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On the exponential Diophantine equation $(18m^2+1)^x+(7m^2-1)^y=(5m)^z$

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Abstract: Let m be a positive integer. We show that the exponential Diophantine equation $(18m^2+1)^x+(7m^2-1)^y=(5m)^z$ has only the positive integer solution $(x,y,z)=(1,1,2)$ except for $m\equiv 23,47,63,87 \pmod{120}$. For $m\not\equiv 0 \pmod 5$ we use some elementary methods and linear forms in two logarithms. For $m\equiv 0 \pmod 5$ we apply a result for linear forms in p-adic logarithms.

Key words: Exponential Diophantine equations, linear forms in the logarithms

1. Introduction

Let a,b,c be relatively prime fixed positive integers greater than one. The exponential Diophantine equation

$$a^x+b^y=c^z \tag{1}$$

in positive integers x,y,z has been studied by a number of authors. Although it is known that the number of solutions (x,y,z) of the equation (1) is finite [9], some conjectures are still completely unproved related to the question of whether or not there exist any other solution (x,y,z) of the equation (1) whenever (x_0,y_0,z_0) is a solution. One of them is a conjecture on Pythagorean triples, i.e. positive integers a,b,c satisfying $a^2+b^2=c^2$. In 1956, Sierpinski proved that $(x,y,z)=(2,2,2)$ is the only positive integer solution of equation $3^x+4^y=5^z$ [13]. The same year, Jeśmanowicz conjectured that if a,b,c are Pythagorean triples the equation has only the positive integer solution $(x,y,z)=(2,2,2)$ [5]. There exist many positive results for Jeśmanowicz's conjecture; see for example [8, 11, 12, 17]. A similar conjecture is proposed by Terai that states that if a,b,c,p,q,r are fixed positive integers satisfying $a^p+b^q=c^r$ with $a,b,c,p,q,r\geq 2$ and $gcd(a,b)=1$ then equation (1) has only the positive integer solution $(x,y,z)=(p,q,r)$ except for a handful of triples (a,b,c) [14, 15]. Exceptional cases are listed explicitly in [19]. Although this conjecture is proved to be true in many special cases, for example [3, 4, 10, 16, 18, 20], it remains an unsolved problem. In this paper we consider the exponential Diophantine equation

$$(18m^2+1)^x+(7m^2-1)^y=(5m)^z \tag{2}$$

where m is a positive integer, and we prove the following.

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Theorem 1 *Let m be a positive integer with $m \not\equiv 23, 47, 63, 87 \pmod{120}$. Then equation (2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

The proof of the above theorem consists of the following main steps. At first we use some elementary methods such as congruences and properties of the Jacobi symbol to reduce the solution to the case $y = 1$ when $m \not\equiv 0 \pmod{5}$, and then apply a lower bound for linear forms in two logarithms due to Laurent [7]. For the case $m \equiv 0 \pmod{5}$, we use a result on linear forms in p-adic logarithm due to Bugeaud [2].

2. Preliminaries

For any nonzero algebraic numbers α of degree d over Q, the absolute logarithmic height of α is defined by the formula

$$h(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log \max\{1, |\alpha^{(i)}|\}\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over Z, and the $\alpha^{(i)}s$ are the conjugates of α.

Let α_1, α_2 be two real algebraic numbers with $|\alpha_1|$, $|\alpha_2| \geq 1$ and b_1, b_2 be positive integers. Consider the linear form

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Put $D = [Q(\alpha_1, \alpha_2) : Q]$. We set

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1},$$

where A_1 and A_2 are real numbers greater than 1 such that

$$\log A_i \geq \max\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\} \ (i = 1, 2).$$

The following proposition is a particular version of [7, Corollary 2] by choosing $m = 10$ and $C_2 = 25.2$ from Table 1 [7, page 328].

Proposition 2 *[7, Corollary 2] Let $\Lambda, \alpha_i, D, A_i$, and b' be as above with $\alpha_i > 1$ for $i \in 1, 2$. Suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log|\Lambda| \geq -25.2D^4 \left(\max\{\log b' + 0.38, \frac{10}{D}, 1\}\right)^2 \log A_1 \log A_2.$$

Next we cite a result from [2]. Here we just use a special case $y_1 = y_2 = 1$ of [2, Theorem 2]. Before stating it, we recall some notations. Let p be an odd prime and v_p denote the p-adic valuation normalized by $v_p(p) = 1$. Let a_1, a_2 be two nonzero integers and g denote the smallest positive integer such that

$$v_p(a_1^g - 1) > 0, \quad v_p(a_2^g - 1) > 0.$$

Assume that there exists a real number E such that

$$v_p(a_1^g - 1) \geq E > \frac{1}{p-1}.$$

The following theorem gives an explicit upper bound for the p-adic valuation of

$$\Lambda = a_1^{b_1} - a_2^{b_2},$$

where b_1 and b_2 are positive integers.

Proposition 3 *[2, Theorem 2] Let $A_1 > 1, A_2 > 1$ be real numbers such that*

$$\log A_i \geq \max\{\log|a_i|, E\log p\}, \quad i = 1, 2$$

and put

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

If a_1 and a_2 are multiplicatively independent then we have the upper estimates

$$v_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4}\left(\max\{\log b' + \log(E\log p) + 0.4, 6E\log p, 5\}\right)^2 \log A_1 \log A_2.$$

3. Proof of Theorem 1

The proof of the theorem follows in a series of lemmas.

Lemma 4 *If $m = 1$ or $m = 2$ then equation (2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

Proof If $m = 1$ then we have the following equation:

$$19^x + 6^y = 5^z.$$

Taking this equation modulo 5 implies that $(-1)^x + 1 \equiv 0 \pmod 5$, and so x is odd. Taking the equation modulo 4 implies that $3 + 2^y \equiv 1 \pmod 4$, and so $y = 1$. Then the equation

$$5^z - 19^x = 6$$

has only the positive integer solution $z = 2,\ x = 1$ [1, Theorem 1.5]. For $m = 2$ the equation

$$73^x + 27^y = 10^z$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$ since the congruence $1 + 3^y \equiv 2^z \pmod 8$ implies that $z = 1$ or $z = 2$. □

Lemma 5 *If (x, y, z) is a positive integer solution of equation (2) then y is odd.*

Proof By Lemma 4 we may assume that $m \geq 3$. Thus taking (2) modulo m implies that $1 + (-1)^y \equiv 0 \pmod m$ and hence y is odd. □

Lemma 6 *If m is even then equation (2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

Proof Clearly $(x,y,z)=(1,1,2)$ is the only positive integer solution of (2) for $z\leq 2$. Assume that $z\geq 3$. By taking (2) modulo m^3 implies that

$$1+18m^2x-1+7m^2y\equiv 0 \pmod{m^3},$$

and so

$$18x+7y\equiv 0 \pmod{m},$$

which is a contradiction, since m is even and y is odd by Lemma 5. Hence for $z\geq 3$, equation (2) has no positive integer solution when m is even. □

3.1. The case $5\not| m$

Lemma 7 *Let (x,y,z) be a positive integer solution of equation (2). If $m\not\equiv 0 \pmod 5$ then x is odd.*

Proof By Lemma 6 we may consider only the case m is odd, and so $m\equiv \pm 1,\pm 3 \pmod{10}$. If $m\equiv \pm 1 \pmod{10}$ then taking equation (2) modulo 10 implies that

$$(-1)^x+6\equiv 5 \pmod{10},$$

and so x is odd. If $m\equiv \pm 3 \pmod{10}$ then again taking equation (2) modulo 10 implies that

$$3^x+2^y\equiv 5 \pmod{10}.$$

Since y is odd by Lemma 5, we have either $2^y\equiv 2 \pmod{10}$ or $2^y\equiv 8 \pmod{10}$, which implies that x is odd. □

Lemma 8 *Let (x,y,z) be a positive integer solution of equation (2). If $m\not\equiv 0 \pmod 5$ then $y=1$ except for $m\equiv 23,47,63,87 \pmod{120}$.*

Proof By Lemma 4 and Lemma 6 we consider only the case $m>2$ is odd. Moreover, from Lemma 5 and Lemma 7 we know that both x and y are odd. Thus

$$\left(\frac{7m^2-1}{18m^2+1}\right)=\left(\frac{25m^2}{18m^2+1}\right)=1$$

and

$$\begin{aligned}\left(\frac{5m}{18m^2+1}\right)&=\left(\frac{5}{18m^2+1}\right)\left(\frac{m}{18m^2+1}\right)\\ &=\left(\frac{18m^2+1}{5}\right)\left(\frac{18m^2+1}{m}\right)(-1)^{\frac{m-1}{2}}=\left(\frac{18m^2+1}{5}\right)(-1)^{\frac{m-1}{2}},\end{aligned}$$

where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. If $m\equiv \pm 1 \pmod 5$ and $m\equiv 3 \pmod 4$ then $\left(\frac{5m}{18m^2+1}\right)=-1$, since $\left(\frac{18m^2+1}{5}\right)=\left(\frac{4}{5}\right)=1$, and similarly if $m\equiv \pm 2 \pmod 5$ and $m\equiv 1 \pmod 4$ then $\left(\frac{5m}{18m^2+1}\right)=-1$, since $\left(\frac{18m^2+1}{5}\right)=\left(\frac{3}{5}\right)=-1$. Hence for $m\equiv 11,13,17,19 \pmod{20}$ we see that z is even. Now assume that $y\geq 3$.

For these values of m, taking equation (2) modulo 8 we get $3^x \equiv 1 \pmod 8$, which implies that x is even, a contradiction. Hence $y = 1$ for $m \equiv 11, 13, 17, 19 \pmod{20}$. If $m \equiv 1, 9 \pmod{20}$ then $m^2 \equiv 1 \pmod{20}$ and

$$(-1)^x + 6^y \equiv 5 \pmod{20}$$

from (2). If $y \neq 1$ then $(-1)^x + 16 \equiv 5 \pmod{20}$, which is impossible. Thus $y = 1$ for $m \equiv 1, 9 \pmod{20}$. For the case $m \equiv 3, 7 \pmod{20}$, first we take the equation (2) modulo 20 and get $3^x + 2^y \equiv 15^z \pmod{20}$. From this z is odd since both x and y are odd and $y > 1$. If $m \equiv 3, 27 \pmod{40}$ then from (2) we get the equation $3^x + 22^y \equiv 15^z \equiv 15 \pmod{40}$ which has not a solution when $y > 1$. So $y = 1$ when $m \equiv 3, 27 \pmod{40}$. If $m \equiv 7, 103 \pmod{120}$ then $m^2 \equiv 49 \pmod{120}$ and so from (2) we have that

$$43^x + 102^y \equiv 35^z \pmod{120}.$$

One can see that this congruence also has no solution in positive integers when x, y and z are odd, since $y \geq 3$. Therefore, we conclude that $y = 1$ for $m \equiv 7, 103 \pmod{120}$. □

Lemma 9 *Let (x, y, z) be a positive integer solution of equation (2). If $y = 1$ then $x < 2521 \log 5m$.*

Proof We consider the equation

$$(18m^2 + 1)^x + (7m^2 - 1) = (5m)^z. \tag{3}$$

If $x = 1$ then clearly $z = 2$. Assume that $x \geq 2$. Then $z > 2$ from (2). For simplicity we set the following notation $a = 18m^2 + 1$, $b = 7m^2 - 1$, $c = 5m$ and consider the linear form of two logarithms

$$\Lambda = z \log c - x \log a.$$

Since

$$0 < \Lambda < e^{\Lambda} - 1 = \frac{c^z}{a^x} - 1 = \frac{b}{a^x}, \tag{4}$$

we get

$$\log \Lambda < \log b - x \log a. \tag{5}$$

From Proposition 2, we write

$$\log \Lambda \geq -25.2 D^4 \left(\max\{\log b' + 0.38, 10\}\right)^2 \log a \log c, \tag{6}$$

where

$$b' = \frac{x}{\log c} + \frac{z}{\log a}.$$

Note that $a^{x+1} - c^z = aa^x - c^z = a(c^z - b) - c^z = (a-1)c^z - ab > 18m^2 \cdot 25m^2 - (18m^2+1)(7m^2-1) > 0$, since $z > 2$. Thus $\frac{x+1}{\log c} > \frac{z}{\log a}$ and therefore $b' < \frac{2x+1}{\log c}$. Write $M = \frac{x}{\log c}$, and so $b' < 2M + \frac{1}{\log c}$. Now combining (5) and (6) we get

$$x \log a < \log b + 25.2 \left(\max\{\log(2M + \frac{1}{\log c}) + 0.38, 10\}\right)^2 \log a \log c.$$

Since $\frac{\log b}{\log a \log c} < 1$ and $\log c = \log 5m > 2$ for $m \geq 3$ we may write

$$M < 1 + 25.2 \left(\max\{\log(2M + 0.5) + 0.38, 10\}\right)^2 .$$

If $\log(2M + 0.5) + 0.38 > 10$ then $M \geq 7532$. However, the inequality $M < 1 + 25.2 \left(\log(2M + 0.5) + 0.38\right)^2$ gives $M \leq 1867$. Thus $\max\{\log(2M + 0.5) + 0.38, 10\} = 10$ implies $M < 2521$ and hence $x < 2521 \log c$. □

For possible solution (x, y, z) of the exponential Diophantine equation $\left(am^2 + 1\right)^x + \left(bm^2 - 1\right)^y = (cm)^z$ we can find an upper and lower bound for z depending on $\max\{x, y\}$ and m.

Lemma 10 *Let a, b, c and $m > 1$ be positive integers such that $a + b = c^2$ and (x, y, z) be a positive integer solution of the exponential Diophantine equation $(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z$. If $\max\{x, y\} = x$ then*

$$\left(2 - \frac{\log(\frac{c^2}{a})}{\log(cm)}\right) x < z \leq 2x,$$

and if $\max\{x, y\} = y$ then

$$\left(2 - \frac{\log\left(\frac{c^2 m^2}{bm^2 - 1}\right)}{\log(cm)}\right) y < z \leq 2y.$$

In particular, if $M = \max\{x, y\} > 1$ then

$$\left(2 - \frac{\log\left(\frac{c^2}{\min\{a, b - \frac{1}{m^2}\}}\right)}{\log(cm)}\right) M < z < 2M.$$

Proof $z \leq 2M$ follows from the fact that

$$(cm)^z = (am^2 + 1)^x + (bm^2 - 1)^y \leq (c^2 m^2 - bm^2 + 1)^M + (bm^2 - 1)^M \leq (cm)^{2M}.$$

If $M = x$ then

$$(cm)^{\left(2 - \frac{\log(\frac{c^2}{a})}{\log(cm)}\right) x} = (am^2)^x < (am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z.$$

Similarly, if $M = y$ then

$$(cm)^{\left(2 - \frac{\log\left(\frac{c^2 m^2}{bm^2 - 1}\right)}{\log(cm)}\right) y} = (bm^2 - 1)^y < (am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z.$$

□

Lemma 11 *Let (x,y,z) be a positive integer solution of equation (2). If $y=1$ then the only positive integer solution of (2) is $(x,y,z)=(1,1,2)$.*

Proof Clearly $z=2$ if $x=1$ and the result follows from Lemma 4 for $m\leq 2$. Assume that $x\geq 2$ and $m\geq 3$. From Lemma 10 we have the inequality

$$1.87x<\left(2-\frac{\log(\frac{25}{18})}{\log(15)}\right)x<z<2x.$$

Thus (3) has no solution in positive integer for $z\leq 6$. Assume that $z>6$. We consider (3) modulo m^4 and modulo m^6. First taking equation (3) modulo m^4, we have that $18m^2x+7m^2\equiv 0 \pmod{m^4}$. In other words $18x+7\equiv 0 \pmod{m^2}$. So there exist a positive integer t such that

$$x=\frac{tm^2-7}{18} \tag{7}$$

Taking equation (3) modulo m^6, we obtain

$$18x+18^2m^2\frac{x(x-1)}{2}+7\equiv 0 \pmod{m^4}.$$

Combining this and (7), we see that $2t+175\equiv 0 \pmod{m^2}$. Therefore, there exists a positive integer s such that

$$t=\frac{sm^2-175}{2}. \tag{8}$$

Therefore, from (7), (8), x is of the form

$$x=\frac{sm^4-175m^2-14}{36}. \tag{9}$$

Hence by Lemma 9 we have the inequality

$$m^4-175m^2-14<36\cdot 2521\log 5m$$

which implies that $m\leq 27$. Now with the same notation as in Lemma 9, from (4) we have that

$$\frac{z}{x}-\frac{\log a}{\log c}<\frac{b}{xa^x\log c},$$

and hence $\left|\frac{\log a}{\log c}-\frac{z}{x}\right|<\frac{b}{xa^x\log c}$. Since $\frac{a^x\log c}{b}>\frac{18m^2x}{7m^2}>2x$, we have the inequality

$$\left|\frac{\log a}{\log c}-\frac{z}{x}\right|<\frac{1}{2x^2},$$

which means that $\frac{z}{x}$ is a convergent of the simple continued fraction expansion of $\frac{\log a}{\log c}$. Let $\frac{z}{x}=\frac{p_k}{q_k}$, where $\frac{p_k}{q_k}$ is the $k-th$ convergent of the simple continued fraction expansion of $\frac{\log a}{\log c}$. Here $q_k\leq x$, since $(p_k,q_k)=1$

and therefore we have an upper bound for q_k as $q_k < 2521 \log 5m$ from Lemma 9. Since any such convergent $\frac{p_k}{q_k}$ satisfies the inequality

$$\frac{1}{q_k\left(q_k+q_{k+1}\right)} < \left|\frac{\log a}{\log c} - \frac{p_k}{q_k}\right|,$$

by taking $q_{k+1} = a_{k+1}q_k + q_{k-1}$ we obtain

$$\frac{1}{q_k^2\left(a_{k+1}+2\right)} < \left|\frac{\log a}{\log c} - \frac{p_k}{q_k}\right| < \frac{b}{xa^x \log c} < \frac{b}{q_k a^{q_k} \log c}$$

where a_k is the $k-th$ partial quotient of the simple continued fraction expansion of $\frac{\log a}{\log c}$; see for example [6, Page 36]. Thus q_k and a_{k+1} satisfy the inequality

$$a_{k+1} + 2 > \frac{a^{q_k} \log c}{bq_k}. \tag{10}$$

As a final step of the proof we checked that with a short computer program in Maple there do not exist any convergent $\frac{p_k}{q_k}$ of $\frac{\log a}{\log c}$ satisfying (10) when $q_k < 2521 \log 5m$ in the range $1 \leq m \leq 27$ and it took not more than a few seconds. This completes the proof. □

3.2. The case $5 \mid m$

For the case $m \equiv 0 \pmod 5$ we have a result from [4] that gives an upper bound for m.

Proposition 12 *[4, Theorem 1.1] Let a, b, c, m be positive integers such that $a+b=c^2$, $2 \mid a$, $2 \nmid c$, $m>1$. If $c \mid m$ and $m > 36c^3 logc$, then $(am^2+1)^x + (bm^2-1)^y = (cm)^z$ has only the solution $(x,y,z) = (1,1,2)$.*

From the above proposition, we may assume that $m \leq 7242$ when $m \equiv 0 \pmod 5$. Therefore, what we need is a restriction on x, y, and z. In what follows we will obtain such an upper bound for x, y and z.

Lemma 13 *Let (x,y,z) be a positive integer solution of equation (2). Suppose that $m \equiv 0 \pmod 5$. Then the only positive integer solution of (2) is $(x,y,z) = (1,1,2)$.*

Proof Clearly $(1,1,2)$ is the only solution of (2) for $M = \max\{x,y\} = 1$. Suppose that $M > 1$. From Lemma 10 for $m \geq 5$ we have that

$$1.55M < \left(2 - \frac{\log(\frac{25}{6})}{\log(25)}\right) M < z < 2M.$$

From this we deduce that $z \geq 5$. Since y is odd from Lemma 7 we set $a_1 = 18m^2+1$, $a_2 = 1-7m^2$, $b_1 = x$, $b_2 = y$, and

$$\Lambda = a_1^{b_1} - a_2^{b_2}$$

Thus if we choose $p = 5$, then $g = 1$ satisfies the condition mentioned before Proposition 3 and therefore we can take $E = 2$. Now we apply Proposition 3 and we get

$$2z \leq \frac{36.1}{8\,(\log 5)^4}\,(\max\{\log b' + \log(2\log 5) + 0.4, 12\log 5, 5\})^2 \log(18m^2+1)\log(7m^2-1). \tag{11}$$

Here

$$b' = \frac{x}{\log(7m^2-1)} + \frac{y}{\log(18m^2+1)}.$$

Since $z \geq 5$, taking (2) modulo m^4 we have $18x + 7y \equiv 0 \pmod{m^2}$. Then $M \geq \frac{m^2}{25}$. Since $z \geq \left(2 - \frac{\log(\frac{25}{6})}{\log(5m)}\right) M$ by Lemma 10 and $b' \leq \frac{M}{\log 2m}$ we have that

$$2\left(2 - \frac{\log(\frac{25}{6})}{\log(5m)}\right) M \leq \frac{36.1}{8\,(\log 5)^4}\left(\max\{\log(\frac{M}{\log 2m}) + \log(2\log 5) + 0.4, 12\log 5\}\right)^2 \log(18m^2+1)\log(7m^2-1). \tag{12}$$

Let

$$h = \max\{\log(\frac{M}{\log 2m}) + \log(2\log 5) + 0.4, 12\log 5\}.$$

Suppose $h = \log(\frac{M}{\log 2m}) + \log(2\log 5) + 0.4 \geq 12\log 5$. Then the inequality $\log(M) \geq 12\log 5 - \log(2\log 5) - 0.4$ implies that $M > 50841461$. On the other hand, from (12) we have that

$$2M \leq (0.68)\,(\log M + 1.57)^2 \log(18 \cdot 7242^2 + 1)\log(7 \cdot 7242^2 - 1),$$

which implies that $M < 17889$, a contradiction. Hence $h = 12\log 5$ and therefore from (12) we have the inequality

$$\frac{2m^2}{25}\left(2 - \frac{\log(\frac{25}{6})}{\log(5m)}\right) \leq 251\log(18m^2+1)\log(7m^2-1).$$

This implies that $m \leq 636$. Hence $M < \dfrac{251\log(18m^2+1)\log(7m^2-1)}{2\left(2 - \frac{\log(\frac{25}{6})}{\log(5m)}\right)}$ and therefore all x, y, z are bounded.

We wrote a program in Maple. It took a few hours to run the program and we found no positive integer solutions (m, x, y, z) of (2) under consideration. Hence there is no positive integer solution of (2) other than $(x, y, z) = (1, 1, 2)$ when $5 \mid m$. □

Proof [Proof of Theorem 1] The result follows from Lemma 8 and Lemma 11 if $m \not\equiv 0 \pmod 5$ and from Lemma 13 if $m \equiv 0 \pmod 5$. □

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