Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1571 – 1589

doi:10.3906/mat-1604-83

Neumann boundary value problems in fan-shaped domains

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Received: 21.04.2016 • **Accepted/Published Online:** 18.12.2017 • **Final Version:** 24.07.2018

Abstract: In this article we give the solvability conditions and the integral representations of the solutions of the Neumann boundary value problem for the Cauchy–Riemann operator and the Beltrami operator with constant coefficient in a disc sector with angle $\vartheta = \frac{\pi}{n}, n \in \mathbb{N}$. Moreover, the Neumann problem for second-order operators with the Bitsadze/Laplace operator as the main part is studied. Classical results of complex analysis are used to obtain the expressions of the solvability conditions and the integral representations for the solutions explicitly.

Key words: Cauchy–Riemann equation, Beltrami equation, Bitsadze/Poisson equation, Neumann problem, Neumann series method

1. Introduction

The Neumann problem is a basic boundary value problem in complex analysis. It is well known that a Neumann problem consists of finding a solution to a specified partial differential equation in the interior of a given domain in terms of prescribed boundary values of its normal derivative. The Neumann or flux boundary condition is typical for elliptic partial differential equations. Its need can be shown physically. For example, in heat diffusion, because the flux is proportional to the temperature gradient, a Neumann condition can inform how the heat flows across the boundary.

The aim of this paper is to investigate the Neumann problems for the Cauchy–Riemann operator, the Beltrami operator, and the Bitsadze/Laplace operator in a disc sector. We obtain the integral representations for the solutions. The resulting functions in general fail to satisfy the respective boundary conditions. Therefore necessary and sufficient solvability conditions are described. In order to get a unique solution the normalization conditions are appended.

This work is also concerned with the continued development of basic boundary value problems for complex partial differential equations; see, e.g.,[1, 2, 4, 6–9, 18, 24]. The purpose of studying these problems lies in the importance and amplitude of their applications. The Neumann problems arise in many areas including crack theory, diffraction theory, electrostatics, elasticity theory, general relativity, heat transfer and diffusion, Hele–Shaw flow, hydrodynamics, magneto statics, optical tomography, porous media, power electromagnetic, and structural mechanics; see, e.g., [10–16, 19–21, 23].

This introductory section is continued to give basic mathematical tools and known results, given without proof. More details one can find in the references provided below. In Section 2 we study the Neumann problem

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2000 *AMS Mathematics Subject Classification:* 30E25; 31A30; 35J05; 35J15 for both homogeneous and inhomogeneous Cauchy–Riemann equations. To do that we use the Dirichlet problem in the sector studied in [24]. Section 3 is devoted to the Neumann problem for the Beltrami equation. Applying the results obtained in Section 2, we reduce the Neumann problem for the Beltrami equation to a singular integral equation. Using the Neumann series method, we get the solution of the problem. In Section 4 we obtain the solvability of the Neumann problem for the Bitsadze/Laplace operator. We split the problem into two boundary value problems: a Neumann problem for the inhomogeneous Cauchy–Riemann equation and a Neumann problem for the inhomogeneous Beltrami equation. These boundary value problems are solved in Sections 2 and 3, respectively. Substituting the solution of the second problem into the solvability condition and the integral representation of the solution of the first one, we get a solution for the main problem of Section 4.

Let S be a bounded sector in the complex plane $\mathbb{C}$ defined by

$$S = \{z \in \mathbb{C} : |z| < 1, 0 < \arg z < \frac{\pi}{n},\ n \in \mathbb{N}\} \tag{1.1}$$

with three corner points at $0, 1, \omega = e^{i\pi/n}$, where n is a fixed positive integer. The boundary $\partial S = [0,1] \cup L \cup [\omega, 0]$ is oriented counterclockwise, where $[0,1]$ is the straight segment from 0 to 1 and $[\omega, 0]$ is the segment from ω to 0. The oriented circular arc L, from 1 to ω, is parameterized by $L : \tau \to e^{i\tau}, \tau \in [0, \frac{\pi}{n}]$. By rotations, one defines the domains

$$S_k = \omega^{2k} S = \{\omega^{2k} z, z \in S\},\ k = 0, 1, ..., n-1, \tag{1.2}$$

where $S_0 = S$ is the sector defined by (1.1). Reflections on the real axis define

$$E_k = \{\bar{z}, z \in S_k\},\ k = 0, 1, ..., n-1. \tag{1.3}$$

Moreover, reflections on the unit circumference define

$$\mathcal{S}_k = \{\bar{z}^{-1} : z \in S_k\},\ \mathcal{E}_k = \{\bar{z}^{-1} : z \in E_k\},\ k = 0, 1, ..., n-1. \tag{1.4}$$

It is clear that $S_k, \mathcal{S}_k, E_k, \mathcal{E}_k,\ k = 0, 1, ..., n-1$ are disjoint domains and

$$\mathbb{C} = \bigcup_{k=0}^{n-1} \left(\overline{S}_k \cup \overline{\mathcal{S}}_k \cup \overline{E}_k \cup \overline{\mathcal{E}}_k\right).$$

Moreover,

$$\bigcup_{k=0}^{n-1} (\overline{S}_k \cup \overline{E}_k) = \{z \in \mathbb{C} : |z| \le 1\},\ \bigcup_{k=0}^{n-1} (\overline{\mathcal{S}}_k \cup \overline{\mathcal{E}}_k) = \{z \in \mathbb{C} : |z| \ge 1\}.$$

Obviously, the following Lemma holds.

Lemma 1.1 *If $z \in S$, then for $k = 0, 1, .., n-1$,*

$$z\omega^{2k} \in S_k,\ \bar{z}\omega^{-2k} \in E_k,\ \bar{z}^{-1}\omega^{2k} \in \mathcal{S}_k,\ z^{-1}\omega^{-2k} \in \mathcal{E}_k,$$

where $S_k, \mathcal{S}_k, E_k, \mathcal{E}_k$ are defined by (1.1)–(1.4), respectively.

The fundamental tool for solving boundary value problems for partial differential equations in complex analysis is the Cauchy–Pompeiu formula [5, 23].

Theorem 1.1 *If $w \in C^1(S;\mathbb{C}) \cap C(\overline{S};\mathbb{C})$, then*

$$\frac{1}{2\pi i}\int_{\partial S} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_S w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \begin{cases} w(z), & z \in S, \\ 0, & z \notin \overline{S}, \end{cases}$$

and

$$-\frac{1}{2\pi i}\int_{\partial S} w(\zeta)\frac{d\bar\zeta}{\overline{\zeta - z}} - \frac{1}{\pi}\int_S w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} = \begin{cases} w(z), & z \in S, \\ 0, & z \notin \overline{S}, \end{cases}$$

where S is the sector defined by (1.1) and $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbb{R}$.

Applying Theorem 1.1 as well for $z \in S$, as for all its symmetric points, given in Lemma 1.1, gives the following modified Cauchy–Pompeiu-type formula for the sector S [24].

Theorem 1.2 *Any $w \in C^1(S;\mathbb{C}) \cap C(\overline{S};\mathbb{C})$ can be represented as*

$$w(z) = \frac{1}{2\pi i}\int_{\partial S} w(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)d\zeta$$

$$-\frac{1}{\pi}\int_S w_{\bar\zeta}(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)d\xi d\eta,\ z \in S.$$

For the required boundary behavior of the solution we consider the following lemma. Let

$$K(z,\zeta) = \sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta - \bar z\omega^{-2k}} + \frac{z}{z\zeta - \omega^{2k}} - \frac{\bar z}{\bar z\zeta - \omega^{-2k}}\right). \tag{1.5}$$

Lemma 1.2 *If $\gamma \in C(\partial S;\mathbb{C})$, then*

$$\lim_{z\in S, z\to t}\frac{1}{2\pi i}\int_{\partial S}\gamma(\zeta)K(z,\zeta)d\zeta = \gamma(t),\ t \in \partial S, \tag{1.6}$$

where $K(z,\zeta)$ is defined by (1.5).

Remark 1 *Lemma 1.2 appears as Proposition 3.1 in [24].*

Now we recall the Dirichlet problem for the homogeneous Cauchy–Riemann equation in the sector S (see [24, Theorem 4.1]), that is used in order to solve the Neumann problem for the Cauchy–Riemann operator (see Section 2).

Lemma 1.3 *If $\gamma \in C(\partial S;\mathbb{C})$, then the Dirichlet problem*

$$w_{\bar z} = 0,\ z \in S,\ w^+(\zeta) = \gamma(\zeta),\ \zeta \in \partial S,$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial S}\gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta=0,\ z\in S,$$

and its solution is uniquely given as

$$w(z)=\frac{1}{2\pi i}\int_{\partial S}\gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{z}{\omega^{2k}-z\zeta}\right)d\zeta,\ \ z\in S.$$

2. Neumann problem for the Cauchy–Riemann operator in the sector

If n_ζ is the outward normal vector to the boundary ∂S, the outward normal derivative

$$\partial_{n_\zeta}=-i[\zeta'\partial_\zeta-\bar{\zeta}'\partial_{\bar{\zeta}}],\ \zeta\in\partial S,$$

takes the form

$$\partial_{n_\zeta}=\begin{cases}\zeta\partial_\zeta+\frac{1}{\zeta}\partial_{\bar{\zeta}}, & \zeta\in L,\\ i\omega\partial_\zeta-\frac{i}{\omega}\partial_{\bar{\zeta}}, & \zeta\in[\omega,0],\\ -i\partial_\zeta+i\partial_{\bar{\zeta}}, & \zeta\in[0,1].\end{cases} \tag{2.1}$$

Theorem 2.1 *Let $\gamma\in C(\partial S;\mathbb{C})$. The Neumann problem*

$$\begin{aligned}&w_{\bar{z}}(z)=0,\ z\in S,\\ &\partial_{n_\zeta}w=\gamma(t),\ t\in\partial S,\ w(\alpha)=c,\end{aligned} \tag{2.2}$$

where $\alpha\in S$ and $c\in\mathbb{C}$ are fixed constants, is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial S}\Gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta=0,\ z\in S, \tag{2.3}$$

where

$$\Gamma(\zeta)=\begin{cases}\frac{1}{\zeta}\gamma(\zeta), & \zeta\in L,\\ -\frac{i}{\omega}\gamma(\zeta), & \zeta\in[\omega,0],\\ i\gamma(\zeta), & \zeta\in[0,1].\end{cases} \tag{2.4}$$

The solution is

$$\begin{aligned}w(z)\ &=\ c+\frac{1}{2\pi i}\int_{\partial S}\Gamma(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)\right.\\ &\left.-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\zeta\alpha-\omega^{2k}}{\zeta z-\omega^{2k}}\right)+\frac{1}{\zeta}(z-\alpha)\right\}d\zeta,\end{aligned} \tag{2.5}$$

where $\alpha\in S$ is a fixed constant.

Proof If w is a solution to the Neumann problem (2.2), then $W = w'$ is a solution to the Dirichlet problem

$$\begin{aligned} W_{\bar{z}}(z) = 0,\ z \in S, \\ W = \Gamma(\zeta),\ \zeta \in \partial S, \end{aligned} \tag{2.6}$$

where Γ is defined by (2.4). According to Lemma 1.3 the Dirichlet problem (2.6) is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial S}\Gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta = 0,\ z\in S,$$

and its solution is

$$W(z) = \frac{1}{2\pi i}\int_{\partial S}\Gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{z}{\omega^{2k}-z\zeta}\right)d\zeta, z\in S. \tag{2.7}$$

For a fixed constant $\alpha \in S$, integrating along any simple path from α to z, lying in S, leads to

$$\begin{aligned} w(z) &= w(\alpha) + \frac{1}{2\pi i}\int_{\partial S}\Gamma(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)\right. \\ &\qquad \left. + \frac{z-\alpha}{\zeta} - \frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\zeta\alpha-\omega^{2k}}{\zeta z-\omega^{2k}}\right)\right\}d\zeta, \end{aligned}$$

which is the desired formula (2.5). Clearly, $w(z)$ given by (2.5) is analytic in S. That this function satisfies the Neumann boundary condition in (2.2) can be seen as follows. Differentiating on both sides of (2.5) with respect to z and subtracting (2.3) from the resulting equation give

$$w'(z) = \frac{1}{2\pi i}\int_{\partial S}\Gamma(\zeta)K(z,\zeta)d\zeta,\ z\in S,$$

where $K(z,\zeta)$ and $\Gamma(\zeta)$ are defined by (1.5) and (2.4), respectively. By Lemma 1.2, $\Gamma(\zeta)$ can be attained as the boundary limit of w'. Hence, using (2.1) one can get

$$\partial_{n_\zeta} w = \zeta w'(\zeta) = \zeta\Gamma(\zeta) = \gamma(\zeta),\ \zeta\in L.$$

Similarly, for $\zeta \in [0,1]\cup[\omega,0]$, one can obtain $\partial_{n_\zeta} w = \gamma(\zeta)$. □

Next, we consider the inhomogeneous Neumann problem

$$\begin{aligned} w_{\bar{z}} = f(z),\ z\in S, \\ \partial_{n_\zeta} w = \gamma(\zeta),\ \zeta\in\partial S,\ w(\alpha) = c, \end{aligned} \tag{2.8}$$

where $f \in L_p(S;\mathbb{C})\cap C^1(S;\mathbb{C}), p>2,\ \gamma\in C(\partial S;\mathbb{C})$ and $\alpha\in S$ and $c\in\mathbb{C}$ are fixed constants.

Theorem 2.2 *Let $f \in L_p(S;\mathbb{C})\cap C^1(S;\mathbb{C}), p>2,\ \gamma\in C(\partial S;\mathbb{C})$. Then the Neumann problem (2.8) is uniquely solvable if and only if*

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\partial S}\Gamma_1(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta \\ &= \frac{1}{\pi}\int_S f_\zeta(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\xi d\eta, \end{aligned} \tag{2.9}$$

where

$$\Gamma_1(\zeta) = \begin{cases} \frac{1}{\zeta}[\gamma(\zeta) - \frac{1}{\zeta} f(\zeta)], & \zeta \in L, \\ -\frac{i}{\omega}[\gamma(\zeta) + \frac{i}{\omega} f(\zeta)], & \zeta \in [\omega, 0], \\ i[\gamma(\zeta) - i f(\zeta)], & \zeta \in [0,1]. \end{cases} \tag{2.10}$$

The solution is

$$\begin{aligned} w(z) \;=\; & c + \frac{1}{2\pi i} \int_{\partial S} \Gamma_1(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right) \right. \\ & \left. + \frac{z-\alpha}{\zeta} - \frac{\omega^{2k}}{\zeta^2} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right) \right\} d\zeta \\ & - \frac{1}{\pi} \int_S f_\zeta(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right) \right. \\ & \left. + \frac{z-\alpha}{\zeta} - \frac{\omega^{2k}}{\zeta^2} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right) \right\} d\xi d\eta, \; z \in S, \end{aligned} \tag{2.11}$$

where $\alpha \in S$ *and* $c \in \mathbb{C}$ *are fixed constants.*

Proof Representing $w = \varphi + T[f](z)$, where φ is analytic in S and

$$T[f](z) = -\frac{1}{\pi} \int_S f(\zeta) \frac{d\xi d\eta}{\zeta - z}, \; z \in S, \tag{2.12}$$

is the classical Pompeiu operator [5, 23], implies that

$$\partial_{n_\zeta} \varphi = \partial_{n_\zeta} w(\zeta) - \partial_{n_\zeta} T[f](\zeta), \; \zeta \in \partial S. \tag{2.13}$$

For $f \in L_p(S;\mathbb{C}), p > 2$, $T[f](z)$ satisfies, in the Sobolev sense [5, 23], the following equalities

$$\partial_{\bar z} T[f](z) = f(z), \; \partial_z T[f](z) = \Pi[f](z), \; z \in S, \tag{2.14}$$

where

$$\Pi[f](z) = -\frac{1}{\pi} \int_S f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2}, \; z \in S, \tag{2.15}$$

is the Ahlfors–Beurling operator [5, 23]. One can see

$$\partial_{n_\zeta} T[f](\zeta) = \begin{cases} \zeta \Pi[f]^+(\zeta) + \frac{1}{\zeta} f(\zeta), & \zeta \in L, \\ i\omega \Pi[f]^+(\zeta) - \frac{i}{\omega} f(\zeta), & \zeta \in [\omega, 0], \\ -i\Pi[f]^+(\zeta) + i f(\zeta), & \zeta \in [0,1]. \end{cases} \tag{2.16}$$

Then if w is a solution to the Neumann problem (2.8), the function $\varphi(z) = w(z) - T[f](z)$ solves the Dirichlet problem

$$\begin{aligned} & \varphi'_{\bar z} = 0, \; z \in S, \\ & \varphi'(\zeta) = \Gamma_1(\zeta) - \Pi[f]^+(\zeta), \; \zeta \in \partial S, \end{aligned} \tag{2.17}$$

where Γ_1 is given by (2.10). By Lemma 1.3 the Dirichlet problem (2.17) for the analytic function $\varphi'(z)$ is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial S}[\Gamma_1(\zeta)-\Pi[f]^+(\zeta)]\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta=0,\ z\in S, \tag{2.18}$$

and its solution is given by

$$\varphi'(z)=\frac{1}{2\pi i}\int_{\partial S}[\Gamma_1(\zeta)-\Pi[f]^+(\zeta)]\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{z}{\omega^{2k}-z\zeta}\right)d\zeta,\ z\in S. \tag{2.19}$$

The later expressions can be simplified as follows. From (2.15), see [5, 15, 23], we get

$$\Pi[f](z)=-\frac{1}{2\pi i}\int_{\partial S}f(\tau)\frac{d\tau}{\tau-z}-\frac{1}{\pi}\int_S f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-z},\ z\in S. \tag{2.20}$$

Taking the limit as $z\to\zeta\in\partial S$ implies

$$\Pi[f]^+(\zeta)=-\frac{1}{2}f(\zeta)-\frac{1}{2\pi i}\int_{\partial S}f(\tau)\frac{d\tau}{\tau-z}-\frac{1}{\pi}\int_S f_{\tilde{\zeta}}(\tilde{\zeta})\frac{d\tilde{\xi}d\tilde{\eta}}{\tilde{\zeta}-\zeta},\ z\in S. \tag{2.21}$$

Here, the Plemelj–Sokhotski formulae [5, 15, 23] are applied. Therefore, for $z\in S$,

$$\begin{aligned}\frac{1}{2\pi i}&\int_{\partial S}\Pi[f]^+(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{z}{\omega^{2k}-z\zeta}\right)d\zeta\\ &=-\frac{1}{2\pi i}\int_{\partial S}f(\zeta)\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int_S f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-z}\\ &\quad+\frac{1}{\pi}\int_S f_{\zeta}(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{z}{\omega^{2k}-z\zeta}\right)d\xi d\eta.\end{aligned} \tag{2.22}$$

Similarly, one obtains

$$\begin{aligned}\frac{1}{2\pi i}&\int_{\partial S}\Pi[f]^+(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta\\ &=\frac{1}{\pi}\int_S f_{\zeta}(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\xi d\eta.\end{aligned} \tag{2.23}$$

From (2.19) and (2.22) we obtain

$$\begin{aligned}\varphi'(z) &= \frac{1}{2\pi i}\int_{\partial S}\Gamma_1(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)d\zeta \\ &\quad + \frac{1}{2\pi i}\int_{\partial S} f(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{\pi}\int_S f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta - z} \\ &\quad - \frac{1}{\pi}\int_S f_\zeta(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)d\xi d\eta \\ &= \frac{1}{2\pi i}\int_{\partial S}\Gamma_1(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)d\zeta - \Pi[f](z) \\ &\quad - \frac{1}{\pi}\int_S f_\zeta(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)d\xi d\eta. \end{aligned} \tag{2.24}$$

Integrating on both sides of (2.24), along a path from α to z, lying in S, gives

$$\begin{aligned} w(z) &= w(\alpha) + \frac{1}{2\pi i}\int_{\partial S}\Gamma_1(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right)\right. \\ &\quad \left. + \frac{z-\alpha}{\zeta} - \frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right)\right\}d\zeta \\ &\quad - \frac{1}{\pi}\int_S f_\zeta(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right)\right. \\ &\quad \left. + \frac{z-\alpha}{\zeta} - \frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right)\right\}d\xi d\eta, \ z \in S. \end{aligned} \tag{2.25}$$

Applying the side condition $w(\alpha) = c$, we get (2.11). Obviously the solvability condition (2.9) follows at once by substituting (2.23) in (2.18).

Next, we verify that $w(z)$ given by (2.11) under the condition (2.9) satisfies the Neumann boundary value problem (2.8). In (2.11), the term

$$\int_S f_\zeta(\zeta)\log(\zeta - z)d\xi d\eta$$

is the only one that is not analytic in S. Therefore, we have

$$\begin{aligned} w_{\bar z} &= \frac{1}{\pi}\partial_{\bar z}\int_S f_\zeta(\zeta)\log(\zeta - z)d\xi d\eta \\ &= -\frac{1}{\pi}\partial_{\bar z}\left\{\frac{1}{2i}\int_{\partial S} f(\zeta)\log(\zeta - z)d\bar\zeta + \int_S f(\zeta)\frac{d\xi d\eta}{\zeta - z}\right\} \\ &= f(z). \end{aligned}$$

This shows that the function, given by (2.11), solves the inhomogeneous Cauchy–Riemann equation (2.8). On the other hand, differentiating (2.11) with respect to z gives

$$\begin{aligned} w_z &= \frac{1}{2\pi i}\int_{\partial S}\Gamma_1(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)d\zeta \\ &\quad -\frac{1}{\pi}\int_S f_\zeta(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)d\xi d\eta, \end{aligned}$$

which, after subtracting (2.9), gives

$$w_z = \frac{1}{2\pi i}\int_{\partial S}\Gamma_1(\zeta)K(z,\zeta)d\zeta - \frac{1}{\pi}\int_S f_\zeta(\zeta)K(z,\zeta)d\xi d\eta, \tag{2.26}$$

where $K(z,\zeta)$ and Γ_1 are defined by (1.5) and (2.10), respectively.

Since, from (1.5), $K(z,\zeta) = 0$, for all $(z,\zeta) \in \partial S \times S$, then

$$\lim_{z\in S, z\to t}\frac{1}{\pi}\int_S f_\zeta(\zeta)K(z,\zeta)d\xi d\eta = 0,\ t \in \partial S.$$

Therefore, taking the limit on both sides of (2.26) when z tends to $t \in \partial S$, by Lemma 1.2 we get

$$\lim_{z\in S,\, z\to t} w_z(z) = \lim_{z\in S, z\to t}\frac{1}{2\pi i}\int_{\partial S}\Gamma_1(\zeta)K(z,\zeta)d\zeta = \Gamma_1(t),\ t \in \partial S,$$

where Γ_1 is defined by (2.10).

Hence, on L, where $\zeta = e^{i\vartheta}$, $\vartheta \in (0, \frac{\pi}{n})$, $n \in \mathbb{N}$

$$\begin{aligned} \partial_{n_\zeta} w &= -i\{\zeta' w_\zeta - \bar{\zeta}' w_{\bar{\zeta}}\} \\ &= \zeta\Gamma_1(\zeta) + \bar{\zeta}f(\zeta) \\ &= \gamma(\zeta),\quad \zeta \in L. \end{aligned}$$

Similarly, one finds

$$\partial_{n_\zeta} w = \gamma(\zeta),\quad \zeta \in [\omega, 0] \cup [0, 1].$$

Hence, $w(z)$, given by (2.11), satisfies the Neumann condition (2.8), that is, $\partial_{n_\zeta} w = \gamma(\zeta), \zeta \in \partial S$. This completes the proof of Theorem 2.2. □

3. Neumann problem for the Beltrami operator in the sector

First, we consider the Neumann problem for the homogeneous Beltrami equation in the sector S

$$w_{\bar{z}} + q w_z = 0,\ \ z \in S \tag{3.1}$$

$$\partial_{n_\zeta} w = \gamma(\zeta),\ \ \zeta \in \partial S \tag{3.2}$$

where $\gamma \in C(\partial S;\mathbb{C})$ and $q \in \mathbb{C}$, $|q| < 1$.

Write (3.1) and (3.2) in the form

$$w_{\bar{z}} = -qw_z,\ z \in S,\ \partial_{n_\zeta} w = \gamma(\zeta),\ \zeta \in \partial S. \tag{3.3}$$

Since

$$\partial_{n_\zeta} w = -i[\zeta' w_\zeta - \bar{\zeta}' w_{\bar{\zeta}}],\ \zeta \in \partial S,$$

then one can obtain

$$w_\zeta = \frac{i\gamma(\zeta)}{\zeta' + q\bar{\zeta}'},\ \zeta \in \partial S. \tag{3.4}$$

By Theorem 2.2 the Neumann problem (3.3) is solvable if and only if

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\partial S} \Gamma_2(\zeta) \sum_{k=0}^{n-1} \left(\frac{1}{\zeta - \bar{z}\omega^{2k}} - \frac{\bar{z}}{\omega^{2k} - \bar{z}\zeta} \right) d\zeta \\ &= -\frac{1}{\pi}\int_S (qw_\zeta)_\zeta \sum_{k=0}^{n-1} \left(\frac{1}{\zeta - \bar{z}\omega^{2k}} - \frac{\bar{z}}{\omega^{2k} - \bar{z}\zeta} \right) d\xi d\eta\end{aligned} \tag{3.5}$$

where

$$\Gamma_2(\zeta) = \begin{cases} \frac{1}{\zeta}[\gamma(\zeta) + \frac{q}{\zeta} w_\zeta], & \zeta \in L, \\ -\frac{i}{\omega}[\gamma(\zeta) - \frac{iq}{\omega} w_\zeta], & \zeta \in [\omega, 0], \\ i[\gamma(\zeta) + iqw_\zeta], & \zeta \in [0,1], \end{cases} \tag{3.6}$$

and its solution is

$$\begin{aligned} w(z) \;=\; & w(\alpha) + \frac{1}{2\pi i}\int_{\partial S} \Gamma_2(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}} \right) \right. \\ & \left. + \frac{z-\alpha}{\zeta} - \frac{\omega^{2k}}{\zeta^2} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}} \right) \right\} d\zeta \\ & + \frac{q}{\pi}\int_S w_{\zeta\zeta}(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}} \right) \right. \\ & \left. + \frac{z-\alpha}{\zeta} - \frac{\omega^{2k}}{\zeta^2} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}} \right) \right\} d\xi d\eta,\ z \in S, \end{aligned} \tag{3.7}$$

where $\alpha \in S$ is a fixed constant. Applying Gauss's theorem [5] gives

$$\begin{aligned}&-\frac{q}{\pi}\int_S w_{\zeta\zeta} \sum_{k=0}^{n-1} \left(\frac{1}{\zeta - \bar{z}\omega^{2k}} - \frac{\bar{z}}{\omega^{2k} - \bar{z}\zeta} \right) d\xi d\eta \\ &= \frac{q}{2\pi i}\int_{\partial S} w_\zeta \sum_{k=0}^{n-1} \left(\frac{1}{\zeta - \bar{z}\omega^{2k}} - \frac{\bar{z}}{\omega^{2k} - \bar{z}\zeta} \right) d\bar{\zeta} \\ &-\frac{q}{\pi}\int_S w_\zeta \sum_{k=0}^{n-1} \left(\frac{1}{(\zeta - \bar{z}\omega^{2k})^2} + \frac{\bar{z}^2}{(\omega^{2k} - \bar{z}\zeta)^2} \right) d\xi d\eta,\end{aligned} \tag{3.8}$$

and

$$\frac{q}{\pi}\int_S w_{\zeta\zeta}\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)+\frac{z-\alpha}{\zeta}\right.$$
$$\left.-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\xi d\eta$$
$$=-\frac{q}{2\pi i}\int_{\partial S} w_{\zeta}\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)+\frac{z-\alpha}{\zeta}\right.$$
$$\left.-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\bar{\zeta}$$
$$-\frac{q}{\pi}\int_S w_{\zeta}\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\left(\frac{1}{\zeta-\alpha\omega^{2k}}-\frac{1}{\zeta-z\omega^{2k}}\right)-\frac{z-\alpha}{\zeta^2}\right.$$
$$\left.-\frac{\omega^{2k}}{\zeta^2}\left(\frac{\alpha}{\alpha\zeta-\omega^{2k}}-\frac{z}{z\zeta-\omega^{2k}}\right)+\frac{2\omega^{2k}}{\zeta^3}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\xi d\eta. \tag{3.9}$$

Then, using (3.8), the solvability condition (3.5) takes the form

$$\frac{1}{2\pi i}\int_{\partial S}\Gamma_3(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta$$
$$=-\frac{q}{\pi}\int_S w_{\zeta}\sum_{k=0}^{n-1}\left(\frac{1}{(\zeta-\bar{z}\omega^{2k})^2}+\frac{\bar{z}^2}{(\omega^{2k}-\bar{z}\zeta)^2}\right)d\xi d\eta, \tag{3.10}$$

where

$$\Gamma_3(\zeta)=\begin{cases}\frac{\zeta+q\bar{\zeta}}{\zeta(\zeta-q\bar{\zeta})}\gamma(\zeta), & \zeta\in L,\\ \frac{-i[\omega-q\bar{\omega}]}{\omega[\omega+q\bar{\omega}]}\gamma(\zeta), & \zeta\in[\omega,0],\\ \frac{i(1-q)}{1+q}\gamma(\zeta), & \zeta\in[0,1].\end{cases} \tag{3.11}$$

Furthermore, from (3.7) and (3.9) we find

$$w(z) = w(\alpha)+\frac{1}{2\pi i}\int_{\partial S}\Gamma_3(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)\right.$$
$$\left.+\frac{z-\alpha}{\zeta}-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\zeta$$
$$-\frac{q}{\pi}\int_S w_{\zeta}(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\left(\frac{1}{\zeta-\alpha\omega^{2k}}-\frac{1}{\zeta-z\omega^{2k}}\right)\right.$$
$$-\frac{z-\alpha}{\zeta^2}-\frac{\omega^{2k}}{\zeta^2}\left(\frac{\alpha}{\alpha\zeta-\omega^{2k}}-\frac{z}{z\zeta-\omega^{2k}}\right)$$
$$\left.+\frac{2\omega^{2k}}{\zeta^3}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\xi d\eta,\ z\in S, \tag{3.12}$$

where Γ_3 is defined by (3.11). Differentiating (3.12) with respect to z gives

$$w_z(z) = \varphi_1(z) + q\Pi_1[w_z](z),\ \ z \in S \tag{3.13}$$

where

$$\varphi_1(z) = \frac{1}{2\pi i}\int_{\partial S}\Gamma_3(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} + \frac{z}{z\zeta-\omega^{2k}}\right)d\zeta,\ \ z\in S, \tag{3.14}$$

and

$$\Pi_1[w_z](z) = \frac{1}{\pi}\int_S w_\zeta(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{(\zeta - z\omega^{2k})^2} + \frac{z^2}{(z\zeta-\omega^{2k})^2}\right)d\xi d\eta,\ \ z\in S. \tag{3.15}$$

The last equation defines a singular integral operator. It is just the sum of the $\Pi-$operator, defined by (2.15), and a bounded operator on $L_p(S;\mathbb{C})$, $p>2$ defined by the integral

$$\frac{1}{\pi}\int_S f(\zeta)\left\{\frac{z^2}{(z\zeta-1)^2} + \sum_{k=1}^{n-1}\left(\frac{1}{(\zeta - z\omega^{2k})^2} + \frac{z^2}{(z\zeta-\omega^{2k})^2}\right)\right\}d\xi d\eta,\ \ z\in S,$$

for any $f\in L_p(S;\mathbb{C})$, $p>2$. Thus, according to [3, Lemma 3.1], see also [4, 5, 23], under the conditions on q and p we have

$$|q|\|\Pi_1\|_{L_p(S;\mathbb{C})} < 1. \tag{3.16}$$

Moreover, $I+q\Pi_1$ is a perturbation of the invertible operator $I+q\Pi$ with a bounded one. Using the bounded index stability theorem [17], the operator $I+q\Pi_1$ is invertible, and hence the Ferdholm alternative can be applied to the singular integral equation (3.13). Its solution can be given by the following Neumann series, for which (3.16) is a sufficient condition to converge (see [22]),

$$w_z(z) = \sum_{l=0}^{\infty}(-1)^l(q\Pi_1)^l[\varphi_1](z),\ \ z\in S. \tag{3.17}$$

Substituting (3.17) in (3.12), the solution to problem (3.3) takes the form

$$\begin{aligned} w(z) \ &= \ w(\alpha) + \frac{1}{2\pi i}\int_{\partial S}\Gamma_3(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right)\right.\\ &\quad \left. + \frac{z-\alpha}{\zeta} - \frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\zeta \\ &\quad -\frac{q}{\pi}\int_S\sum_{l=0}^{\infty}(-1)^{l+1}(q\Pi_1)^l[\varphi_1](\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\left(\frac{1}{\zeta-\alpha\omega^{2k}} - \frac{1}{\zeta - z\omega^{2k}}\right)\right.\\ &\quad -\frac{z-\alpha}{\zeta^2} - \frac{\omega^{2k}}{\zeta^2}\left(\frac{\alpha}{\alpha\zeta-\omega^{2k}} - \frac{z}{z\zeta-\omega^{2k}}\right)\\ &\quad \left. + \frac{2\omega^{2k}}{\zeta^3}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\xi d\eta,\ \ z\in S, \end{aligned} \tag{3.18}$$

where Γ_3, φ_1, and Π_1 are defined by (3.11), (3.14), and (3.15), respectively, and $\alpha \in S$ is a fixed constant. The solvability condition (3.10) takes the form

$$\frac{1}{2\pi i}\int_{\partial s}\Gamma_3(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta$$
$$=\frac{q}{\pi}\int_s\sum_{l=0}^{\infty}(-1)^{l+1}(q\Pi_1)^l[\varphi_1](\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{(\zeta-\bar{z}\omega^{2k})^2}+\frac{\bar{z}^2}{(\omega^{2k}-\bar{z}\zeta)^2}\right)d\xi d\eta \tag{3.19}$$

where Γ_3, φ_1, and Π_1 are defined by (3.11), (3.14), and (3.15), respectively. Thus, we have proved the following result.

Theorem 3.1 *The Neumann problem (3.1) and (3.2) is solvable if and only if (3.19) is satisfied. A unique solution, up to some additive complex constant $w(\alpha) = c$, can be given by (3.18).*

Next, we deal with the inhomogeneous Neumann problem

$$\begin{aligned} &w_{\bar{z}} + qw_z = f(z), \;\; z \in S \\ &\partial_{n_\zeta} w = \gamma(\zeta), \;\; \zeta \in \partial S. \end{aligned} \tag{3.20}$$

As before, we write (3.20) as

$$\begin{aligned} &w_{\bar{z}} = f(z) - qw_z, \;\; z \in S, \\ &\partial_{n_\zeta} w = \gamma(\zeta), \;\; \zeta \in \partial S, \end{aligned} \tag{3.21}$$

and thus we can obtain

$$w_\zeta = \frac{\gamma(\zeta) - i\bar{\zeta}' f(\zeta)}{-i[\zeta' + q\bar{\zeta}']}, \;\zeta \in \partial S,$$

or

$$w_\zeta(\zeta) = \begin{cases} \frac{\gamma(\zeta)-\bar{\zeta}f(\zeta)}{\zeta - q\bar{\zeta}}, & \zeta \in L, \\ \frac{\gamma(\zeta)+i\bar{\omega}f(\zeta)}{i[\omega+q\bar{\omega}]}, & \zeta \in [\omega, 0], \\ \frac{i[\gamma(\zeta)-if(\zeta)]}{1+q}, & \zeta \in [0,1]. \end{cases} \tag{3.22}$$

By Theorem 2.2 the Neumann problem (3.21) is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial S}\Gamma_4(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta$$
$$=\frac{1}{\pi}\int_S(f-qw_\zeta)_\zeta(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\xi d\eta, \tag{3.23}$$

where

$$\Gamma_4(\zeta) = \begin{cases} \frac{1}{\zeta}[\gamma(\zeta) - \frac{1}{\zeta}(f(\zeta) - qw_\zeta(\zeta))], & \zeta \in L, \\ -\frac{i}{\omega}[\gamma(\zeta) + \frac{i}{\omega}(f(\zeta) - qw_\zeta(\zeta))], & \zeta \in [\omega, 0], \\ i[\gamma(\zeta) - i(f(\zeta) - qw_\zeta(\zeta))], & \zeta \in [0,1], \end{cases} \tag{3.24}$$

and its solution is

$$\begin{aligned} w(z) \;=\; & w(\alpha) + \frac{1}{2\pi i}\int_{\partial S} \Gamma_4(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right) + \frac{z-\alpha}{\zeta} \right. \\ & \left. - \frac{\omega^{2k}}{\zeta^2} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right)\right\} d\zeta \\ & - \frac{1}{\pi}\int_S (f - qw_\zeta)_\zeta(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right) + \frac{z-\alpha}{\zeta} \right. \\ & \left. - \frac{\omega^{2k}}{\zeta^2} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right)\right\} d\xi d\eta, \;\; z \in S. \end{aligned} \tag{3.25}$$

Applying Gauss's theorem gives

$$\begin{aligned} & \frac{1}{\pi}\int_S (f - qw_\zeta)_\zeta(\zeta) \sum_{k=0}^{n-1} \left(\frac{1}{\zeta - \bar{z}\omega^{2k}} - \frac{\bar{z}}{\omega^{2k} - \bar{z}\zeta} \right) d\xi d\eta \\ & = -\frac{1}{2\pi i}\int_{\partial S} (f - qw_\zeta)(\zeta) \sum_{k=0}^{n-1} \left(\frac{1}{\zeta - \bar{z}\omega^{2k}} - \frac{\bar{z}}{\omega^{2k} - \bar{z}\zeta} \right) d\bar{\zeta} \\ & + \frac{1}{\pi}\int_S (f - qw_\zeta)(\zeta) \sum_{k=0}^{n-1} \left(\frac{1}{(\zeta - \bar{z}\omega^{2k})^2} + \frac{\bar{z}^2}{(\omega^{2k} - \bar{z}\zeta)^2} \right) d\xi d\eta, \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{\pi}\int_S (f - qw_\zeta)_\zeta(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right) + \frac{z-\alpha}{\zeta} \right. \\ & \left. - \frac{\omega^{2k}}{\zeta^2} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right)\right\} d\xi d\eta \\ & = \frac{1}{2\pi i}\int_{\partial S} (f - qw_\zeta)(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \log\left(\frac{\zeta - \alpha\omega^{2k}}{\zeta - z\omega^{2k}}\right) + \frac{z-\alpha}{\zeta} \right. \\ & \left. - \frac{\omega^{2k}}{\zeta^2} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right)\right\} d\bar{\zeta} \\ & + \frac{1}{\pi}\int_S (f - qw_\zeta)(\zeta) \sum_{k=0}^{n-1} \left\{ \frac{1}{\omega^{2k}} \left(\frac{1}{\zeta - \alpha\omega^{2k}} - \frac{1}{\zeta - z\omega^{2k}} \right) - \frac{z-\alpha}{\zeta^2} \right. \\ & \left. + \frac{2\omega^{2k}}{\zeta^3} \log\left(\frac{\alpha\zeta - \omega^{2k}}{z\zeta - \omega^{2k}}\right) - \frac{\omega^{2k}}{\zeta^2} \left(\frac{\alpha}{\alpha\zeta - \omega^{2k}} - \frac{z}{z\zeta - \omega^{2k}} \right) \right\} d\xi d\eta, \;\; z \in S. \end{aligned}$$

Using these results, the solvability condition (3.23) takes the form

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial S}\Gamma_5(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta\\
&-\frac{1}{\pi}\int_{S}(f-qw_{\zeta})(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{(\zeta-\bar{z}\omega^{2k})^2}+\frac{\bar{z}^2}{(\omega^{2k}-\bar{z}\zeta)^2}\right)d\xi d\eta\\
&=0, \quad z\in S,
\end{aligned}
\tag{3.26}
$$

where

$$
\Gamma_5(\zeta)=\begin{cases}\frac{\gamma(\zeta)[\zeta+q\bar{\zeta}]}{\zeta[\zeta-q\bar{\zeta}]}-\frac{2f(\zeta)}{\zeta[\zeta-q\bar{\zeta}]}, & \zeta\in L,\\ \frac{-i[\omega-q\bar{\omega}]\gamma(\zeta)}{\omega[\omega+q\bar{\omega}]}+\frac{2f(\zeta)}{\omega[\omega+q\bar{\omega}]}, & \zeta\in[\omega,0],\\ \frac{i[1-q]\gamma(\zeta)}{1+q}+\frac{2f(\zeta)}{1+q}, & \zeta\in[0,1].\end{cases}
\tag{3.27}
$$

Moreover, the implicit expression (3.25) of the solution can be reduced to the following form:

$$
\begin{aligned}
w(z) \;=\; & w(\alpha)+\frac{1}{2\pi i}\int_{\partial S}\Gamma_5(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)\right.\\
&\left.+\frac{z-\alpha}{\zeta}-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\zeta\\
&+\frac{1}{\pi}\int_{S}(f-qw_{\zeta})(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\left(\frac{1}{\zeta-\alpha\omega^{2k}}-\frac{1}{\zeta-z\omega^{2k}}\right)\right.\\
&-\frac{z-\alpha}{\zeta^2}-\frac{\omega^{2k}}{\zeta^2}\left(\frac{\alpha}{\alpha\zeta-\omega^{2k}}-\frac{z}{z\zeta-\omega^{2k}}\right)\\
&\left.+\frac{2\omega^{2k}}{\zeta^3}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\xi d\eta, \; z\in S.
\end{aligned}
\tag{3.28}
$$

Differentiating (3.28) with respect to z gives

$$
w_z(z)=\varphi_2(z)-\Pi_1[f-qw_z](z), \quad z\in S,
\tag{3.29}
$$

where Π_1 is defined by (3.15), and

$$
\varphi_2(z)=\frac{1}{2\pi i}\int_{\partial S}\Gamma_5(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-z\omega^{2k}}+\frac{z}{z\zeta-\omega^{2k}}\right)d\zeta, \; z\in S,
\tag{3.30}
$$

with Γ_5 as defined by (3.27). Again, under the condition (3.16) applying the Neumann series method to the singular integral equation (3.29) gives

$$
f(z)-qw_z(z)=\sum_{l=0}^{\infty}(-1)^l(q\Pi_1)^l[f-q\varphi_2](z), \; z\in S.
\tag{3.31}
$$

Hence, substituting (3.31) in both (3.26) and (3.28) gives the necessary and sufficient condition for problem (3.20) to be solvable

$$\frac{1}{2\pi i}\int_{\partial S}\Gamma_5(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta$$
$$-\frac{1}{\pi}\int_S\sum_{l=0}^{\infty}(-1)^l(q\Pi_1)^l(f-q\varphi_2)(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{(\zeta-\bar{z}\omega^{2k})^2}+\frac{\bar{z}^2}{(\omega^{2k}-\bar{z}\zeta)^2}\right)d\xi d\eta$$
$$=0,\quad z\in S, \tag{3.32}$$

and the solution of the Neumann problem (3.20) is expressed as

$$\begin{aligned} w(z) \;=\; & w(\alpha)+\frac{1}{2\pi i}\int_{\partial S}\Gamma_5(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)\right. \\ & \left.+\frac{z-\alpha}{\zeta}-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\zeta \\ & +\frac{1}{\pi}\int_S\sum_{l=0}^{\infty}(-1)^l(q\Pi_1)^l(f-q\varphi_2)(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\left(\frac{1}{\zeta-\alpha\omega^{2k}}-\frac{1}{\zeta-z\omega^{2k}}\right)\right. \\ & -\frac{z-\alpha}{\zeta^2}-\frac{\omega^{2k}}{\zeta^2}\left(\frac{\alpha}{\alpha\zeta-\omega^{2k}}-\frac{z}{z\zeta-\omega^{2k}}\right) \\ & \left.+\frac{2\omega^{2k}}{\zeta^3}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\xi d\eta,\; z\in S, \end{aligned}\tag{3.33}$$

where Γ_5, φ_2, and Π_1 are defined by (3.27), (3.30), and (3.15), respectively. This proves the following theorem.

Theorem 3.2 *Let $f\in L_p(S;\mathbb{C}), p>2, \gamma\in C(\partial S;\mathbb{C})$, and $q\in\mathbb{C}, |q|<1$. The Neumann problem (3.20) is solvable if and only if the condition (3.32) is satisfied. The solution is unique up to some additive complex constant $w(\alpha)=c$, and it can be of the form (3.33).*

4. Neumann problem for the Bitsadze/Laplace operator in the sector

In this section we investigate a Neumann problem for a second-order partial differential equation with Bitsadze/Laplace operator as main part in the sector S, defined by (1.1).

Theorem 4.1 *Let $f\in L_p(S;\mathbb{C})\cap C^1(S;\mathbb{C}), p>2$, $\gamma_0,\gamma_1\in C(\partial S;\mathbb{C})$, and $q\in\mathbb{C}$, $|q|<1$. Then the Neumann problem*

$$\begin{aligned} & w_{\bar{z}\bar{z}}+qw_{z\bar{z}}=f, z\in S, \\ & \partial_{n_\zeta}w=\gamma_0(\zeta),\; \partial_{n_\zeta}w_{\bar{z}}=\gamma_1(\zeta),\; \zeta\in\partial S \\ & w(\alpha)=c_0,\; w_{\bar{z}}=c_1. \end{aligned}\tag{4.1}$$

where $\alpha \in S$ and $c_0,\, c_1 \in \mathbb{C}$ are fixed constants, is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial S}\tilde{\Gamma}_1(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta$$

$$=\frac{1}{\pi}\int_{S}u_{\zeta}(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\xi d\eta, \tag{4.2}$$

where

$$\tilde{\Gamma}_1(\zeta)=\begin{cases}\frac{1}{\zeta}[\gamma_0(\zeta)-\frac{1}{\zeta}u(\zeta)], & \zeta\in L,\\ -\frac{i}{\omega}[\gamma_0(\zeta)+\frac{i}{\omega}u(\zeta)], & \zeta\in[\omega,0],\\ i[\gamma_0(\zeta)-iu(\zeta)], & \zeta\in[0,1],\end{cases} \tag{4.3}$$

and

$$\begin{aligned} u(z) \;=\; & c_1+\frac{1}{2\pi i}\int_{\partial S}\Gamma_5(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)\right.\\ & \left.+\frac{z-\alpha}{\zeta}-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\zeta\\ & +\frac{1}{\pi}\int_S\sum_{l=0}^{\infty}(-1)^l(q\Pi_1)^l(f-q\varphi_2)(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\left(\frac{1}{\zeta-\alpha\omega^{2k}}-\frac{1}{\zeta-z\omega^{2k}}\right)\right.\\ & -\frac{z-\alpha}{\zeta^2}-\frac{\omega^{2k}}{\zeta^2}\left(\frac{\alpha}{\alpha\zeta-\omega^{2k}}-\frac{z}{z\zeta-\omega^{2k}}\right)\\ & \left.+\frac{2\omega^{2k}}{\zeta^3}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\xi d\eta,\ z\in S, \end{aligned} \tag{4.4}$$

with Γ_5,φ_2, and Π_1 defined by (3.27), (3.30), and (3.15), respectively, and $\alpha\in S$, $c_1\in\mathbb{C}$ are fixed constants. The solution to problem (4.1) is

$$\begin{aligned} w(z) \;=\; & c_0+\frac{1}{2\pi i}\int_{\partial S}\tilde{\Gamma}_1(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)\right.\\ & \left.+\frac{z-\alpha}{\zeta}-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\zeta\\ & -\frac{1}{\pi}\int_S u_{\zeta}(\zeta)\sum_{k=0}^{n-1}\left\{\frac{1}{\omega^{2k}}\log\left(\frac{\zeta-\alpha\omega^{2k}}{\zeta-z\omega^{2k}}\right)\right.\\ & \left.+\frac{z-\alpha}{\zeta}-\frac{\omega^{2k}}{\zeta^2}\log\left(\frac{\alpha\zeta-\omega^{2k}}{z\zeta-\omega^{2k}}\right)\right\}d\xi d\eta,\ z\in S, \end{aligned} \tag{4.5}$$

where $\tilde{\Gamma}_1$ and u are defined by (4.3) and (4.4), respectively, and $\alpha\in S$ is a fixed constant.

Proof The problem (4.1) is equivalent to the system

$$\begin{cases} w_{\bar{z}}=u(z),\ z\in S,\\ \partial_{n_\zeta}w(\zeta)=\gamma_0(\zeta),\ \zeta\in\partial S,\ w(\alpha)=c_0,\end{cases} \tag{4.6}$$

$$\begin{cases} u_{\bar{z}} + q u_z = f(z), \ z \in S, \\ \partial_{n_\zeta} u = \gamma_1(\zeta), \ \zeta \in \partial S, \ u(\alpha) = c_1. \end{cases} \tag{4.7}$$

By Theorem 3.2 the Neumann problem (4.7) is uniquely solvable if and only if

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\partial S}\Gamma_5(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)d\zeta \\ &-\frac{1}{\pi}\int_S\sum_{l=0}^{\infty}(-1)^l(q\Pi_1)^l(f-q\varphi_2)(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{(\zeta-\bar{z}\omega^{2k})^2}+\frac{\bar{z}^2}{(\omega^{2k}-\bar{z}\zeta)^2}\right)d\xi d\eta, \\ &= 0, \quad z \in S, \end{aligned} \tag{4.8}$$

where Γ_5, φ_2, and Π_1 are defined by (3.27), (3.30), and (3.15), respectively, and its solution is given by (4.4). Here the Neumann series in (4.8) is covergent under the sufficient condition (3.16).

According to Theorem 2.2 the Neumann problem (4.6) is uniquely solvable if and only if the condition (4.2) is satisfied, and the solution is given by (4.5). □

Acknowledgment

The authors thank the referees for helpful suggestions to improve the readability of the article.

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