Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 2045 – 2060

doi:10.3906/mat-1711-53

On ordered hypersemigroups given by a table of multiplication and a figure

Niovi KEHAYOPULU*
Department of Mathematics, University of Athens, Panepistimiopolis, Greece

Received: 12.11.2017 • **Accepted/Published Online:** 23.05.2018 • **Final Version:** 24.07.2018

Abstract: The aim is to show that from every example of a regular, intraregular, left (right) regular, left (right) quasiregular, semisimple, left (right) simple, simple, or strongly simple ordered semigroup given by a table of multiplication and an order, a corresponding example of regular, intraregular, left (right) regular, left (right) quasiregular, semisimple, left (right) simple, simple, or strongly simple ordered hypersemigroup can be constructed having the same left (right) ideals, bi-ideals, quasi-ideals, or interior ideals. On this occasion, some further related results have also been given.

Key words: Ordered semigroup, ordered hypersemigroup, right (left) ideal, bi-ideal, quasi-ideal, interior ideal, regular, right (left) regular, intraregular, right (left) quasiregular, semisimple, right (left) simple, simple

1. Introduction

A very important subject in the theory of ordered hypersemigroups is the determination of right (left) ideals, bi-ideals, and quasi-ideals, which play an essential role in the investigation. This is because very often we need counterexamples that clearly are impossible to make by hand and it is difficult to write such programs as well. Examples of some types of ordered hypersemigroups (such as regular, intraregular, and completely regular) are also very useful, and again it is difficult to write programs. To overcome this difficulty we use examples that come from ordered semigroups for which we already have such programs. The examples of ordered semigroups given by a table of multiplication and an order play an essential role in it. The paper in [2; p. 104. l. 12-20] was on hypersemigroups (without order) and it has been proved in it that if $(S, \cdot, \leq)$ is an ordered semigroup and endow it with the hyperoperation $a \circ b := \{x \in S \mid x \leq ab\}$, then $(S, \circ)$ is a hypersemigroup. Here we prove that this is not only a hypersemigroup but an ordered hypersemigroup as well. It may be mentioned that if $(S, \cdot, \leq)$ is an ordered groupoid, then the hypergroupoid $(S, \circ)$ with the same order "$\leq$" of S is an ordered hypergroupoid. Thus, from every ordered groupoid (ordered semigroup) $(S, \cdot, \leq)$ given by a table of multiplication and an order, an ordered hypergroupoid (ordered hypersemigroup) $(S, \circ, \leq)$ can be constructed. Moreover, a set A is a right (left) ideal or quasi-ideal of the ordered groupoid $(S, \cdot, \leq)$ if and only if it is a right (left) ideal or quasi-ideal of the ordered hypergroupoid $(S, \circ, \leq)$; a set A is a bi-ideal or interior ideal of the ordered semigroup $(S, \cdot, \leq)$ if and only if it is a bi-ideal or interior ideal of the ordered hypersemigroup $(S, \circ, \leq)$. In addition, if an ordered groupoid $(S, \cdot, \leq)$ is left (resp. right) simple, then the ordered hypergroupoid $(S, \circ, \leq)$ is so. An ordered semigroup $(S, \cdot, \leq)$ is left (resp. right) simple or simple if

*Correspondence: nkehayop@math.uoa.gr
2010 *AMS Mathematics Subject Classification:* 06F99, 06F05

and only if the ordered hypersemigroup $(S,\circ,\leq)$ is so. An ordered semigroup $(S,\cdot,\leq)$ is regular, intraregular, left (right) regular, left (right) quasiregular, semisimple, simple, or strongly regular if and only if the ordered hypersemigroup $(S,\circ,\leq)$ is, respectively, so. Hence, from every example of a regular, intraregular, left (right) regular, left (right) quasiregular, semisimple, left (right) simple, simple, or strongly regular ordered semigroup given by a table of multiplication and an order, a corresponding example of an ordered hypersemigroup can be constructed having the same right (left) ideals, bi-ideals, quasi-ideals, and interior ideals. For the sake of completeness, we will give some definitions already given in [20].

2. Prerequisites

A groupoid is a nonempty set S with a binary operation (called multiplication) "$\cdot$" on S. An ordered groupoid (*po*-groupoid) is a groupoid $(S,\cdot)$ with an order relation "$\leq$" on S such that $a\leq b$ implies $ac\leq bc$ and $ca\leq cb$ for every $c\in S$. If the multiplication on S is associative, then $(S,\cdot,\leq)$ is called an ordered semigroup (*po*-semigroup) [1]. For an ordered groupoid $(S,\cdot,\leq)$ and a subset A of S, we denote by $(A]$ the subset of S defined by $(A]=\{t\in S\mid t\leq a \text{ for some } a\in A\}$ [6]. If $(S,\cdot,\leq)$ is an ordered groupoid, a nonempty subset A of S is called a right (left) ideal of S [6] if (1) $AS\subseteq A$ (resp. $SA\subseteq A$) and (2) if $a\in A$ and $S\ni b\leq a$, then $b\in A$ –condition (2) is equivalent to $(A]=A$; it is called a quasi-ideal of S if $(AS]\cap(SA]\subseteq A$ and (2) if $a\in A$ and $S\ni b\leq a$, then $b\in A$ [14]. A nonempty subset A of an ordered semigroup S is called a bi-ideal of S if (1) $ASA\subseteq A$ and (2) if $a\in A$ and $S\ni b\leq a$, then $b\in A$ [9]; an interior ideal of S if (1) $SAS\subseteq A$ and (2) if $a\in A$ and $S\ni b\leq a$, then $b\in A$ [17]. An ordered groupoid S is called right (resp. left) simple if S is the only right (resp. left) ideal of S; it is called simple if S is the only ideal of S. If S is an ordered groupoid satisfying the relation $(aS]=S$ for every $a\in S$ or $(AS]=S$ for every $A\subseteq S$, then S is right simple; if $(Sa]=S$ for every $a\in S$ or $(SA]=S$ for every $A\subseteq S$, then S is left simple. In particular, if S is an ordered semigroup, then S is right (resp. left) simple if and only if $(aS]=S$ for every $a\in S$ or $(AS]=S$ for every $A\subseteq S$ (resp. $(Sa]=S$ for every $a\in S$ or $(SA]=S$ for every $A\subseteq S$); it is simple if and only if $(SaS]=S$ for every $a\in S$, equivalently if $(SAS]=S$ for every $A\subseteq S$ [8]. If an ordered semigroup is right simple or left simple, then it is simple. An ordered semigroup $(S,\cdot,\leq)$ is called *regular* [10] if for every $a\in S$ there exists $x\in S$ such that $a\leq axa$, that is if $a\in(aSa]$ for every $a\in S$ or $A\subseteq(ASA]$ for every $A\subseteq S$; it is called left (resp. right) regular [7] if for every $a\in S$ there exists $x\in S$ such that $a\leq xa^2$ (resp. $a\leq a^2x$) that is, if $a\in(Sa^2]$ for every $a\in S$ or $A\subseteq(SA^2]$ for every $A\subseteq S$ (resp. $a\in(a^2S]$ for every $a\in S$ or $A\subseteq(A^2S]$ for every $A\subseteq S$); intraregular [11] if for every $a\in S$ there exist $x,y\in S$ such that $a\leq xa^2y$, equivalently $a\in(Sa^2S]$ for every $a\in S$ or $A\subseteq(SA^2S]$ for every $A\subseteq S$. An ordered semigroup $(S,\cdot,\leq)$ is called right quasiregular if for every $a\in S$ there exist $x,y\in S$ such that $a\leq axay$, equivalently for every $a\in S$ we have $a\in(aSaS]$ or $A\subseteq(ASAS]$ for every $A\subseteq S$; left quasiregular if for every $a\in S$ there exist $x,y\in S$ such that $a\leq xaya$, equivalently $a\in(SaSa]$ for every $a\in S$ or $A\subseteq(SASA]$ for every $A\subseteq S$; and semisimple if for every $a\in S$, there exist $x,y,z\in S$ such that $a\leq xayaz$, that is $a\in(SaSaS]$ for every $a\in S$ or $A\subseteq(SASAS]$ for every $A\subseteq S$ [18]. The right quasiregular and the left quasiregular ordered semigroups are semisimple. If an ordered semigroup is right simple and left simple, then it is regular [23]. An ordered semigroup $(S,\cdot,\leq)$ is called strongly regular [16] if for every $a\in S$ there exists $x\in S$ such that $a\leq axa$ and $ax=xa$.

We denote by $\mathcal{P}^*(S)$ the set of nonempty subsets of S; and the notation $A\preceq B$, where A,B are subsets of an ordered set $(S,\leq)$ means that for every $a\in A$ there exists $b\in B$ such that $a\leq b$. An *hypergroupoid* is a nonempty set S with an "operation"

$\circ : S \times S \to \mathcal{P}^*(S) \mid (a,b) \to a \circ b$

on S called hyperoperation (as it maps to each couple a, b of elements of S a nonempty subset $a \circ b$ of S) and an operation

$* : \mathcal{P}^*(S) \times \mathcal{P}^*(S) \to \mathcal{P}^*(S) \mid (A,B) \to A * B$

on $\mathcal{P}^*(S)$ (induced by the hyperoperation of S) such that

$A * B = \bigcup\limits_{(a,b) \in A \times B} (a \circ b)$ for every $A, B \in \mathcal{P}^*(S)$.

A hypergroupoid is called *hypersemigroup* if $\{x\} * (y \circ z) = (x \circ y) * \{z\}$ for every $x, y, z \in S$. If S is a hypergroupoid then, for every $x, y \in S$, we clearly have $\{x\} * \{y\} = x \circ y$. As the operation "$*$" depends on the hyperoperation "$\circ$", for a hypergroupoid S we use the notation $(S, \circ, \leq)$ instead of $(S, \circ, *, \leq)$.
The following two properties, though obvious, play an essential role in the investigation:

(1) If $x \in A * B$, then $x \in a \circ b$ for some $a \in A$, $b \in B$.

(2) If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.

In a hypergroupoid, $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$ for any $A, B, C \in \mathcal{P}^*(S)$. The operation "$*$" on a hypersemigroup S is associative, that is $(\mathcal{P}^*(S), *)$ is a semigroup. The concepts related to ordered groupoids (ordered semigroups) mentioned above are naturally transferred to ordered hypergroupoids (ordered hypersemigroups). A hypergroupoid is called an *ordered hypergroupoid* if there is an order relation "$\leq$" on S such that $a \leq b$ implies $a \circ c \preceq b \circ c$ and $c \circ a \preceq c \circ b$ for every $c \in S$ [3]. In an ordered hypersemigroup the symbol $(A]$ denotes the same set as in an ordered semigroup and has the same properties as in ordered semigroup. A nonempty subset A of an ordered hypergroupoid $(S, \circ, \leq)$ is called a *right* (resp. *left*) *ideal* of S if (1) $A * S \subseteq A$ (resp. $S * A \subseteq A$) and (2) if $a \in A$ and $S \ni t \leq a$, then $t \in A$, that is, if $(A] = A$; it is called an *ideal* of S if it is both a right ideal and a left ideal of S. A nonempty subset Q of an ordered hypergroupoid S is called a *quasi-ideal* of S if (1) $(Q * S] \cap (S * Q] \subseteq Q$ and (2) if $a \in Q$ and $S \ni t \leq a$, then $t \in Q$. A nonempty subset B of an ordered hypersemigroup S is called a *bi-ideal* of S if (1) $B * S * B \subseteq B$ and (2) if $a \in B$ and $S \ni t \leq a$, then $t \in B$; a nonepty subset A of S is called an *interior ideal* of S is (1) $S * A * S \subseteq A$ and (2) if $a \in A$ and $S \ni t \leq a$, then $t \in A$. An ordered hypersemigroup $(S, \circ, \leq)$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $\{a\} \preceq (a \circ x) * \{a\}$; it is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $\{a\} \preceq (a \circ a) * \{x\}$; *left regular* if for every $a \in S$ there exists $x \in S$ such that $\{a\} \preceq \{x\} * (a \circ a)$; *intraregular* if for every $a \in S$ there exist $x, y \in S$ such that $\{a\} \preceq (x \circ a) * (a \circ y)$. As is an ordered semigroup, an ordered hypersemigroup is regular, for example, if and only if $a \in (\{a\} * S * \{a\}]$ for every $a \in S$ or $A \subseteq (A * S * A]$ for every nonempty subset A of S. The other type of ordered hypersemigroups mentioned above can also be characterized in a similar way.

3. Main results

Lemma 1. (cf. also [2; p. 104, l. 12-20]) *Let $(S, \cdot, \leq)$ be an ordered groupoid. We consider the hyperoperation "$\circ$" on S defined by*

$$\circ : S \times S \to \mathcal{P}^*(S) \mid (a,b) \to a \circ b, \text{ where}$$

$$a \circ b := \{x \in S \mid x \leq ab\}.$$

Then $(S, \circ, \leq)$ *is an ordered hypergroupoid. If the multiplication of the groupoid* $(S, \cdot, \leq)$ *is associative, then* $(S, \circ)$ *is a hypersemigroup.*

In the following we denote by $(S, \cdot, \leq)$ the ordered groupoid and by $(S, \circ, \leq)$ the ordered hypergroupoid constructed in Lemma 1.

Corollary 2. *If* $(S, \cdot, \leq)$ *is an ordered semigroup, then* $(S, \circ, \leq)$ *is an ordered hypersemigroup.*

Theorem 3. *A set* A *is a right (left) ideal, ideal, or quasi-ideal of an ordered groupoid* $(S, \cdot, \leq)$ *if and only if* A *is a right (left) ideal, ideal or quasi-ideal, respectively, of the ordered hypergroupoid* $(S, \circ, \leq)$*. In particular, if* $(S, \cdot, \leq)$ *is an ordered semigroup, then* A *is a bi-ideal or interior ideal of* $(S, \cdot, \leq)$ *if and only if* A *is a bi-ideal or interior ideal of* $(S, \circ, \leq)$*.*

Proof Let A be a right ideal of $(S, \cdot, \leq)$. If $t \in A * S$ then $t \in a \circ x$ for some $a \in A$, $x \in S$, then $t \leq ax \in AS \subseteq A$ and so $t \in A$; thus A is a right ideal of $(S, \circ, \leq)$. Similarly, if A is a left ideal of $(S, \cdot, \leq)$, then A is a left ideal of $(S, \circ, \leq)$. As a consequence, if A is an ideal of $(S, \cdot, \leq)$, then A is an ideal of $(S, \circ, \leq)$. Let A be a right ideal of $(S, \circ, \leq)$. If $t \in AS$, then $t = ax$ for some $a \in A$, $x \in S$. We have $ax \in a \circ x \subseteq A * S \subseteq A$ and so $t \in A$. Similarly, if A is a left ideal of $(S, \circ, \leq)$, then A is a left ideal of $(S, \cdot, \leq)$. Hence, if A is an ideal of $(S, \circ, \leq)$, then A is an ideal of $(S, \cdot, \leq)$.

Let Q be a quasi-ideal of $(S, \cdot, \leq)$ and $t \in (Q * S] \cap (S * Q]$. Since $t \in (Q * S]$, we have $t \leq x$ for some $x \in Q * S$; since $t \in (S * Q]$, we have $t \leq y$ for some $y \in S * Q$. Then we have $x \in q \circ s$ for some $q \in Q$, $s \in S$, and $y \in c \circ d$ for some $c \in S$, $d \in Q$. Then we have $t \leq x \leq qs \in QS$ and $t \leq y \leq cd \in SQ$ and then $t \in (QS] \cap (SQ] \subseteq Q$. Thus we have $(Q * S] \cap (S * Q] \subseteq Q$ and so Q is a quasi-ideal of $(S, \circ, \leq)$. Let Q be a quasi-ideal of $(S, \circ, \leq)$ and $t \in (QS] \cap (SQ]$. Since $t \in (QS]$, we have $t \leq qs$ for some $q \in Q$, $s \in S$. Since $t \in SQ$, we have $t \leq cd$ for some $c \in S$, $d \in Q$. Then we have $t \in q \circ s \in Q * S \subseteq (Q * S]$ and $t \in c \circ d \in S * Q \subseteq (S * Q]$. Then $t \in (Q * S] \cap (S * Q] \subseteq Q$ and so Q is a quasi-ideal of $(S, \cdot, \leq)$.

Let A be a bi-ideal of $(S, \cdot, \leq)$ and $t \in (A * S) * A$. Then $t \in x \circ y$ for some $x \in A * S$, $y \in A$ and $x \in a \circ s$ for some $a \in A$, $s \in S$. Then we have $t \leq xy \leq (as)y \in ASA \subseteq A$, and $t \in A$. Thus we have $A * S * A \subseteq A$ and so A is a bi-ideal of $(S, \circ, \leq)$. Let A be a bi-ideal of $(S, \circ, \leq)$ and $t \in ASA$. Then $t = (as)b$ for some $a, b \in A$, $s \in S$. Since $(as)b \leq (as)b$, we have $(as)b \in (as) \circ b = \{as\} * \{b\}$. Since $as \leq as$, we have $as \in a \circ s$, that is $\{as\} \subseteq a \circ s$. Thus we have $t = (as)b \in (a \circ s) * \{b\} = \{a\} * S * \{b\} \subseteq A * S * A \subseteq A$; then $t \in A$ and so A is a bi-ideal of $(S, \cdot, \leq)$.

Let now A be an interior ideal of $(S, \cdot, \leq)$ and $t \in (S * A) * S$. Then $t \in x \circ y$ for some $x \in S * A$, $y \in S$ and $x \in s \circ a$ for some $s \in S$, $a \in A$. Then we have $t \leq xy \leq (sa)y \in SAS \subseteq A$; then $t \in A$. Thus we have $S * A * S \subseteq A$, and A is an interior ideal of $(S, \circ, \leq)$. Finally, let A be an interior ideal of $(S, \circ, \leq)$ and $t \in SAS$. Then $t = (xa)y$ for some $x, y \in S$, $a \in A$. Then $(xa)y \in (xa) \circ y \subseteq (x \circ a) * \{y\} = \{x\} * \{a\} * \{y\} \subseteq S * A * S \subseteq A$; then $t \in A$ and so A is an interior ideal of $(S, \cdot, \leq)$. □

If S is an ordered hypergroupoid, an element $A \in \mathcal{P}^*(S)$ is called *proper* if $A \neq S$; it is called *subidempotent* if $(A * A] \subseteq A$.

Definition 4. An ordered hypergroupoid S is called *right* (resp. *left*) *simple* if S does not contain proper right (resp. left) ideals, that is if A is a right (resp. left) ideal of S, then $A = S$; it is called *simple* if S does not contain proper ideals, that is if A is an ideal of S, then $A = S$.

If S is an ordered hypergroupoid such that $(\{a\} * S] = S$ for every $a \in S$ or $(A * S] = S$ for every

$A \in \mathcal{P}^*(S)$, then S is right simple; if $(S * \{a\}] = S$ for every $a \in S$ or $(S * A] = S$ for every $A \in \mathcal{P}^*(S)$, then S is left simple. An ordered hypersemigroup S is simple if and only if $(S * \{a\} * S] = S$ for every $a \in S$, equivalently if $(S * A * S] = S$ for every $A \in \mathcal{P}^*(S)$; it is right simple if and only if $(\{a\} * S] = S$ for every $a \in S$, equivalently if $(A * S] = S$ for every $A \in \mathcal{P}^*(S)$; left simple if and only if $(S * \{a\}] = S$ for every $a \in S$, equivalently if $(S * A] = S$ for every $A \in \mathcal{P}^*(S)$; and an ordered hypersemigroup that is right simple or left simple, it is simple.

Remark 5. The results on left and right simple ordered semigroups considered in [23] can be naturally transferred to ordered hypersemigoups and Proposition 1 in it can be proved using only sets. In fact, if S is a hypersemigroup that is both right and left simple and A a bi-ideal of S then, since $(A * S] = S$ and $(S * A] = S$, we have $S = (S * A] = ((A * S] * A] = ((A * S) * A] = (A * S * A] \subseteq (A] = A$; then $A = S$ and so S does not contain proper bi-ideals. Conversely, if an ordered hypersemigroup S does not contain proper subidempotent bi-ideals and A is a left ideal of S then, since $S * A \subseteq A$, we have $A * (S * A) \subseteq A * A \subseteq S * A \subseteq A$ and $(A * A] \subseteq (A] = A$ and so A is a subidempotent bi-ideal of S; by hypothesis, we have $A = S$ and so S is left simple. Similarly S is right simple. On the other hand, if an ordered hypersemigroup is left and right simple, then it is regular, and in regular ordered hypersemigroups the bi-ideals and the subidempotent bi-ideals are the same.

Theorem 6. *If an ordered groupoid $(S, \cdot, \leq)$ is left (resp. right) simple, then the ordered hypergroupoid $(S, \circ, \leq)$ is so. An ordered semigroup $(S, \cdot, \leq)$ is left (resp. right) simple or simple if and only if the ordered hypersemigroup $(S, \circ, \leq)$ is so.*

Proof Let $(S, \cdot, \leq)$ be left simple and $a, b \in S$. Since $(Sa] = S$, we have $b \leq xa$ for some $x \in S$. Then we have $b \in x \circ a = \{x\} * \{a\} \subseteq S * \{a\} \subseteq (S * \{a\}]$; then $S = (S * \{a\}]$ and so $(S, \circ, \leq)$ is left simple. Let now $(S, \circ, \leq)$ be a left simple ordered hypersemigroup and $a, b \in S$. Since $(S * \{a\}] = S$, we have $b \leq t$ for some $t \in S * \{a\}$. Then $t \in x \circ a$ for some $x \in S$ and so $t \leq xa$. We have $b \leq t \leq xa \in Sa$; thus $b \in (Sa]$ and so $(S, \cdot, \leq)$ is left simple.

Let $(S, \cdot, \leq)$ be a simple ordered semigroup and $a, b \in S$. Then we have $(SaS] = S$ and so $b \leq (xa)y$ for some $x, y \in S$; thus $b \in (xa) \circ y$. Since $xa \in x \circ a$, we have $b \in (xa) \circ y \subseteq (x \circ a) * \{y\} = \{x\} * \{a\} * \{y\} \subseteq S * \{a\} * S \subseteq (S * \{a\} * S]$; then $S = (S * \{a\} * S]$ and so $(S, \circ, \leq)$ is simple. Conversely, let $(S, \circ, \leq)$ be simple and $a, b \in S$. Then $(S * \{a\} * S] = S$ and $b \leq t$ for some $t \in (S * \{a\}) * S$. Then we have $t \in u \circ s$ for some $u \in S * \{a\}$, $s \in S$, and $u \in v \circ a$ for some $v \in S$. Then $b \leq t \leq us \leq (va)s = vas \in SaS$, $a \in (SaS]$, and $S = (SaS]$ and so $(S, \cdot, \leq)$ is simple. □

Definition 7. An ordered hypersemigroup S is called *left quasiregular* if for every $a \in S$ there exist $x, y \in S$ such that $\{a\} \preceq (x \circ a) * (y \circ a)$. It is called *right quasiregular* if for every $a \in S$ there exist $x, y \in S$ such that $\{a\} \preceq (a \circ x) * (a \circ y)$.

Definition 8. An ordered hypersemigroup S is called *semisimple* if for every $a \in S$ there exist $x, y, z \in S$ such that $\{a\} \preceq (x \circ a) * (y \circ a) * \{z\}$.

Exactly as in ordered semigroups, the right quasiregular and the left quasiregular ordered hypersemigroups are semisimple.

Theorem 9. *An ordered semigroup $(S, \cdot, \leq)$ is regular, right (left) regular, intraregular, right (left) quasiregular, or semisimple if and only if the ordered hypersemigroup $(S, \circ, \leq)$ is, respectively, so.*

Proof Let $(S,\cdot,\leq)$ be regular and $a\in S$. Then there exists $x\in S$ such that $a\leq (ax)a$. Then $a\in (ax)\circ a$. Since $ax\in a\circ x$, we have $ax\circ a\subseteq (a\circ x)*\{a\}$. Since $a\in (a\circ x)*\{a\}$ and $a\leq a$, we have $\{a\}\preceq (a\circ x)*\{a\}$ and so $(S,\circ,\leq)$ is regular. Let $(S,\circ,\leq)$ be regular and $a\in S$. Then there exists $x\in S$ such that $\{a\}\preceq (a\circ x)*\{a\}$. That is, there exist $x,t\in S$ such that $t\in (a\circ x)*\{a\}$ and $a\leq t$. Then $t\in u\circ a$ for some $u\in a\circ x$, from which $t\leq ua$ and $u\leq ax$. Thus we get $a\leq t\leq (ax)a=axa$ and so $(S,\cdot,\leq)$ is regular.

Let $(S,\cdot,\leq)$ be right regular and $a\in S$. Then there exists $x\in S$ such that $a\leq a^2x$. Since $a\leq (aa)x$, we have $a\in aa\circ x$. Since $aa\in a\circ a$, we have $aa\circ x\subseteq (a\circ a)*\{x\}$. Thus we have $a\in (a\circ a)*\{x\}$ and $a\leq a$; then $\{a\}\preceq (a\circ a)*\{x\}$ and so $(S,\circ,\leq)$ is right regular. Let $(S,\circ,\leq)$ be right regular and $a\in S$. Then there exists $x\in S$ such that $\{a\}\preceq (a\circ a)*\{x\}$. Then $a\in u\circ x$ for some $u\in a\circ a$. Since $a\leq ux$ and $u\leq a^2$, we have $a\leq a^2x$ and so $(S,\cdot,\leq)$ is right regular. Similarly $(S,\cdot,\leq)$ is left regular if and only if $(S,\circ,\leq)$ is so.

Let $(S,\cdot,\leq)$ be intraregular and $a\in S$. Then there exist $x,y\in S$ such that $a\leq xa^2y=(xa)(ay)$. Then $a\in (xa)\circ(ay)$ and, since $xa\in x\circ a$ and $ay\in a\circ y$, we have $xa\circ ay\subseteq (x\circ a)*(a\circ y)$. Since $a\in (x\circ a)*(a\circ y)$ and $a\leq a$, then $\{a\}\preceq (x\circ a)*(a\circ y)$, and $(S,\circ,\leq)$ is intraregular. Let $(S,\circ,\leq)$ be intraregular and $a\in S$. Then there exist $x,y,t\in S$ such that $t\in (x\circ a)*(a\circ y)$ and $a\leq t$. Then $t\in u\circ v$ for some $u\in x\circ a$, $v\in a\circ y$ and so $t\leq uv$, $u\leq xa$, and $v\leq ay$. Then we have $a\leq t\leq (xa)(ay)=xa^2y$ and so $(S,\cdot,\leq)$ is intraregular.

Let $(S,\cdot,\leq)$ be right quasiregular and $a\in S$. Then there exist $x,y\in S$ such that $a\leq (ax)(ay)$. Then $a\in (ax)\circ(ay)\subseteq (a\circ x)*(a\circ y)$ and so $a\in (a\circ x)*(a\circ y)$; then $\{a\}\preceq (a\circ x)*(a\circ y)$ and $(S,\circ,\leq)$ is right quasiregular. Let $(S,\circ,\leq)$ be right quasiregular and $a\in S$. Then there exist $x,y\in S$ such that $\{a\}\preceq (a\circ x)*(a\circ y)$. Then there exists $t\in S$ such that $t\in (a\circ x)*(a\circ y)$ and $a\leq t$. Then we have $t\in u\circ v$ for some $u\in a\circ x$, $v\in a\circ y$ and then $t\leq uv$, $u\leq ax$, and $v\leq ay$. Hence $a\leq t\leq (ax)(ay)=axay$, and $(S,\cdot,\leq)$ is right quasiregular. In a similar way, $(S,\cdot,\leq)$ is left quasiregular if and only if $(S,\circ,\leq)$ is so.

Let $(S,\cdot,\leq)$ be semisimple and $a\in S$. Then there exist $x,y,z\in S$ such that $a\leq (xa)(yaz)$. Then we have $a\in (xa)\circ(yaz)$. We also have $xa\in x\circ a$ and $(ya)z\in (ya)\circ z\subseteq (y\circ a)*\{z\}$. We have $a\in (x\circ a)*(y\circ a)*\{z\}$ and $a\leq a$ and so we have $\{a\}\preceq (x\circ a)*(y\circ a)*\{z\}$, and $(S,\circ,\leq)$ is semisimple. Let now $(S,\circ,\leq)$ be semisimple and $a\in S$. Then there exist $x,y,z,t\in S$ such that $t\in ((x\circ a)*(y\circ a))*\{z\}$ and $a\leq t$. Then $t\in u\circ z$ for some $u\in (x\circ a)*(y\circ a)$ and $u\in v\circ w$ for some $v\in x\circ a$, $w\in y\circ a$. We have $t\leq uz$, $u\leq vw$, $v\leq xa$, and $w\leq ya$. Then we have $a\leq t\leq uz\leq (vw)z\leq (xa)(ya)z=xayaz$ and so $(S,\cdot,\leq)$ is semisimple. □

Definition 10. An ordered hypersemigroup $(S,\circ,\leq)$ is called *strongly regular* if for every $a\in S$ there exists $x\in S$ such that $\{a\}\preceq (a\circ x)*\{a\}$ and $a\circ x=x\circ a$.

Lemma 11. *Let $(S,\cdot,\leq)$ be an ordered hypergroupoid and $a,x\in S$. Then $ax=xa$ if and only if $a\circ x=x\circ a$.*

Proof $\Longrightarrow$. Let $ax=xa$. If $t\in a\circ x$, then $t\leq ax=xa$, that is $t\in x\circ a$ and so $a\circ x\subseteq x\circ a$. If $t\in x\circ a$, then $t\leq xa=ax$, so $t\in a\circ x$ and $x\circ a\subseteq a\circ x$ and so $a\circ x=x\circ a$.
$\Longleftarrow$. Let $a\circ x=x\circ a$. Since $ax\in a\circ x=x\circ a$, we have $ax\leq xa$. Since $xa\in x\circ a=a\circ x$, we have $xa\leq ax$. Thus we have $ax=xa$. □

Theorem 12. *An ordered semigroup $(S,\cdot,\leq)$ is strongly regular if and only if the ordered hypersemigroup $(S,\circ,\leq)$ is so.*

As application of the theorems of the paper we give the following examples.

Example 13. (cf. [13; Example 1]) The set $S=\{a,b,c,d,f\}$ with the multiplication "$\cdot$" given by Table 1 and the order "$\leq$" below is an example of an intraregular ordered semigroup.

Table 1. Multiplication table of Example 13.

$\cdot$	a	b	c	d	f
a	b	a	a	a	a
b	a	b	b	b	b
c	a	b	b	b	b
d	a	b	b	d	d
f	a	b	c	d	f

$$\leq := \{(a,a),(b,b),(c,b),(c,c),(d,d),(f,d),(f,f)\}.$$

The covering relation "$\prec$" of $(S,\cdot,\leq)$ is the following:

$$\prec = \{(c,b),(f,d)\};$$

and the figure of $(S,\cdot,\leq)$ is given by Figure 1.

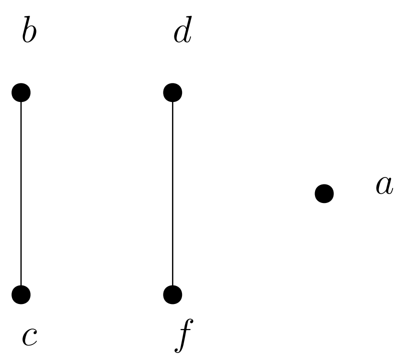

Figure 1. Figure corresponding to the order of Example 13.

The right, left ideals, bi-ideals, and quasi-ideals of $(S,\cdot,\leq)$ are the same and they are the sets $\{a,b,c\}$ and S.

The hypersemigroup $(S,\circ,\leq)$ that corresponds to the ordered semigroup $(S,\cdot,\leq)$ is given by Table 2 of the hyperoperation and the same order as $(S,\cdot,\leq)$.

Table 2. The hyperoperation of Example 13.

$\circ$	a	b	c	d	f
a	$\{b,c\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
c	$\{a\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
d	$\{a\}$	$\{b,c\}$	$\{b,c\}$	$\{d,f\}$	$\{d,f\}$
f	$\{a\}$	$\{b,c\}$	$\{c\}$	$\{d,f\}$	$\{f\}$

According to Theorems 3 and 9, the ordered hypersemigroup $(S,\circ,\leq)$ is intraregular and has the same right, left ideals, bi-ideals, and quasi-ideals as the ordered semigroup $(S,\cdot,\leq)$.

Example 14. (cf. also [15; the Example]) The set $S = \{a, b, c, d, f\}$ with the multiplication "$\cdot$" given by Table 3 and the order below is an example of a regular ordered semigroup.

Table 3. Multiplication table of Example 14.

$\cdot$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,f),(b,b),(c,c),(d,d),(f,f)\}.$$

The covering relation "$\prec$" of $(S, \cdot, \leq)$ is the following:

$$\prec = \{a,b),(a,c),(a,d),(a,f)\};$$

and its figure is given by Figure 2.

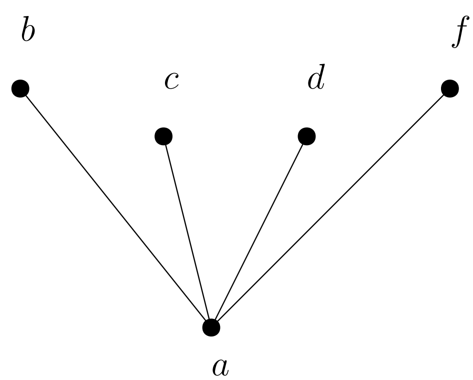

Figure 2. Figure corresponding to the order of Example 14.

The right ideals of $(S, \cdot, \leq)$ are the sets: $\{a\}$, $\{a, b, d\}$, $\{a, c, f\}$, and S.
The left ideals of $(S, \cdot, \leq)$ are the sets: $\{a\}$, $\{a, c, d\}$, $\{a, b, f\}$, and S.
The bi-ideals and the quasi-ideals of $(S, \cdot, \leq)$ are the same and they are the following: $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, f\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{a, b, f\}$, $\{a, c, f\}$, and S.

The ordered hypersemigroup $(S, \circ, \leq)$ that corresponds to $(S, \cdot, \leq)$ is given by Table 4.

According to Theorems 3 and 9, the ordered hypersemigroup $(S, \circ, \leq)$ is regular and has the same right ideals, left ideals, bi-ideals, and quasi-ideals as the ordered semigroup $(S, \cdot, \leq)$.

Example 15. (cf. also [12; Example 1]) The set $S = \{a, b, c, d, e\}$ with the multiplication "$\cdot$" given by Table 5 and the order "$\leq$" below is an ordered semigroup that is regular, right regular, left regular, and intraregular.

Table 4. The hyperoperation of Example 14.

$\circ$	a	b	c	d	f
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,d\}$	$\{a\}$
c	$\{a\}$	$\{a,f\}$	$\{a,c\}$	$\{a,c\}$	$\{a,f\}$
d	$\{a\}$	$\{a,b\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b\}$
f	$\{a\}$	$\{a,f\}$	$\{a\}$	$\{a,c\}$	$\{a\}$

Table 5. Multiplication table of Example 15.

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\begin{aligned}\leq: \;\; = \;\; & \{(a,a),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,e),\\ & (d,d),(d,e),(e,e)\}\end{aligned}$$

The covering relation of $(S,\cdot,\leq)$ is the following:

$$\prec = \{(a,c),(a,d),(c,e),(b,d),(d,e)\};$$

and its figure is given by Figure 3.

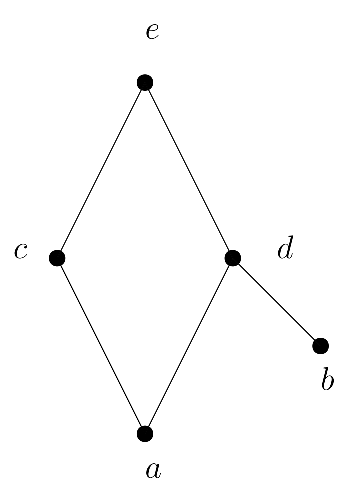

Figure 3. Figure corresponding to the order of Example 15.

The right ideals of $(S,\cdot,\leq)$ are the sets: $\{a,b,d\}$ and S.
The left ideals of $(S,\cdot,\leq)$ are the sets: $\{a\},\{a,c\},\{a,b,d\},\{a,b,c,d\}$, and S.

The bi-ideals and the quasi-ideals of $(S, \cdot, \leq)$ coincide with the left ideals of $(S, \cdot, \leq)$.
The hypersemigroup $(S, \circ, \leq)$ that corresponds to the ordered semigroup
$(S, \cdot, \leq)$ is given by Table 6.

Table 6. The hyperoperation of Example 15.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a,b,d\}$	$\{a\}$	$\{a,b,d\}$	$\{a,b,d\}$
b	$\{a\}$	$\{b\}$	$\{a\}$	$\{a,b,d\}$	$\{a,b,d\}$
c	$\{a\}$	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	S
d	$\{a\}$	$\{a,b,d\}$	$\{a\}$	$\{a,b,d\}$	$\{a,b,d\}$
e	$\{a\}$	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	S

According to Theorems 3 and 9, the ordered hypersemigroup $(S, \circ, \leq)$ is regular, right regular, left regular, and intraregular and the right (left) ideals, bi-ideals, and quasi-ideals of $(S, \circ, \leq)$ are the same as the right (left) ideals, bi-ideals, and quasi-ideals of $(S, \cdot, \leq)$.

Example 16. (cf. also [9; the Example]) We consider the ordered semigroup $S = \{a, b, c, d, e\}$ defined by the multiplication "$\cdot$" given by Table 7 and the order "$\leq$" below.

Table 7. Multiplication table of Example 16.

$\cdot$	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$$\begin{aligned}\leq: \quad = \quad & \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),\\ & (c,c),(c,e),(d,d),(d,e),(e,e)\}.\end{aligned}$$

We give the covering relation of $(S, \cdot, \leq)$; and its figure is given by Figure 4.

$$\prec = \{(a,b),(b,c),(b,d),(c,e),(d,e)\}.$$

This is a simple ordered semigroup, as $(SaS] = S$ for every $a \in S$; it is also right quasiregular and left quasiregular, and so semisimple as well.
The right ideals of $(S, \cdot, \leq)$ are the sets: $\{a, b, c\}$ and S.
The left ideals of $(S, \cdot, \leq)$ are the sets: $\{a, b, d\}$ and S.
The bi-ideals of $(S, \cdot, \leq)$ are the sets: $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$, and S.
The quasi-ideals of $(S, \cdot, \leq)$ are the sets: $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$, and S.

The ordered hypersemigroup $(S, \circ, \leq)$ that corresponds to the ordered semigroup $(S, \cdot, \leq)$ is given by Table 8.

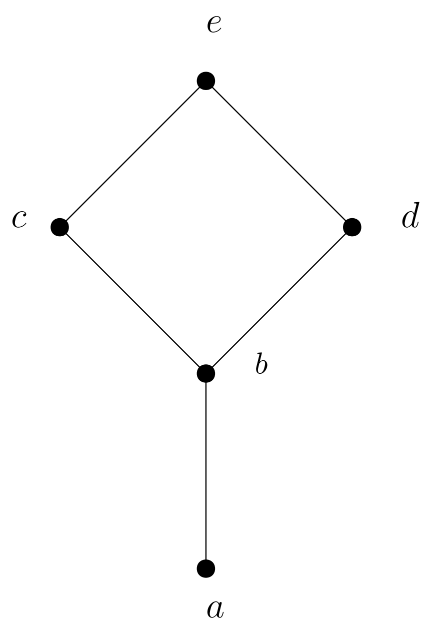

Figure 4. Figure corresponding to the order of Example 16.

Table 8. The hyperoperation of Example 16.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a\}$	$\{a,b,c\}$
b	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a\}$	$\{a,b,c\}$
c	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a\}$	$\{a,b,c\}$
d	$\{a,b,d\}$	$\{a,b,d\}$	S	$\{a,b,d\}$	S
e	$\{a,b,d\}$	$\{a,b,d\}$	S	$\{a,b,d\}$	S

According to Theorems 3, 6, and 9, the ordered hypersemigroup $(S,\circ,\leq)$ is also simple, right quasiregular, and left quasiregular, and has the same right, left ideals, ideals, bi-ideals, and quasi-ideals as the ordered semigroup $(S,\cdot,\leq)$.

Example 17. (cf. also [16; the Example]) The set $S=\{a,b,c,d,e,f\}$ with the multiplication given by Table 9 and the order defined by Figure 5 is a strongly regular ordered semigroup.

Table 9. Multiplication table of Example 17.

$\cdot$	a	b	c	d	e	f
a	b	c	d	d	d	d
b	c	d	d	d	d	d
c	d	d	d	d	d	d
d	d	d	d	d	d	d
e	e	e	e	e	e	e
f	f	f	f	f	f	f

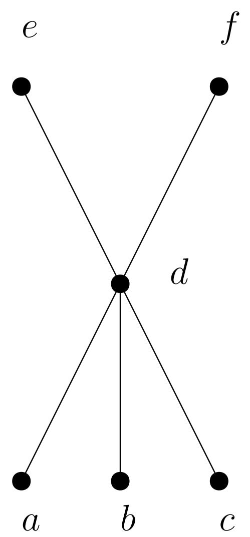

Figure 5. Figure that shows the order of Example 17.

The right ideals of $(S,\cdot,\le)$ are the sets: $\{a,b,c,d\}$, $\{a,b,c,d,e\}$, $\{a,b,c,d,f\}$, and S. The only left ideal of $(S,\cdot,\le)$ is the set S itself. The bi-ideals and the quasi-ideals of $(S,\cdot,\le)$ coincide with the right ideals of $(S,\cdot,\le)$.

The hypersemigroup $(S,\circ,\le)$ that corresponds to the above ordered semigroup is given by Table 10.

Table 10. The hyperoperation of Example 17.

$\circ$	a	b	c	d	e	f
a	$\{b\}$	$\{c\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
b	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
c	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
d	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
e	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$
f	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$	$\{a,b,c,d,f\}$

According to Theorems 3 and 12, $(S,\circ,\le)$ is a strongly regular ordered hypersemigroup having the same right (left), bi-ideals, and quasi-ideals as the ordered semigroup $(S,\cdot,\le)$.

Example 18. (cf. also [22; Theorem 1]) Let us consider the ordered semigroup $(S,\cdot,\le)$ defined by Table 11 and Figure 6.

This is right regular and so intraregular as well, not left regular, right quasiregular, not left quasiregular, semisimple, not regular, right simple, not left simple, not simple, not strongly regular.
The right ideals of $(S,\cdot,\le)$ are the sets: $\{a,b,c,d\}$, $\{a,b,c,d,f\}$, and S.
The left ideals of $(S,\cdot,\le)$ are the sets: $\{a\}$, $\{a,b\}$, $\{a,b,c,d\}$, $\{a,b,c,d,f\}$, and S. The bi-ideals and the quasi-ideals of $(S,\cdot,\le)$ are the same as the left ideals of $(S,\cdot,\le)$.

Table 11. Multiplication table of Example 18.

$\cdot$	a	b	c	d	e	f
a	a	a	a	d	d	d
b	a	a	b	d	d	d
c	a	a	c	d	d	d
d	a	a	d	d	d	d
e	a	a	d	d	e	f
f	a	a	d	d	e	f

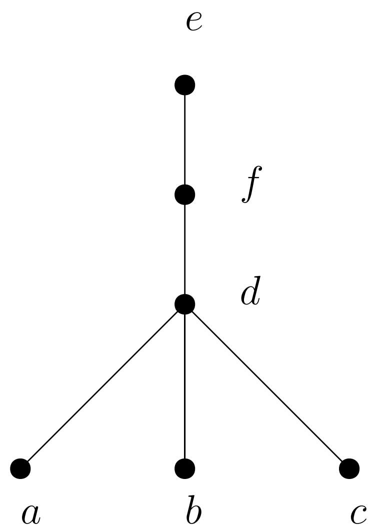

Figure 6. Figure that shows the order of Example 18.

The ordered hypersemigroup $(S, \circ, \leq)$ that corresponds to the ordered semigroup $(S, \cdot, \leq)$ is given by Table 12. It is of the same type as $(S, \cdot, \leq)$ and has the same right, left ideals, bi-ideals, and quasi-ideals as the ordered semigroup $(S, \cdot, \leq)$.

Table 12. The hyperoperation of Example 18.

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
b	$\{a\}$	$\{a\}$	$\{b\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
c	$\{a\}$	$\{a\}$	$\{c\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
d	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
e	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	S	$\{a,b,c,d,f\}$
f	$\{a\}$	$\{a\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	S	$\{a,b,c,d,f\}$

In the above examples the ordered hypersemigroup $(S, \circ, \leq)$ has the same interior ideals as the ordered semigroup $(S, \cdot, \leq)$. One can write a program to find the interior ideals of $(S, \cdot, \leq)$.

Remark 19. By a *poe*-semigroup we mean an ordered semigroup (: *po*-semigroup) having a greatest element "e" (i.e. $e \geq a$ for all $a \in S$). If we have a *poe*-semigroup given by a table of multiplication and a figure and we want to check if it is regular, left (right) regular, intraregular, left (right) quasiregular or semisimple, then we can easily check—by hand—if for every $a \in S$ we have $a \leq aea$, $a \leq ea^2$ ($a \leq a^2e$), $a \leq ea^2e$, $a \leq eaea$ ($a \leq aeae$), or $a \leq eaeae$, respectively (cf. also [4]).

The following question is natural: What about the filters? If $(S, \cdot, \leq)$ is an ordered semigroup and F a filter of $(S, \cdot, \leq)$, then is F a filter of the hypersemigroup $(S, \circ, \leq)$? The following theorem gives the answer.

For the sake of completeness, let us first give the definition of the filter in an ordered semigroup and in an ordered hypersemigroup. If $(S, \cdot, \leq)$ is an ordered semigroup, a nonempty subset F of S is called a filter of S [5] if the following hold: (1) if $a, b \in F$, then $ab \in F$; (2) if $a, b \in S$ such that $ab \in F$, then $a \in F$ and $b \in F$, and (3) if $a \in F$ and $S \ni b \geq a$, then $b \in F$. A nonempty subset F of $(S, \circ, \leq)$ is called a filter of S if the following hold: (1) if $a, b \in F$, then $a \circ b \subseteq F$; (2) if $a, b \in S$ such that $a \circ b \subseteq F$, then $a \in F$ and $b \in F$; (3) for every $a, b \in S$, we have $a \circ b \subseteq F$ or $(a \circ b) \cap F = \emptyset$, and (4) if $a \in F$ and $S \ni b \geq a$, then $b \in F$ [19].

Theorem 20. *If F is a filter of $(S, \circ, \leq)$, then it is a filter of $(S, \cdot, \leq)$. The converse statement does not hold in general.*

Proof Let F be a filter of $(S, \circ, \leq)$ and $a, b \in F$. Then we have $a \circ b \subseteq F$. On the other hand, $ab \in a \circ b$; thus we get $ab \in F$. Let now $a, b \in S$ such that $ab \in F$. Since $ab \in a \circ b$, we have $(a \circ b) \cap F \neq \emptyset$; then $a \circ b \subseteq F$. Since F is a filter of $(S, \circ, \leq)$, we have $a, b \in F$. Hence F is a filter of $(S, \cdot, \leq)$.

For the converse statement we give the following example. □

Example 21. ([21; Example 2]) We consider the ordered semigroup $(S, \cdot, \leq)$ defined by Table 13 and Figure 7.

Table 13. Multiplication table of Example 21.

$\cdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	b	e	e
c	a	b	b	b	e	e
d	a	b	b	c	e	e
e	a	b	b	b	e	e
f	a	b	b	c	e	e

The ordered hypersemigroup $(S, \circ, \leq)$ that corresponds to the ordered semigroup $(S, \cdot, \leq)$ is given by Table 14 and the same order as $(S, \cdot, \leq)$.

Table 14. The hyperoperation of Example 21.

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c,e\}$	$\{a,b,c,e\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c,e\}$	$\{a,b,c,e\}$
d	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,e\}$	$\{a,b,c,e\}$
e	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c,e\}$	$\{a,b,c,e\}$
f	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,e\}$	$\{a,b,c,e\}$

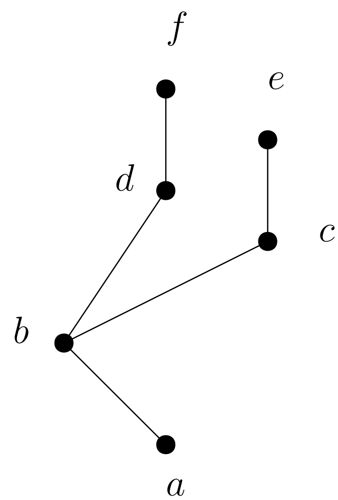

Figure 7. Figure that shows the order of Example 21.

The set $\{b, c, d, e, f\}$ is a filter of $(S, \cdot, \leq)$ but it is not a filter of $(S, \circ, \leq)$ ($\because$ $b \circ c = \{a, b\} \nsubseteq \{b, c, d, e, f\}$).

Note. In the above examples the part related to ordered semigroups has been implemented using our computer programs. Clearly it was not possible to construct such examples by hand. Having examples on ordered semigroups given by a table of multiplication and a figure, we get the corresponding examples on ordered hypersemigroups using the theorems given in the present paper.

I would like to thank the two anonymous referees for their time to read the paper –something lately not very usual.

References

[1] Birkhoff G. Lattice Theory. Amer Math Soc Coll Publ Vol 25. Corrected reprint of the 1967 third edition. Providence, RI, USA: American Mathematical Society, 1979.

[2] Corsini P, Shabir M, Mahmood T. Semisimple semihypergroups in terms of hyperideals and fuzzy hyperideals. Iran J Fuzzy Systems 2011; 8: 95-111.

[3] Heidari D, Davvaz B. On ordered hyperstructures. UPB Sci Bull Series A 2011; 73: 85-96.

[4] Kehayopulu N. On intra-regular $\vee e$-semigroups. Semigroup Forum 1980; 19: 111-121.

[5] Kehayopulu N. On weakly commutative *poe*-semigroups. Semigroup Forum 1987; 34: 367-370.

[6] Kehayopulu N. On weakly prime ideals of ordered semigroups. Math Japon 1990; 35: 1051-1056.

[7] Kehayopulu N. On right regular and right duo ordered semigroups. Math Japon 1991; 36: 201-206.

[8] Kehayopulu N. Note on Green's relations in ordered semigroups. Math Japon 1991; 36: 211-214.

[9] Kehayopulu N. On completely regular *poe*-semigroups. Math Japon 1992; 37: 123-130.

[10] Kehayopulu N. On regular duo ordered semigroups. Math Japon 1992; 37: 535-540.

[11] Kehayopulu N. On prime, weakly prime ideals in ordered semigroups. Semigroup Forum 1992; 44: 341-346.

[12] Kehayopulu N. On semilattices of simple *poe*-semigroups. Math Japon 1993; 38: 305-318.

[13] Kehayopulu N. On intra-regular ordered semigroups. Semigroup Forum 1993; 46: 271-278.

[14] Kehayopulu N. On regular, intra-regular ordered semigroups. Pure Math Appl 1993; 4: 447-461.

[15] Kehayopulu N. On regular ordered semigroups. Math Japon 1997; 3: 549-553.

[16] Kehayopulu N. A note on strongly regular ordered semigroups. Sci Math Jpn 1998; 1: 33-36.

[17] Kehayopulu N. Note on interior ideals, ideal elements in ordered semigroups. Sci Math Jpn 1999; 2: 407-409.

[18] Kehayopulu N. Characterization of left quasi-regular and semisimple ordered semigroups in terms of fuzzy sets. Int J Algebra 2012; 6: 747-755.

[19] Kehayopulu N. Fuzzy sets in $\leq$-hypergroupoids. Sci Math Jpn 2017; 80: 307-314.

[20] Kehayopulu N. Left regular and intra-regular ordered hypersemigroups in terms of semiprime and fuzzy semiprime subsets. Sci Math Jpn 2017; 80: 295-305.

[21] Kehayopulu N, Lajos S, Tsingelis M. A note on filters in ordered semigroups. Pure Math Appl 1997; 8: 83-93.

[22] Kehayopulu N, Lepouras G. On right regular and right duo *poe*-semigroups. Math Japon 1998; 47: 281-285.

[23] Kehayopulu N, Ponizovskii JS, Tsingelis, M. Note on bi-ideals in ordered semigroups and in ordered groups. Zap Nauchn Sem S-Peterburg. Otdel Mat Inst Steklov (POMI) 265 (1999), Vopr Teor Predst Algebr i Grupp 6, 198-201, 327 (2000); translation in J Math Sci (New York) 112 (2002), no 4, 4353-4354.