

Sandwich theorems for a class of p -valent meromorphic functions involving the Erdélyi–Kober-type integral operators

Hari SRIVASTAVA^{1,2} , Rabha EL-ASHWAH³ , Wafaa KOTA^{3,*} 

¹Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, Canada

²Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, Republic of China

³Department of Mathematics, Faculty of Science, Damietta University, New Damietta, Egypt

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Abstract: In this paper, the authors study some subordination and superordination properties for classes of p -valent meromorphic, analytic, and univalent functions associated with a linear operator $\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)$ of the Erdélyi–Kober type. Connections with several earlier results are also pointed out.

Key words: Analytic functions, univalent functions, Erdélyi–Kober-type integral operator, subordination properties, superordination properties, p -valent meromorphic functions, integral operators

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and suppose that $\mathcal{H}[b, n]$ denotes a subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form

$$f(z) = b + b_n z^n + b_{n+1} z^{n+1} + \dots \quad (b \in \mathbb{C}; n \in \mathbb{N} = \{1, 2, 3, \dots\}).$$

Now let \mathcal{A}_n be the class of the form

$$\mathcal{A}_n = \{f : f \in \mathcal{H}(\mathbb{U}) \text{ and } f(z) = z + b_{n+1} z^{n+1} + \dots\}.$$

If we put $n = 1$, we obtain the class of $\mathcal{A}_1 = \mathcal{A}$ of normalized analytic functions in \mathbb{U} .

Definition 1 For $f(z)$ and $g(z)$ analytic in \mathbb{U} , we say that the function $f(z)$ is subordinate to $g(z)$ in \mathbb{U} , written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$ that is analytic in \mathbb{U} , satisfying the following conditions:

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

*Correspondence: wafaa_kota@yahoo.com

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In particular, if the function $g(z)$ is univalent in \mathbb{U} , we have the following equivalence (see [14, 25, 27]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let Σ_p be the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=-p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \tag{1}$$

which are analytic and univalent in the *punctured* unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}.$$

For $m \in \mathbb{Z}$, $\ell > 0$ and $\lambda \geq 0$, El-Ashwah (see [15, 16]) defined the multiplier transformations $\mathfrak{L}_p^m(\ell, \lambda)$ as follows:

$$\mathfrak{L}_p^m(\ell, \lambda) = \frac{1}{z^p} + \sum_{k=-p+1}^{\infty} \left(\frac{\ell + \lambda(k+p)}{\ell} \right)^m b_k z^k.$$

For $\mu > 0$, $a, c \in \mathbb{C}$ such that $\Re(c - a) \geq 0$, $\Re(a) \geq \mu p$ ($p \in \mathbb{N}$) and for $f(z) \in \Sigma_p$ given by (1), El-Ashwah and Hassan [20] introduced the integral operator

$$\mathcal{J}_{p,\mu}^{a,c} : \Sigma_p \rightarrow \Sigma_p$$

given by

- For $\Re(c - a) > 0$,

$$\mathcal{J}_{p,\mu}^{a,c} f(z) = \frac{\Gamma(c - \mu p)}{\Gamma(a - \mu p)\Gamma(c - a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} f(zt^\mu) dt;$$

- For $a = c$,

$$\mathcal{J}_{p,\mu}^{a,a} f(z) = f(z).$$

It is easily seen that the operator $\mathcal{J}_{p,\mu}^{a,c} f(z)$ can be expressed as follows:

$$\mathcal{J}_{p,\mu}^{a,c} f(z) = \frac{1}{z^p} + \frac{\Gamma(c - \mu p)}{\Gamma(a - \mu p)} \sum_{k=-p+1}^{\infty} \frac{\Gamma(a + \mu k)}{\Gamma(c + \mu k)} a_k z^k,$$

where $\mu > 0$, $a, c \in \mathbb{C}$, $\Re(c - a) \geq 0$, $\Re(a) \geq \mu p$ ($p \in \mathbb{N}$).

We now consider the linear operator $\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) : \Sigma_p \rightarrow \Sigma_p$ given by

$$\begin{aligned} &\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) \\ &= \frac{1}{z^p} + \frac{\Gamma(c - \mu p)}{\Gamma(a - \mu p)} \sum_{k=-p+1}^{\infty} \left(\frac{\ell + \lambda(k+p)}{\ell} \right)^m \frac{\Gamma(a + \mu k)}{\Gamma(c + \mu k)} b_k z^k, \end{aligned} \tag{2}$$

where $m \in \mathbb{Z}$, $\ell > 0$, $\lambda \geq 0$, $\mu > 0$, $a, c \in \mathbb{C}$, $\Re(c - a) \geq 0$, and $\Re(a) \geq \mu p$ ($p \in \mathbb{N}$). It is readily verified from (2) that

$$z(\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z))' = \frac{\ell}{\lambda} \mathfrak{L}_{p,\lambda}^{m+1,\ell}(a, c, \mu)f(z) - \left(p + \frac{\ell}{\lambda}\right) \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z) \quad (\lambda > 0) \quad (3)$$

and

$$z(\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z))' = \frac{a - \mu p}{\mu} \mathfrak{L}_{p,\lambda}^{m,\ell}(a + 1, c, \mu)f(z) - \frac{a}{\mu} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z). \quad (4)$$

The above-defined operator includes several simpler operators. We point out here some of these special cases as follows:

- (a) Putting $\ell = 1$ and $a = c$, we obtain $D_\lambda^m f(z)$, which was studied by Al-Oboudi and Al-Zkeri [6];
- (b) Putting $m = -\alpha$, $\lambda = 1$, $\ell = 1$, and $a = c$, we obtain $P^\alpha f(z)$, which was studied by Aqlan et al. [7];
- (c) Putting $a = c$, we obtain $I_p^m(\lambda, \ell)f(z)$, which was studied by El-Ashwah (see [15, 16]);
- (d) Putting $\mu = 1$, $a = a + p$, $c = c + p$, and $m = 0$, we obtain $\ell_p(a, c)f(z)$ ($a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0, 1, 2, \dots\}, p \in \mathbb{N}$), which was studied by Liu and Srivastava [24];
- (e) Putting $\mu = 1$, $a = n + 2p$, $c = p + 1$, and $m = 0$, we obtain $D^{n+p-1}f(z)$ (n is an integer, $n > -p$ and $p \in \mathbb{N}$), which was studied by Aouf [3] (see also [5, 35]);
- (f) Putting $\mu = 1$, $c = a + 1$, and $m = 0$, we obtain $J_p^\alpha f(z)$ ($\Re(a) > p; p \in \mathbb{N}$), which was studied by Kumar and Shukla [22];
- (g) Putting $\mu = 1$, $a = \beta + p$, $c = \alpha + \beta - \gamma + 1 + p$, and $m = 0$, we obtain $Q_{\alpha,\beta,\gamma}^{p,1}f(z)$ ($\beta > 0, \alpha > \gamma - 1, \gamma > 0, p \in \mathbb{N}$), which was studied by El-Ashwah et al. [18];
- (h) Putting $\mu = 1$, $a = \beta + p$, $c = \alpha + \beta + p$, and $m = 0$, we obtain $Q_{\alpha,\beta}^p f(z)$ ($\beta > 0, \alpha > 0, p \in \mathbb{N}$), which was studied by Aqlan et al. [7] (see also Aouf et al. [4]);
- (i) Putting $p = 1$, $m = \alpha$, $\lambda = 1$, $\ell = \beta$, and $a = c$, we obtain $P_\beta^\alpha f(z)$, which was studied by Lashin [23];
- (j) Putting $p = 1$, $\lambda = 1$, and $a = c$, we obtain $I(m, \ell)f(z)$, which was studied by Cho et al. (see [10, 11]);
- (k) Putting $p = 1$, $\lambda = 1$, $\ell = 1$, and $a = c$, we obtain $I^m f(z)$, which was studied by Uralegaddi and Somanatha [34];
- (l) Putting $p = 1$ and $m = 0$, we obtain $I_\mu(a, c)f(z)$, which was studied by El-Ashwah [17];

Recently, based on various linear operators, some subordination results have been studied in [1, 2] and [8] (see also [9, 12, 13, 17, 19, 29, 31–33, 36]). In the present paper, the authors study some subordination and superordination properties for classes of p -valent meromorphic, analytic, and univalent functions associated with a linear operator $\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)$. The linear operator $\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z)$ is convolution between the linear integral operator $\mathcal{J}_{p,\mu}^{a,c}f(z)$ and the multiplier transforms operator $\mathfrak{L}_p^m(\ell, \lambda)f(z)$.

2. A set of preliminaries

To prove our main theorems, we need several lemmas and definitions, which are presented in this section.

Definition 2 ([27, p. 21, Definition 2.2b]) Let Q be the class of functions g that are analytic and injective on $\bar{\mathbb{U}} \setminus E(g)$, where

$$E(g) = \{\xi \in \partial\mathbb{U} : \lim_{z \rightarrow \xi} g(z) = \infty\}$$

and $g'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(g)$.

Definition 3 ([27, p. 16]) For $h, k \in H(\mathbb{U})$, let $\varphi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C}$ and let $h(z)$ be univalent in \mathbb{U} . If $k(z)$ satisfies the first-order differential subordination

$$\varphi(k(z), zk'(z); z) \prec h(z), \tag{5}$$

then $k(z)$ is a solution of the differential subordination (5). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (5), if $k(z) \prec q(z)$ for all the functions $k(z)$ satisfying (5). A univalent dominant $\tilde{q}(z)$ is said to be the best dominant of (5) if $\tilde{q}(z) \prec q(z)$ for all dominant $q(z)$.

Definition 4 (see [28]) Let $\varphi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C}$ and suppose that the functions $k(z)$ and $\varphi(k(z), zk'(z); z)$ are univalent in \mathbb{U} . If $k(z)$ satisfies the first-order differential superordination

$$h(z) \prec \varphi(k(z), zk'(z); z), \tag{6}$$

then $k(z)$ is a solution of the differential superordination (6). The univalent function $q(z)$ is called a subordinant of the solutions of the differential superordination, if $q(z) \prec k(z)$ for all the functions $k(z)$ satisfying (6). A subordinant $\tilde{q}(z)$ is said to be the best subordinant of (6) if $q(z) \prec \tilde{q}(z)$ for all the subordinants $q(z)$.

Definition 5 A function $L(z, t) : \mathbb{U} \times [0, \infty) \rightarrow \mathbb{C}$ is called a Löwner chain (subordination), if $L(\cdot, t)$ is analytic and univalent in \mathbb{U} and $L(z, s) \prec L(z, t)$, for all $t \geq 0$ and $0 \leq s \leq t$.

Lemma 1 (see [26]) Let $\vartheta, \zeta \in \mathbb{C}$ and $\zeta \neq 0$ and let $h(z) \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If

$$\Re(\zeta h(z) + \vartheta) > 0 \quad (z \in \mathbb{U}),$$

then the solution of the following differential equation:

$$p(z) + \frac{zp'(z)}{\zeta p(z) + \vartheta} = h(z) \quad (p(0) = c)$$

has analytic solution in \mathbb{U} that satisfies

$$\Re(\zeta p(z) + \vartheta) > 0 \quad (z \in \mathbb{U}).$$

Lemma 2 (see [30]) Let

$$L(z; t) = b_1(t)z + b_2(t)z^2 + \dots$$

with

$$b_1(t) \neq 0 \quad (\forall t \geq 0) \quad \text{and} \quad \lim_{t \rightarrow \infty} |b_1(t)| = \infty.$$

Assume that $L(\cdot; t)$ is analytic in \mathbb{U} and $(\forall t \geq 0)$, $L(z; \cdot)$ is continuously differentiable on $[0, \infty)$ $(\forall z \in \mathbb{U})$. If $L(z; t)$ satisfies

$$\Re \left(z \frac{\partial L / \partial z}{\partial L / \partial t} \right) > 0 \quad (z \in \mathbb{U}; t \geq 0)$$

and

$$|L(z; t)| \leq K_0 |b_1(t)| \quad (|z| < r_0 < 1; t \geq 0)$$

for some positive constants K_0 and r_0 , then $L(z; t)$ is a subordination chain.

Lemma 3 (see [27] and [25]) Suppose that $H : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the following condition:

$$\Re(H(i\rho, t)) \leq 0 \quad (\forall \rho, t \in \mathbb{C})$$

with

$$t \leq -\frac{1}{2} j(1 + \rho^2) \quad (j \in \mathbb{N}).$$

If the function $q(z) = 1 + q_j z^j + \dots$ is analytic in \mathbb{U} and

$$\Re \{H(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then

$$\Re(q(z)) > 0 \quad (z \in \mathbb{U}).$$

Lemma 4 (see [28]) Let $\mathbf{p} \in H[b, 1]$, $\Psi : \mathbb{C}^2 \rightarrow \mathbb{C}$, and $\Psi(\mathbf{p}(z), z\mathbf{p}'(z)) = h(z)$. If $L(z; t) = \Psi(\mathbf{p}(z), tz\mathbf{p}'(z))$ is a subordination chain and $g \in H[b, 1] \cap Q$, then

$$h(z) \prec \Psi(g(z), zg'(z)) \quad \text{implies that} \quad \mathbf{p}(z) \prec g(z).$$

Furthermore, if $\Psi(\mathbf{p}(z), z\mathbf{p}'(z)) = h(z)$ has a univalent solution $\mathbf{p} \in Q$, then g is the best subordinator.

Lemma 5 ([27]) Let $q \in Q$ with $q(0) = b$ and

$$\mathbf{p}(z) = b + b_k z^k + b_{k+1} z^{k+1} + \dots$$

be analytic in \mathbb{U} with $\mathbf{p}(z) \neq b$ for $k \in \mathbb{N}$. If \mathbf{p} is not subordinate to q , then there exist the points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(f)$ such that $\mathbf{p}(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$, $\mathbf{p}(z_0) = q(\zeta_0)$ and $z_0 \mathbf{p}'(z_0) = j \zeta_0 q'(\zeta_0)$ $(j \geq k)$, where

$$\mathbb{U}_{r_0} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < r_0\}.$$

3. The main results

Unless otherwise mentioned, we suppose that $m \in \mathbb{Z}$, $\ell > 0$, $\lambda \geq 0$, $\mu > 0$, $a, c \in \mathbb{C}$, $\Re(c - a) \geq 0$, $0 \leq \beta \leq p$, $\Re(a) \geq \mu p$, and $p \in \mathbb{N}$.

We first prove the following subordination theorem for the linear operator $\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)$.

Theorem 1 *Let*

$$\eta = \frac{\ell p(a - \mu p)}{(p - \beta)\lambda(a - \mu p) + \mu\beta\ell} \tag{7}$$

be such that $\Re(\eta) \geq 1$. Suppose for $f(z) \in \Sigma_p$ that

$$\chi_1(z) = \frac{z^{p+1}}{p} \left[(p - \beta)\mathfrak{L}_{p,\lambda}^{m+1,\ell}(a, c, \mu)f(z) + \beta\mathfrak{L}_{p,\lambda}^{m,\ell}(a + 1, c, \mu)f(z) \right] \tag{8}$$

satisfies the following condition:

$$\Re \left(1 + \frac{z\chi_1''(z)}{\chi_1'(z)} \right) > -v \quad (z \in \mathbb{U}). \tag{9}$$

If $\Re(\eta) = 1$, then $v = 0$ and if $\Re(\eta) > 1$, then

$$v \leq \begin{cases} \frac{\Re(\eta) - 1}{2} & (1 < \Re(\eta) \leq 2) \\ \frac{1}{2[\Re(\eta) - 1]} & (\Re(\eta) > 2) \end{cases} \tag{10}$$

and

$$[\mathfrak{S}(\eta)]^2 \leq [\Re(\eta) - 1 - 2v] \left(\frac{1}{2v} - \Re(\eta) + 1 \right). \tag{11}$$

Equations (10) and (11) occur only when $\mathfrak{S}(\eta) = 0$. If $g(z) \in \Sigma_p$ satisfies the following condition:

$$\begin{aligned} & \frac{z^{p+1}}{p} \left[(p - \beta)\mathfrak{L}_{p,\lambda}^{m+1,\ell}(a, c, \mu)g(z) + \beta\mathfrak{L}_{p,\lambda}^{m,\ell}(a + 1, c, \mu)g(z) \right] \\ & \prec \chi_1(z) \quad (z \in \mathbb{U}), \end{aligned} \tag{12}$$

then

$$z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)g(z) \prec z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z) \quad (z \in \mathbb{U}) \tag{13}$$

and the function $z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z)$ is the best dominant of (12).

Proof Put

$$G(z) = z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)g(z) \quad \text{and} \quad \mathfrak{A}(z) = z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z). \tag{14}$$

First of all, we prove that the function $\mathfrak{A}(z)$ is convex univalent in \mathbb{U} . Let

$$s(z) = 1 + \frac{z\mathfrak{A}''(z)}{\mathfrak{A}'(z)} \quad (z \in \mathbb{U}). \tag{15}$$

For $f(z) \in \Sigma_p$, by using equations (3) and (4), we obtain

$$\chi_1(z) = \frac{z\mathfrak{A}'(z)}{\eta} + \left(1 - \frac{1}{\eta}\right)\mathfrak{A}(z), \tag{16}$$

where η is given by (7). Differentiating (16) and using (15), we have

$$\frac{\chi_1'(z)}{\mathfrak{A}'(z)} = \left(1 - \frac{1}{\eta}\right) + \frac{s(z)}{\eta},$$

which, upon differentiating once again and using equation (15), yields

$$1 + \frac{z\chi_1''(z)}{\chi_1'(z)} = s(z) + \frac{zs'(z)}{s(z) + \eta - 1} = h(z). \tag{17}$$

From (9) and (10), we have

$$\Re\{h(z) + \eta - 1\} > 0 \quad (z \in \mathbb{U}).$$

Thus, by using Lemma 1, we conclude that equation (17) has a solution $s(z) \in \mathcal{H}(\mathbb{U})$ with

$$s(0) = h(0) = 1.$$

We will now use Lemma 3 to prove that the inequality

$$\Re(s(z)) > 0 \quad (z \in \mathbb{U})$$

is true. Let

$$H(x, y) = x + \frac{y}{x + \eta - 1} + v, \tag{18}$$

where v is given by (10). From equations (9), (17), and (18), we have

$$\Re\left(H(s(z), zs'(z))\right) > 0 \quad (z \in \mathbb{U}). \tag{19}$$

We proceed to show that $\Re(H(i\varrho, t)) \leq 0$ for all $\varrho, t \in \mathbb{C}$ with

$$t \leq -\frac{1}{2}(1 + \varrho^2).$$

Using (18), we find that

$$\begin{aligned} \Re(H(i\varrho, t)) &= \Re\left(i\varrho + \frac{t}{i\varrho + \eta - 1} + v\right) \\ &= \frac{(\Re(\eta) - 1)t}{|i\varrho + \eta - 1|^2} + v. \end{aligned} \tag{20}$$

If we take $\Re(\eta) = 1$ and $v = 0$, we obtain $\Re(H(i\varrho, t)) = 0$. If we take $\Re(\eta) > 1$, we get

$$\Re(H(i\varrho, t)) \leq -\frac{\varpi(\varrho, \eta, v)}{2|i\varrho + \eta - 1|^2}, \tag{21}$$

where

$$\varpi(\varrho, \eta, \nu) = [\Re(\eta) - 1](1 + \varrho^2) - 2\nu|i\varrho + \eta - 1|^2.$$

By taking $\Re(\eta) - 1 = x$ and $\Im(\eta) = y$, we can rewrite $\varpi(\varrho, \eta, \nu)$ as follows:

$$\varpi(\varrho, \eta, \nu) = (x - 2\nu)\varrho^2 - 4\nu y\varrho + x - 2\nu(x^2 + y^2).$$

If we set $y = 0$ and use equation (10), we obtain

$$\varpi(\varrho, \eta, \nu) = (x - 2\nu)\varrho^2 + (1 - 2\nu)x \geq 0.$$

If $y \neq 0$ and we assume that $x - 2\nu > 0$ for any $x > 0$, then we have

$$\begin{aligned} \varpi(\varrho, \eta, \nu) &= (x - 2\nu) \left(\varrho - \frac{2\nu y}{x - 2\nu} \right)^2 - \frac{4\nu^2 y^2}{x - 2\nu} + x - 2\nu(x^2 + y^2) \\ &= (x - 2\nu) \left(\varrho - \frac{2\nu y}{x - 2\nu} \right)^2 + x \left[1 - 2\nu \left(x + \frac{y^2}{x - 2\nu} \right) \right] \geq 0, \end{aligned}$$

which, in light of (10), yields that $\varpi(\varrho, \eta, \nu) \geq 0$ for all $\varrho \in \mathbb{R}$. Thus, from equations (20) and (21), we obtain $\Re(H(i\varrho, t)) \leq 0$ for all $\varrho \in \mathbb{R}$ and $t \leq -(1 + \varrho^2)/2$. Thus, by using Lemma 3, we find that $\Re(s(z)) > 0$ for all $z \in \mathbb{U}$, which proves that the function $\mathfrak{A}(z)$ is convex univalent for all $z \in \mathbb{U}$.

Secondly, we prove that

$$G(z) \prec \mathfrak{A}(z) \quad (z \in \mathbb{U}), \tag{22}$$

if condition (12) is true. We define a function $\mathcal{L}(z, t)$ by

$$\mathcal{L}(z; t) = \left(1 - \frac{1}{\eta} \right) \mathfrak{A}(z) + \frac{(1+t)}{\eta} z \mathfrak{A}'(z) \quad (z \in \mathbb{U}; t \geq 0) \tag{23}$$

and

$$\frac{\partial \mathcal{L}(z; t)}{\partial z} \Big|_{z=0} = \mathfrak{A}'(0) \left(1 + \frac{t}{\eta} \right) = 1 + \frac{t}{\eta} \neq 0 \quad (t \geq 0). \tag{24}$$

This shows that the function

$$\mathcal{L}(z; t) = b_1(t)z + b_2(t)z^2 + \dots$$

with $b_1(t) = 1 + \frac{t}{\eta} \neq 0$ for all $t \geq 0$ and $\lim_{t \rightarrow \infty} |b_1(t)| = \infty$. Using (24), we can deduce the following equality:

$$\Re \left(z \frac{\partial \mathcal{L} / \partial z}{\partial \mathcal{L} / \partial t} \right) = \Re(\eta) - 1 + (1+t) \Re \left(1 + \frac{z \mathfrak{A}''(z)}{\mathfrak{A}'(z)} \right).$$

By the inequalities $\Re(s(z)) > 0$ and $\Re(\eta) > 1$, the above relation yields

$$\Re \left(z \frac{\partial \mathcal{L} / \partial z}{\partial \mathcal{L} / \partial t} \right) > 0 \quad (\forall z \in \mathbb{U}; \forall t \geq 0).$$

Since the function $\mathfrak{A}(z)$ is convex and normalized in \mathbb{U} , we have the following growth and distortion sharp bounds (see [21]):

$$\frac{r}{1+r} \leq |\mathfrak{A}(z)| \leq \frac{r}{1-r}, \quad |z| \leq r < 1,$$

and

$$\frac{1}{(1+r)^2} \leq |\mathfrak{A}'(z)| \leq \frac{1}{(1-r)^2}, \quad |z| \leq r < 1.$$

From equations (23) and (7), we have

$$\begin{aligned} \frac{|\mathcal{L}(z; t)|}{|b_1(t)|} &\leq \frac{|\eta - 1|}{|\eta + t|} |\mathfrak{A}(z)| + \frac{|1 + t|}{|\eta + t|} |z\mathfrak{A}'(z)| \\ &\leq |\mathfrak{A}(z)| + |z\mathfrak{A}'(z)| \leq \frac{r}{1-r} + \frac{r}{(1-r)^2} \\ &\leq \frac{r}{(1-r)^2} \quad (|z| \leq r < 1; t \geq 0). \end{aligned}$$

Hence, the second assumptions of Lemma 2 hold true. Hence, the function $\mathcal{L}(z; t)$ is a subordination chain.

Now we assume that $G(z)$ and $\mathfrak{A}(z)$ are analytic and univalent in $\bar{\mathbb{U}}$ and $\mathfrak{A}'(\zeta) \neq 0$ for $|\zeta| = 1$. Otherwise, we replace G by $G_r(z) = G(rz)$ and \mathfrak{A} by $\mathfrak{A}_r(z) = \mathfrak{A}(rz)$, where $r \in (0, 1)$. This function satisfies the conditions of Theorem 1 on $\bar{\mathbb{U}}$. We thus need to prove that $G_r(z) \prec \mathfrak{A}_r(z)$ for all $r \in (0, 1)$, which enables us to prove (22) by letting $r \rightarrow 1^-$. Suppose that $G(z)$ is not subordinate to $\mathfrak{A}(z)$. Then, by Lemma 5, there exist points $z_0 \in \mathbb{U}$ and $\varsigma_0 \in \partial\mathbb{U}$, and the number $t \geq 0$, such that

$$G(z_0) = \mathfrak{A}(\varsigma_0) \quad \text{and} \quad z_0 G'(z_0) = (1+t)\varsigma_0 \mathfrak{A}'(\varsigma_0).$$

Thus, from the above two relations and the condition (12), we obtain

$$\begin{aligned} \mathcal{L}(\varsigma_0; t) &= \left(1 - \frac{1}{\eta}\right) \mathfrak{A}(\varsigma_0) + \frac{1+t}{\eta} \varsigma_0 \mathfrak{A}'(\varsigma_0) \\ &= \left(1 - \frac{1}{\eta}\right) G(z_0) + \frac{1+t}{\eta} z_0 G'(z_0) \\ &= \frac{z_0^{p+1}}{p} \left[(p-\beta) \mathfrak{L}_{p,\lambda}^{m+1,\ell}(a, c, \mu) g(z_0) + \beta \mathfrak{L}_{p,\lambda}^{m,\ell}(a+1, c, \mu) g(z_0) \right] \\ &\in \chi_1(\mathbb{U}), \end{aligned}$$

which contradicts the above observation that $\mathcal{L}(\varsigma; t) \notin \chi_1(\mathbb{U})$. Thus, the subordination condition (12) must imply the subordination given by (22). Considering $G(z) \prec \mathfrak{A}(z)$, we see that $\mathfrak{A}(z)$ is the best dominant. This completes the proof of Theorem 1. □

Remark 1 For $p = 1$ in Theorem 1, we obtain the result that was obtained by El-Ashwah [17].

We next prove a superordination theorem for the linear operator $\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)$.

Theorem 2 Suppose that η given by (7) is such that $\Re(\eta) > 1$ and that, for $f(z) \in \Sigma_p$,

$$\chi_2(z) = \frac{z^{p+1}}{p} \left[(p-\beta) \mathfrak{L}_{p,\lambda}^{m+1,\ell}(a, c, \mu) f(z) + \beta \mathfrak{L}_{p,\lambda}^{m,\ell}(a+1, c, \mu) f(z) \right]$$

satisfies the following condition:

$$\Re \left(1 + \frac{z\chi_2''(z)}{\chi_2'(z)} \right) > -v \quad (z \in \mathbb{U}),$$

where v is given by (10). If $g(z) \in \Sigma_p$, let

$$\frac{z^{p+1}}{p} \left[(p - \beta)\mathfrak{L}_{p,\lambda}^{m+1,\ell}(a, c, \mu)g(z) + \beta\mathfrak{L}_{p,\lambda}^{m,\ell}(a + 1, c, \mu)g(z) \right] \quad (z \in \mathbb{U}),$$

be univalent in \mathbb{U} and

$$z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)g(z) \in \mathcal{H}[0, 1] \cap Q.$$

Then the condition given by

$$\chi_2(z) \prec \frac{z^{p+1}}{p} \left[(p - \beta)\mathfrak{L}_{p,\lambda}^{m+1,\ell}(a, c, \mu)g(z) + \beta\mathfrak{L}_{p,\lambda}^{m,\ell}(a + 1, c, \mu)g(z) \right] \quad (25)$$

$$(z \in \mathbb{U})$$

implies that

$$z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z) \prec z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)g(z) \quad (z \in \mathbb{U}), \quad (26)$$

and the function $z^{p+1}\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f(z)$ is the best subordinant of (25).

Proof By using the same method as in the proof of Theorem 1, we can prove that $\Re(s(z)) > 0$ for all $z \in \mathbb{U}$. Secondly, we prove that subordination (25) implies that

$$\mathfrak{A}(z) \prec G(z) \quad (z \in \mathbb{U}),$$

where $G(z)$ and $\mathfrak{A}(z)$ are defined by (14). We now define a function $\mathcal{L}(z; t)$ by

$$\mathcal{L}(z; t) = \left(1 - \frac{1}{\eta} \right) \mathfrak{A}(z) + \frac{t}{\eta} z\mathfrak{A}'(z) \quad (z \in \mathbb{U}; t \geq 0) \quad (27)$$

and

$$\left. \frac{\partial \mathcal{L}(z; t)}{\partial z} \right|_{z=0} = \mathfrak{A}'(0) \left(1 - \frac{1-t}{\eta} \right) = 1 - \frac{1-t}{\eta} \neq 0 \quad (t \geq 0).$$

This shows that the function

$$\mathcal{L}(z; t) = b_1(t)z + b_2(t)z^2 + \dots$$

with

$$b_1(t) = 1 - \frac{1-t}{\eta} \neq 0$$

for all $t \geq 0$ and $\lim_{t \rightarrow \infty} |b_1(t)| = \infty$. Using (27) and (7), we have

$$\begin{aligned} \frac{|\mathcal{L}(z; t)|}{|b_1(t)|} &\leq \left| \frac{\left(1 - \frac{1}{\eta} \right) \mathfrak{A}(z) + \frac{t}{\eta} z\mathfrak{A}'(z)}{1 - \frac{1-t}{\eta}} \right| \\ &\leq \frac{\left| 1 - \frac{1}{\eta} \right| |\mathfrak{A}(z)| + \left| \frac{t}{\eta} \right| |z\mathfrak{A}'(z)|}{\left| 1 - \frac{1-t}{\eta} \right|} \quad (|z| \leq r < 1; t \geq 0). \end{aligned}$$

Since the function $\mathfrak{A}(z)$ is convex and normalized in \mathbb{U} , we obtain

$$\begin{aligned} \frac{|\mathcal{L}(z; t)|}{|b_1(t)|} &\leq \frac{|\eta - 1|}{|\eta - 1 + t|} |\mathfrak{A}(z)| + \frac{|t|}{|\eta - 1 + t|} |z\mathfrak{A}'(z)| \\ &\leq |\mathfrak{A}(z)| + |z\mathfrak{A}'(z)| \leq \frac{r}{1-r} + \frac{r}{(1-r)^2} \\ &\leq \frac{r}{(1-r)^2} \quad (|z| \leq r < 1; t \geq 0). \end{aligned}$$

We can thus deduce the equality:

$$\Re \left(z \frac{\partial \mathcal{L} / \partial z}{\partial \mathcal{L} / \partial t} \right) = \Re(\eta) - 1 + t \Re \left(1 + \frac{z\mathfrak{A}''(z)}{\mathfrak{A}'(z)} \right).$$

By the inequalities $\Re(s(z)) > 0$ and $\Re(\eta) > 1$, the above relation yields

$$\Re \left(z \frac{\partial \mathcal{L} / \partial z}{\partial \mathcal{L} / \partial t} \right) > 0 \quad (\forall z \in \mathbb{U}; \forall t \geq 0).$$

Hence, the second assumptions of Lemma 2 hold true. Thus, the function $\mathcal{L}(z; t)$ is a subordination chain. Therefore, according to Lemma 4, we conclude that superordination (25) implies superordination (26). Furthermore, equation (26) has the univalent solution \mathfrak{A} , which is the best subordinant of the given differential superordination. This completes the proof of Theorem 2. \square

Combining the above results involving differential subordination and differential superordination, we state the following sandwich-type theorem.

Theorem 3 Suppose that η given by (7)

$$\eta = \frac{\ell p(a - \mu p)}{(p - \beta)\lambda(a - \mu p) + \mu\beta\ell}$$

is such that $\Re(\eta) > 1$ and that, for $f_j(z) \in \Sigma_p$ ($j = 1, 2$),

$$\chi_j(z) = \frac{z^{p+1}}{p} \left[(p - \beta)\mathfrak{S}_{p,\lambda}^{m+1,\ell}(a, c, \mu)f_j(z) + \beta\mathfrak{S}_{p,\lambda}^{m,\ell}(a + 1, c, \mu)f_j(z) \right]$$

satisfy the following condition:

$$\Re \left[1 + \frac{z\chi_j''(z)}{\chi_j'(z)} \right] > -v \quad (z \in \mathbb{U}),$$

where v is given by (10). If $g(z) \in \Sigma_p$, let

$$\frac{z^{p+1}}{p} \left[(p - \beta)\mathfrak{S}_{p,\lambda}^{m+1,\ell}(a, c, \mu)g(z) + \beta\mathfrak{S}_{p,\lambda}^{m,\ell}(a + 1, c, \mu)g(z) \right] \quad (z \in \mathbb{U}),$$

be univalent in \mathbb{U} and

$$z^{p+1}\mathfrak{S}_{p,\lambda}^{m,\ell}(a, c, \mu)g(z) \in \mathcal{H}[0, 1] \cap \mathcal{Q}.$$

Then the condition

$$\begin{aligned} \chi_1(z) &< \frac{z^{p+1}}{p} \left[(p - \beta) \mathfrak{L}_{p,\lambda}^{m+1,\ell}(a, c, \mu)g(z) + \beta \mathfrak{L}_{p,\lambda}^{m,\ell}(a + 1, c, \mu)g(z) \right] \\ &< \chi_2(z) \quad (z \in \mathbb{U}) \end{aligned} \tag{28}$$

implies that

$$\begin{aligned} z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f_1(z) &< z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)g(z) \\ &< z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f_2(z) \quad (z \in \mathbb{U}) \end{aligned} \tag{29}$$

and the functions $z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f_1(z)$ and $z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu)f_2(z)$ are the best subdominant and the best dominant of (28), respectively.

Corollary 1 Let $k \in \Sigma_p$ and suppose that

$$\begin{aligned} \phi(z) &= \frac{z^{p+1}}{p} \left(\left(p - \beta \left[1 - \frac{\mu\ell}{\lambda(a - \mu p)} \right] \right) \mathfrak{L}_p^{m+1}(\ell, \lambda)k(z) \right. \\ &\quad \left. + \beta \left[1 - \frac{\mu\ell}{\lambda(a - \mu p)} \right] \mathfrak{L}_p^m(\ell, \lambda)k(z) \right) \end{aligned} \tag{30}$$

satisfies the following condition:

$$\Re \left(1 + \frac{z\phi''(z)}{\phi'(z)} \right) > -v \quad (z \in \mathbb{U}),$$

where

- (1) if $\Re(\eta) = 1$, then $v = 0$;
- (2) if $\Re(\eta) > 1$, then v is given by (10) and (11).

If $f \in \Sigma_p$ and

$$\begin{aligned} \frac{z^{p+1}}{p} \left(\left(p - \beta \left[1 - \frac{\mu\ell}{\lambda(a - \mu p)} \right] \right) \mathfrak{L}_p^{m+1}(\ell, \lambda)f(z) \right. \\ \left. + \beta \left[1 - \frac{\mu\ell}{\lambda(a - \mu p)} \right] \mathfrak{L}_p^m(\ell, \lambda)f(z) \right) < \phi(z), \end{aligned} \tag{31}$$

then

$$z^{p+1} \mathfrak{L}_p^m(\ell, \lambda)f(z) < z^{p+1} \mathfrak{L}_p^m(\ell, \lambda)k(z) \quad (z \in \mathbb{U})$$

and the function $z^{p+1} \mathfrak{L}_p^m(\ell, \lambda)k(z)$ is the best dominant of (31).

Remark 2 Putting $\beta = 0$ in Corollary 1, we obtain the following consequence.

Corollary 2 Let $g, f \in \Sigma_p$ and suppose that

$$\psi_0(z) = z^{p+1} \mathfrak{L}_p^{m+1}(\ell, \lambda)g(z)$$

satisfies the condition:

$$\Re \left(1 + \frac{z\psi_0''(z)}{\psi_0'(z)} \right) > -\tau \quad (z \in \mathbb{U}),$$

where

(1) if $\frac{\ell}{\lambda} = 1$, then $\tau = 0$;

(2) if $\frac{\ell}{\lambda} > 1$, then

$$\tau \leq \begin{cases} \frac{\ell - \lambda}{2\lambda} & \left(1 < \frac{\ell}{\lambda} < 2 \right) \\ \frac{\lambda}{2(\ell - \lambda)} & \left(\frac{\ell}{\lambda} > 2 \right). \end{cases}$$

If

$$z^{p+1} \mathfrak{L}_p^{m+1}(\ell, \lambda)f(z) \prec \psi_0(z),$$

then

$$z^{p+1} \mathfrak{L}_p^m(\ell, \lambda)f(z) \prec z^{p+1} \mathfrak{L}_p^m(\ell, \lambda)g(z) \quad (z \in \mathbb{U})$$

and the function $z^{p+1} \mathfrak{L}_p^m(\ell, \lambda)g(z)$ is the best dominant.

Corollary 3 Let $k \in \Sigma_p$ and suppose that

$$\begin{aligned} \phi_1(z) = & \frac{z^{p+1}}{p} \left((p - \beta) \left[1 - \frac{\lambda(a - \mu p)}{\mu \ell} \right] \mathcal{J}_{p, \mu}^{a, c} k(z) \right. \\ & \left. + \left(p - (p - \beta) \left[1 - \frac{\lambda(a - \mu p)}{\mu \ell} \right] \right) \mathcal{J}_{p, \mu}^{a+1, c} k(z) \right) \end{aligned} \quad (32)$$

satisfies the following condition:

$$\Re \left(1 + \frac{z\phi_1''(z)}{\phi_1'(z)} \right) > -v \quad (z \in \mathbb{U}),$$

where

(1) if $\Re(\eta) = 1$, then $v = 0$;

(2) if $\Re(\eta) > 1$, then v is given by (10) with (11).

If $f \in \Sigma_p$ and

$$\begin{aligned} & \frac{z^{p+1}}{p} \left((p - \beta) \left[1 - \frac{\lambda(a - \mu p)}{\mu \ell} \right] \mathcal{J}_{p,\mu}^{a,c} f(z) \right. \\ & \left. + \left(p - (p - \beta) \left[1 - \frac{\lambda(a - \mu p)}{\mu \ell} \right] \right) \mathcal{J}_{p,\mu}^{a+1,c} f(z) \right) \prec \phi_1(z), \end{aligned} \tag{33}$$

then

$$z^{p+1} \mathcal{J}_{p,\mu}^{a,c} f(z) \prec z^{p+1} \mathcal{J}_{p,\mu}^{a,c} k(z) \quad (z \in \mathbb{U})$$

and the function $z^{p+1} \mathcal{J}_{p,\mu}^{a,c} k(z)$ is the best dominant of (33).

Remark 3 Putting $\beta = p$ in Corollary 3, we obtain the following result.

Corollary 4 Let $g, f \in \Sigma_p$ and suppose that

$$\psi(z) = z^{p+1} \mathcal{J}_{p,\mu}^{a+1,c} g(z)$$

satisfies the following condition:

$$\Re \left(1 + \frac{z\psi''(z)}{\psi'(z)} \right) > -\tau \quad (z \in \mathbb{U}),$$

where

(1) if $\Re \left(\frac{a}{\mu p} \right) = 2$, then $\tau = 0$;

(2) if $\Re \left(\frac{a}{\mu p} \right) > 2$, then

$$\tau \leq \begin{cases} \frac{\Re \left(\frac{a}{\mu p} \right) - 2}{2} & \left(2 < \Re \left(\frac{a}{\mu p} \right) < 3 \right) \\ \frac{1}{2 \left[\Re \left(\frac{a}{\mu p} \right) - 2 \right]} & \left(\Re \left(\frac{a}{\mu p} \right) > 3 \right) \end{cases}$$

and

$$\left[\Im \left(\frac{a}{\mu p} \right) \right]^2 \leq \left[\Re \left(\frac{a}{\mu p} \right) - 2 - 2\tau \right] \left[\frac{1}{\tau} - \Re \left(\frac{a}{\mu p} \right) + 2 \right].$$

The equality in the above equations holds true when $\Im(a) = 0$. Then

$$z^{p+1} \mathcal{J}_{p,\mu}^{a,c} f(z) \prec z^{p+1} \mathcal{J}_{p,\mu}^{a,c} g(z) \quad (z \in \mathbb{U})$$

and the function $z^{p+1} \mathcal{J}_{p,\mu}^{a,c} g(z)$ is the best dominant.

4. Subordination and superordination properties involving the integral operator $F_{\nu,p}$

In this section, we consider the integral operator $F_{\nu,p}$ defined by (see [22])

$$F_{\nu,p}f(z) = \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t) dt \quad (f(z) \in \Sigma_p; \nu > 0; p \in \mathbb{N}). \tag{34}$$

From equation (34), it is easily verified that

$$\begin{aligned} & z \left(\mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p}f(z) \right)' \\ &= \nu \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) f(z) - (\nu + p) \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p}f(z). \end{aligned} \tag{35}$$

By using (35), we can prove the following theorem.

Theorem 4 *Let $\nu > 0$ and $f_j \in \Sigma_p$ ($j = 1, 2$) and suppose that*

$$\psi_j(z) = z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) f_j(z) \quad (j = 1, 2), \tag{36}$$

satisfies the following condition:

$$\Re \left(1 + \frac{z\psi_j''(z)}{\psi_j'(z)} \right) > -\nu \quad (z \in \mathbb{U}; j = 1, 2),$$

where ν is given by

$$\nu \leq \begin{cases} \frac{\nu - 1}{2} & (1 < \nu \leq 2) \\ \frac{1}{2(\nu - 1)} & (\nu > 2). \end{cases} \tag{37}$$

For $g(z) \in \Sigma_p$, if we suppose that

$$z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) g(z) \quad (z \in \mathbb{U}),$$

is univalent in \mathbb{U} and that

$$z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p}g(z) \in \mathcal{H}[0, 1] \cap \mathcal{Q},$$

then the following condition:

$$\psi_1(z) \prec z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) g(z) \prec \psi_2(z) \quad (z \in \mathbb{U}), \tag{38}$$

implies that

$$\begin{aligned} & z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p}f_1(z) \prec z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p}g(z) \\ & \prec z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p}f_2(z) \quad (z \in \mathbb{U}) \end{aligned} \tag{39}$$

and the functions $z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p}f_1(z)$ and $z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p}f_2(z)$ are the best subordinant and the best dominant of (38), respectively.

Proof Let us set

$$G(z) = z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p} g(z) \tag{40}$$

and

$$K_j(z) = z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p} f_j(z) \quad (j = 1, 2). \tag{41}$$

From equation (35) in combination with (36), (40), and (41), we obtain

$$\nu \psi_j(z) = (\nu - 1) K_j(z) + z K_j'(z). \tag{42}$$

Putting

$$p_j(z) = 1 + \frac{z K_j''(z)}{K_j'(z)} \quad (j = 1, 2)$$

and differentiating equation (42), we obtain

$$1 + \frac{z \psi_j''(z)}{\psi_j'(z)} = p_j(z) + \frac{z p_j'(z)}{p_j(z) + \nu - 1}.$$

The remaining part of the proof of Theorem 4 is similar to that of Theorem 3 (a combined proof of Theorems 1 and 2) and we omit the details involved. □

Corollary 5 Let $\nu > 0$ and $f_j \in \Sigma_p$ ($j = 1, 2$) and suppose that

$$\psi_j(z) = z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) f_j(z) \quad (j = 1, 2) \tag{43}$$

satisfies the following condition:

$$\Re \left(1 + \frac{z \psi_j''(z)}{\psi_j'(z)} \right) > -\nu \quad (z \in \mathbb{U}),$$

where ν is given by (37). If $g(z) \in \Sigma_p$, then the condition:

$$\psi_1(z) \prec z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) g(z) \prec \psi_2(z) \quad (z \in \mathbb{U}) \tag{44}$$

implies that

$$\begin{aligned} z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p} f_1(z) &\prec z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p} g(z) \\ &\prec z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p} f_2(z) \quad (z \in \mathbb{U}) \end{aligned} \tag{45}$$

and the functions $z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p} f_1(z)$ and $z^{p+1} \mathfrak{L}_{p,\lambda}^{m,\ell}(a, c, \mu) F_{\nu,p} f_2(z)$ are the best subordinant and the best dominant of (44), respectively.

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