \quad (\text{for } 2 \leq m \leq n-2).$$

Moreover, it is also clear that $G(n,n-1) = H(n,n-1) \cong S_{n-1}$. Thus, it follows that

$$G(n,m) = \langle (1\ 2), (1\ 2\ \cdots\ m), (m+1\ m+2), (m+1\ m+2\ \cdots\ n) \rangle \tag{2}$$

for all $2 \leq m \leq n-2$. Since $(2\ 3\ \cdots\ m)(1\ 2) = (1\ 2\ \cdots\ m)$ for all $3 \leq m \leq n-1$, we also have that $S_n = \langle (1\ 2), (2\ 3\ \cdots\ n) \rangle$, and so

$$\begin{aligned} G(n,m) &= \langle\ (1\ 2), (2\ 3\ \cdots\ m), (m+1\ m+2), (m+1\ m+2\ \cdots\ n)\ \rangle & (3) \\ &= \langle\ (1\ 2), (1\ 2\ \cdots\ m), (m+1\ m+2), (m+2\ m+3\ \cdots\ n)\ \rangle & (4) \\ &= \langle\ (1\ 2), (2\ 3\ \cdots\ m), (m+1\ m+2), (m+2\ m+3\ \cdots\ n)\ \rangle. & (5) \end{aligned}$$

Lemma 3 *For* $1 \leq m \leq n-1$,

$$\operatorname{rank}(G(n,m)) = \begin{cases} 1 & \textit{if } (n,m) = (2,1),\ (3,1) \textit{ or } (3,2) \\ 2 & \textit{otherwise.} \end{cases}$$

Proof Since $G(n,n-1) \cong G(n,1) \cong S_{n-1}$, it follows that $\operatorname{rank}(G(n,1)) = 1$ for $n = 2,3$, $\operatorname{rank}(G(3,2)) = 1$, and that $\operatorname{rank}(G(n,1)) = \operatorname{rank}(G(n,n-1)) = 2$ for $n \geq 4$.

First notice that if $(n,m) \notin \{(2,1),(3,1),(3,2)\}$, then $G(n,m)$ is not a cyclic group, and so $\operatorname{rank}(G(n,m)) \geq 2$. If $m = 2$ and $n = 4$, then $G(4,2) \cong S_2 \times S_2$ (the Klein-4 group), and so $\operatorname{rank}(G(4,2)) = 2$ ($G(4,2) = \langle (1\ 2), (3\ 4) \rangle$).

Before we consider the other cases, we consider the following cycles in S_n:

$$\begin{aligned} \alpha_1 &= (1\ 2), & \alpha_2 &= (m+1\ m+2), \\ \beta_1 &= (1\ 2\ \cdots\ m), & \beta_2 &= (m+1\ m+2\ \cdots\ n), \\ \gamma_1 &= (2\ 3\ \cdots\ m), & \gamma_2 &= (m+2\ m+3\ \cdots\ n), \end{aligned}$$

and moreover, the permutations S_n:

$$\delta = \alpha_1\beta_2, \quad \eta = \alpha_1\gamma_2, \quad \varepsilon = \alpha_2\beta_1, \quad \text{and} \quad \zeta = \alpha_2\gamma_1.$$

Suppose that $m = 2$ and $n \geq 5$. If $n-2$ is an odd number, then since

$$\delta^{n-2} = \alpha_1 \quad \text{and} \quad \delta^{n-1} = \beta_2,$$

it follows from Eq. (2) that $G(n,2) = \langle \delta, \alpha_2 \rangle$. If $n-2$ is an even number, then since

$$\eta^{n-3} = \alpha_1 \quad \text{and} \quad \eta^{n-2} = \gamma_2,$$

it follows from Eq. (4) that $G(n,2)=\langle\alpha_2,\eta\rangle$.

Suppose that $n-m=2$ and $m\geq 3$. If m is an odd number, then since

$$\varepsilon^m=\alpha_2 \quad \text{and} \quad \varepsilon^{m+1}=\beta_1,$$

it follows from Eq. (2) that $G(n,m)=\langle\alpha_1,\varepsilon\rangle$. If m is an even number, then since

$$\zeta^{m-1}=\alpha_2 \quad \text{and} \quad \zeta^m=\gamma_1,$$

it follows from Eq. (3) that $G(n,m)=\langle\alpha_1,\zeta\rangle$.

Finally, suppose that $m\geq 3$ and $n-m\geq 3$. If both m and $n-m$ are odd numbers, then since

$$\delta^{n-m}=\alpha_1, \quad \delta^{n-m+1}=\beta_2, \quad \varepsilon^m=\alpha_2, \quad \text{and} \quad \varepsilon^{m+1}=\beta_1,$$

it follows from Eq. (2) that $G(n,m)=\langle\delta,\varepsilon\rangle$. If m is an even number and if $n-m$ is an odd number, then since

$$\delta^{n-m}=\alpha_1, \quad \delta^{n-m+1}=\beta_2, \quad \zeta^{m-1}=\alpha_2, \quad \text{and} \quad \zeta^m=\gamma_1,$$

it follows from Eq. (3) that $G(n,m)=\langle\delta,\zeta\rangle$. If m is an odd number and if $n-m$ is an even number, then since

$$\eta^{n-m-1}=\alpha_1, \quad \eta^{n-m}=\gamma_2, \quad \varepsilon^m=\alpha_2, \quad \text{and} \quad \varepsilon^{m+1}=\beta_1,$$

it follows from Eq. (4) that $G(n,m)=\langle\varepsilon,\eta\rangle$. If both m and $n-m$ are even numbers, then since

$$\eta^{n-m-1}=\alpha_1, \quad \eta^{n-m}=\gamma_2, \quad \zeta^{m-1}=\alpha_2, \quad \text{and} \quad \zeta^m=\gamma_1,$$

it follows from Eq. (5) that $G(n,m)=\langle\zeta,\eta\rangle$. □

Therefore, in the above theorem, we have found the rank of the direct product of two symmetric groups.

Lemma 4 *(i)* $T_{(n,m)}\setminus G(n,m)$ *is an ideal of* $T_{(n,m)}$ *for* $1\leq m\leq n-1$.

(ii) $H(n,m)\setminus G(n,m)$ *is an ideal of* $H(n,m)$ *for* $1\leq m\leq n-2$.

(iii) $T_{(n,m)}\setminus H(n,m)$ *is a subsemigroup of* $T_{(n,m)}$ *for* $1\leq m\leq n-1$.

Proof (i) Let $\alpha\in T_{(n,m)}$ and $\beta\in T_{(n,m)}\setminus G(n,m)$. First notice that $|\mathrm{im}\,(\beta)|\leq n-1$ and that all elements in $G(n,m)$ are permutations. Since $\mathrm{im}\,(\alpha\beta)\subseteq\mathrm{im}\,(\beta)$, it follows that $|\mathrm{im}\,(\alpha\beta)|\leq|\mathrm{im}\,(\beta)|\leq n-1$. Since $\ker(\beta\alpha)\supseteq\ker(\beta)$, it follows that $|\mathrm{im}\,(\beta\alpha)|\leq|\mathrm{im}\,(\beta)|\leq n-1$. Therefore, both $\alpha\beta$ and $\beta\alpha$ are elements of $T_{(n,m)}\setminus G(n,m)$, as required.

(ii) Similar to the proof of (i).

(iii) Let $\alpha,\beta\in T_{(n,m)}\setminus H(n,m)$. Then there exists at least one i such that $m+1\leq i\leq n$ and $i\alpha\in X_m$. Thus, we have $i(\alpha\beta)=(i\alpha)\beta\in X_m$, and so $\alpha\beta\in T_{(n,m)}\setminus H(n,m)$, as required. □

Proposition 5 *Let* $\alpha\in G(n,m)$ *and* $\beta\in T_{(n,m)}\setminus H(n,m)$. *Then both* $\alpha\beta$ *and* $\beta\alpha$ *are elements of* $T_{(n,m)}\setminus H(n,m)$.

Proof Let $\alpha \in G(n,m)$ and $\beta \in T_{(n,m)} \setminus H(n,m)$. Similarly, there exists at least one i such that $m+1 \leq i \leq n$ and $i(\beta\alpha) = (i\beta)\alpha \in X_m$. Moreover, for the same i, there exists only one j such that $m+1 \leq j \leq n$ and $j\alpha = i$. Therefore, $j(\alpha\beta) = i\beta \in X_m$, as required. □

Thus, for every generating set A of $T_{(n,m)}$, it follows from Lemma 4 (i) that $A \cap G(n,m)$ is a generating set of $G(n,m)$ $(1 \leq m \leq n-1)$. Therefore, A must include at least 2 elements from $G(n,m)$ when $(n,m) \notin \{(2,1),(3,1),(3,2)\}$ and at least 1 element from $G(n,m)$ when $(n,m) \in \{(2,1),(3,1),(3,2)\}$. Similarly, for every generating set B of $H(n,m)$, it follows from Lemma 4 (ii) that $B \cap G(n,m)$ is a generating set of $G(n,m)$ $(1 \leq m \leq n-2)$.

Theorem 6 *For* $1 \leq m \leq n-1$,

$$\operatorname{rank}(T_{(n,m)}) = \begin{cases} 2 & \text{if } (n,m) = (2,1) \text{ or } (3,2) \\ 3 & \text{if } (n,m) = (3,1) \text{ or } 4 \leq n \text{ and } m = n-1 \\ 4 & \text{if } 4 \leq n \text{ and } 1 \leq m \leq n-2. \end{cases}$$

Proof For $(n,m) = (2,1)$ since $T_{(2,1)} \cong P_1$ and $\operatorname{rank}(P_1) = 2$, it follows that $\operatorname{rank}(T_{(2,1)}) = 2$. For $(n,m) = (3,2)$, since $\operatorname{rank}(G(3,2)) = 1$, it follows from Lemma 4 (i) that $\operatorname{rank}(T_{(3,2)}) \geq 2$. It is easy to show that $\{(1\ 2), \binom{3}{1}\}$ is a generating set of $T_{(3,2)}$, and so $\operatorname{rank}(T_{(3,2)}) = 2$. For $(n,m) = (3,1)$, since $T_{(3,1)} \cong P_2$, it follows from the well-known fact $\operatorname{rank}(P_2) = 3$ that $\operatorname{rank}(T_{(3,1)}) = 3$.

Suppose that $4 \leq n$ and $m = n-1$. Then it follows from Lemma 2 that $\{(1\ 2), (1\ 2\ \cdots\ n-1), \binom{n}{1}\}$ is a generating set of $T_{(n,n-1)}$. From Lemmas 3 and 4 (i) since we have $\operatorname{rank}(T_{(n,n-1)}) \geq 3$, it follows that $\operatorname{rank}(T_{(n,n-1)}) = 3$ for $4 \leq n$ and $m = n-1$.

Suppose that $4 \leq n$ and $1 \leq m \leq n-2$. Let A be any generating set for $T_{(n,m)}$. From Lemmas 3 and 4 (i) notice that $|A \cap G(n,m)| \geq 2$. Since $G(n,m) \neq T_{(n,m)}$ we have $\operatorname{rank}(T_{(n,m)}) \geq 3$.

Now we show that $\operatorname{rank}(T_{(n,m)}) \geq 4$. Assume that $\operatorname{rank}(T_{(n,m)}) = 3$. Then it follows from Lemma 4 (i) that there exists only one element α in $B = A \setminus G(n,m)$. If $\alpha \in H(n,m)$, then it follows from Lemma 4 (ii) that $\langle A \rangle$ must be a subsemigroup of $H(n,m)$, which is a contradiction since $H(n,m) \neq T_{(n,m)}$. If $\alpha \in T_{(n,m)} \setminus H(n,m)$, then similarly it follows from Proposition 5 that

$$\langle A \rangle \text{ is a subsemigroup of } \Big(G(n,m) \cup \big(T_{(n,m)} \setminus H(n,m)\big)\Big) \neq T_{(n,m)},$$

which is again a contradiction. Therefore, we must have $\operatorname{rank}(T_{(n,m)}) \geq 4$. For $4 \leq n$ and $1 \leq m \leq n-2$ the result follows from Lemmas 2, 3, and 4 (i) that $\operatorname{rank}(T_{(n,m)}) = 4$, as required. □

Note that for any generating set $\{\alpha, \beta\}$ of $G(n,m)$, we have just proved that the set

$$\{\alpha, \beta, \binom{m+1}{m+2}, \binom{m+1}{1}\}$$

is a minimal generating set of $T_{(n,m)}$ for $4 \leq n$ and $1 \leq m \leq n-2$. Also notice that $\binom{m+1}{m+2} \in H(n,m) \setminus G(n,m)$ and that $\binom{m+1}{1} \in T_{(n,m)} \setminus H(n,m)$. One can easily prove that if $\gamma \in H(n,m) \setminus G(n,m)$ and $\delta \in T_{(n,m)} \setminus H(n,m)$ are any two idempotents of the same defect 1, then $\{\alpha, \beta, \gamma, \delta\}$ is a minimal generating set of $T_{(n,m)}$ for $4 \leq n$ and $1 \leq m \leq n-2$.

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