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Regular $\mathscr{D}$-classes of the semigroup of $n \times n$ tropical matrices

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Abstract: In this paper we give the characterizations of Green's relations $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{D}$ on the set of matrices with entries in a tropical semiring. An $m \times n$ tropical matrix A is called regular if there exists an $n \times m$ tropical matrix X satisfying $AXA = A$. Furthermore, we study the regular $\mathscr{D}$-classes of the semigroup of all $n \times n$ tropical matrices under multiplication and give a partition of a nonsingular regular $\mathscr{D}$-class.

Key words: Tropical algebra, basis submatrix, nonsingular idempotent matrix, regular matrix

1. Introduction

The set $\mathbb{R}$ of reals extended by adding an infinite negative element $-\infty$ is called a tropical semiring. The tropical operations on $\mathbb{R} \cup \{-\infty\}$ are

$$a \oplus b = max\{a, b\} \text{ and } a \otimes b = a + b.$$

Such algebra is also called the max-plus semiring and is denoted by $\mathbb{T}$. It has been an active area of study in its own right since the 1970s [4] and has broad applications in many different areas of science (see [1–5]). From an algebraic perspective, a key object is the multiplicative semigroup of all square matrices of a given size over the tropical algebra. In particular, Green's relations of the multiplicative semigroup have been studied by some authors in recent years (see [7, 10, 11]). In 2011 Johnson and Kambites [10] studied the algebraic structure of the multiplicative semigroup of all 2×2 tropical matrices. They gave a complete description of Green's relations, idempotents, and maximal subgroups of this semigroup. In 2012 Hollings and Kambites [7] gave a complete description of Green's relation $\mathscr{D}$ for the multiplicative semigroup of all $n \times n$ tropical matrices. Johnson and Kambites [11] studied Green's $\mathscr{J}$-order and $\mathscr{J}$-equivalence for the semigroup of all $n \times n$ matrices over the tropical semiring.

As usual, the set of all $m \times n$ tropical matrices is denoted by $M_{m\times n}(\mathbb{T})$. In particular, we shall use $M_n(\mathbb{T})$ instead of $M_{n\times n}(\mathbb{T})$. The operations $\oplus$ and $\otimes$ on $\mathbb{T}$ induce corresponding operations on tropical matrices in the obvious way. For brevity, we shall write AC in place of $A \otimes C$. It is easy to see that $(M_n(\mathbb{T}), \otimes)$ is a semigroup. Other concepts such as transpose and block matrix are defined in the usual way. For convenience, we refer to a matrix as a tropical matrix in the remainder of this paper. The Green relations $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{D}$

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on the set of matrices with entries in tropical semiring $\mathbb{T}$ are defined as follows:

$$\begin{aligned} A\mathscr{R}B &\Leftrightarrow (\exists\ X,\ Y)\ A = BX,\ B = AY; \\ A\mathscr{L}B &\Leftrightarrow (\exists\ X,\ Y)\ A = XB,\ B = YA; \\ A\mathscr{D}B &\Leftrightarrow (\exists\ C) A\mathscr{R}C,\ C\mathscr{L}B; \\ A\mathscr{H}B &\Leftrightarrow A\mathscr{R}B,\ A\mathscr{L}B; \end{aligned}$$

where A, B, X, Y, C are (possibly rectangular) matrices with entries in $\mathbb{T}$. These relations are classical (and of great importance) in semigroup theory; see, e.g., [8]. Of course, the set of matrices with entries in $\mathbb{T}$ is not a multiplicative semigroup because the product of two matrices is not defined if the size is incompatible. The $\mathscr{R}$-class ($\mathscr{L}$-class, $\mathscr{H}$-class, and $\mathscr{D}$-class resp.) containing matrix A will be written R_A (L_A, H_A, and D_A resp.).

The aim of this paper is to study the nonsingular regular $\mathscr{D}$-classes of the semigroup of $n \times n$ tropical matrices. Some preliminary results are presented in Section 2. In Section 3, we study Green's relation $\mathscr{D}$ on the set of matrices with entries in $\mathbb{T}$. Based on this, we give the characterization of the nonsingular regular $\mathscr{D}$-classes of the semigroup of $n \times n$ tropical matrices in Section 4.

2. Preliminaries

Let $\mathbb{T}^n$ denote the direct product of n copies of $\mathbb{T}$. Then $\mathbb{T}^n$ forms a semiring and can be viewed as a $\mathbb{T}$-semimodule [1]. Each element of this semimodule is called a vector. A vector α in $\mathbb{T}^n$ is called a *linear combination* of a subset $\{\alpha_1, \dots, \alpha_k\}$ of $\mathbb{T}^n$ if there exist $m_1, \dots, m_k \in \mathbb{T}$ such that

$$\alpha = m_1\alpha_1 \oplus \cdots \oplus m_k\alpha_k.$$

For a subset S of $\mathbb{T}^n$, let *span*(S) denote

$$\{\oplus_{i=1}^{k} m_i\alpha_i \mid k \in \mathbb{N}, \alpha_i \in S, m_i \in \mathbb{T}, i = 1, 2, \dots, k\},$$

where $\mathbb{N}$ denotes the set of all natural numbers. Then *span*(S) is a subsemimodule of $\mathbb{T}^n$ generated by S. Recall that the set S is called *weakly linearly dependent* if there exists a vector $\alpha \in S$ such that α is a linear combination of elements in $S \setminus \{\alpha\}$. Otherwise, S is called *weakly linearly independent*. A subset $\{\alpha_i \mid i \in I\}$ of a subsemimodule $\mathcal{V}$ of $\mathbb{T}^n$ is called a *weak basis* of $\mathcal{V}$ if *span*$(\{\alpha_i \mid i \in I\}) = \mathcal{V}$ and $\{\alpha_i \mid i \in I\}$ is weakly linearly independent.

By Theorem 5 in [12] we immediately have the following.

Lemma 2.1 *Let S and S' be weak bases of a subsemimodule $\mathcal{V}$ of $\mathbb{T}^n$. Then for each $\alpha \in S$ there exists a unique $\beta \in S'$ such that $\alpha = m\beta$ for some $m \in \mathbb{R}$.*

Lemma 2.1 tells us that the cardinalities of any two weak bases for a given subsemimodule of $\mathbb{T}^n$ are same. The cardinality is called the *weak dimension* of $\mathcal{V}$ and is denoted by $dim_w\mathcal{V}$. The *column space* (*row space*, resp.) of an $m \times n$ matrix A is the subsemimodule of $\mathbb{T}^m$ ($\mathbb{T}^n$, resp.) spanned by all its columns (rows, resp.) and is denoted by $\mathrm{Col}(A)$ ($\mathrm{Row}(A)$, resp.). The *column rank* (*row rank*, resp.) of A, denoted by $c(A)$ ($r(A)$, resp.), is $dim_w\mathrm{Col}(A)$ ($dim_w\mathrm{Row}(A)$, resp.). An $m \times n$ matrix A is called *nonsingular* if $c(A) = n$ and $r(A) = m$ and otherwise *singular*.

In the sequel, the following notions and notations are needed.

- ⋄ I_n denotes the *identity matrix*, i.e. the $n \times n$ matrix whose diagonal entries are 0 and the other entries are $-\infty$.
- ⋄ An $n \times n$ matrix A is called *invertible* if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, B is called an inverse of A and is denoted by A^{-1}.
- ⋄ An $n \times n$ matrix is called a *monomial matrix* if it has exactly one entry in each row and column that is not equal to $-\infty$.
- ⋄ An $n \times n$ matrix is called a *permutation matrix* if it is formed from the identity matrix by reordering its columns.

It is well known [6] that an $n \times n$ matrix A is invertible if and only if A is monomial. Also, the inverse of a permutation matrix is its transpose. Denote the set of all $n \times n$ monomial matrices by $GL_n(\mathbb{T})$.

For a matrix A, let $\boldsymbol{a}_{i*}$ and $\boldsymbol{a}_{*j}$ denote the ith row and the jth column of A, respectively. As a consequence, it follows from Lemma 2.1 that:

Corollary 2.2 *Let* $\{\boldsymbol{a}_{*i_1}, \ldots, \boldsymbol{a}_{*i_r}\}$ *and* $\{\boldsymbol{a}_{*j_1}, \ldots, \boldsymbol{a}_{*j_k}\}$ *be any two weak bases of Col*(A). *Then* $r = k$, *and there exists an* $r \times r$ *monomial matrix* M *such that*

$$[\boldsymbol{a}_{*i_1} \cdots \boldsymbol{a}_{*i_r}] = [\boldsymbol{a}_{*j_1} \cdots \boldsymbol{a}_{*j_k}]M.$$

In the remainder of this paper, for simplicity, we use the following notation concerning a matrix A without further comment:

$$c(A) = r \text{ and } r(A) = s,$$

- $A^c = [\boldsymbol{a}_{*i_1} \cdots \boldsymbol{a}_{*i_r}]$ is a submatrix of A such that the set $\{\boldsymbol{a}_{*i_1}, \ldots, \boldsymbol{a}_{*i_r}\}$ is a weak basis of $\mathrm{Col}(A)$;
- $A^r = \begin{bmatrix} \boldsymbol{a}_{j_1*} \\ \vdots \\ \boldsymbol{a}_{j_r*} \end{bmatrix}$ is a submatrix of A such that the set $\{\boldsymbol{a}_{j_1*}, \ldots, \boldsymbol{a}_{j_s*}\}$ is a weak basis of $\mathrm{Row}(A)$;
- $\bar{A}$ denotes the $s \times r$ submatrix of A lying in A^c and A^r.

The submatrix $\bar{A}$ is called a *basis submatrix* of A. For any nonzero matrix A we have that $c(A) > 0$ and $r(A) > 0$. It is easy to see that $c(A) = c(\bar{A}) =$ the number of columns of $\bar{A}$, and that $r(A) = r(\bar{A}) =$ the number of rows of $\bar{A}$.

3. Green's $\mathscr{D}$ relations

In this section, we discuss Green's $\mathscr{D}$ relation. First, we need the following result.

Lemma 3.1 *Let* A *and* B *denote two matrices with entries in* $\mathbb{T}$. *Then the following statements are equivalent:*

(i) $A \mathscr{R} B$;

(i) $\mathrm{Col}(A) = \mathrm{Col}(B)$;

(i) $(\exists\ M \in GL_r(\mathbb{T}))\ B^c = A^c M$.

Proof The equivalence of (i) and (ii) was proved by Theorem 100 in "Two lectures on max-plus algebra" (http://amadeus.inria.fr/gaubert).

Suppose that $\mathrm{Col}(A)=\mathrm{Col}(B)$. Then we have that the columns of A^c and B^c are both weak bases of $\mathrm{Col}(A)$. It follows that

$$\mathrm{Col}(A)=\mathrm{Col}(B) \Longleftrightarrow (\exists\ M\in GL_r(\mathbb{T}))\ B^c=A^cM \quad \text{(by Corollary 2.2)}.$$

□

The dual of Lemma 3.1 for relation $\mathscr{L}$ is clear.

Lemma 3.2 *Let A and B denote two matrices with entries in $\mathbb{T}$. Then the following statements are equivalent:*

(i) $A\,\mathscr{D}\,B$;

(i) $\bar{A}\,\mathscr{D}\,\bar{B}$;

(i) $(\exists\ N\in GL_s(\mathbb{T}), M\in GL_r(\mathbb{T}))\ \bar{B}=N\bar{A}M$.

Proof We need to prove it only for any nonzero matrices A and B. Since $(PAQ)\,\mathscr{D}\,A$ for any permutation matrices P and Q, we may assume that

$$A=\begin{bmatrix}\bar{A} & A_1\\ A_2 & A_3\end{bmatrix}, B=\begin{bmatrix}\bar{B} & B_1\\ B_2 & B_3\end{bmatrix},$$

where $\bar{A},\bar{B}$ are basic submatrices and A_i,B_i are of appropriate sizes for $i=1,2,3$.

(i) $\Longleftrightarrow$ (ii). Suppose that $A\,\mathscr{D}\,B$. Since

$$\mathrm{Col}(\begin{bmatrix}\bar{A}\\ A_2\end{bmatrix})=\mathrm{Col}(A),$$

it follows that $A\,\mathscr{R}\begin{bmatrix}\bar{A}\\ A_2\end{bmatrix}$ by Lemma 3.1. Since $\mathrm{Row}(\begin{bmatrix}\bar{A}\\ A_2\end{bmatrix})=\mathrm{Row}(\bar{A})$, we have that $\begin{bmatrix}\bar{A}\\ A_2\end{bmatrix}\mathscr{L}\,\bar{A}$ by the dual of Lemma 3.1. Thus, $A\,\mathscr{D}\,\bar{A}$. Similarly, $B\,\mathscr{D}\,\bar{B}$. Therefore, $A\,\mathscr{D}\,B$ is equivalent to $\bar{A}\,\mathscr{D}\,\bar{B}$¡£

(ii) $\Leftrightarrow$ (iii). (ii) is equivalent to

$$\bar{A}\,\mathscr{R}\,C \text{ and } C\,\mathscr{L}\,\bar{B} \tag{3.1}$$

for some nonsingular matrix C.

Suppose that (3.1) holds. Then (3.1) implies that

$$\bar{A}=CM^{-1} \text{ and } \bar{B}=NC$$

for some monomial matrix M and some monomial matrix N by Lemma 3.1 and its dual. Thus, we obtain that $N\bar{A}M=\bar{B}$.

If (iii) holds, then there exist two monomial matrices N and M such that $N\bar{A}M=\bar{B}$. Hence, (3.1) holds for $C=\bar{A}M$. □

As a consequence, we have the following.

Corollary 3.3 *Let A denote a matrix with entries in $\mathbb{T}$. If $A\,\mathscr{D}\,B$, then $r(A)=r(B)$ and $c(A)=c(B)$.*

An $m \times n$ matrix A is *regular* if there exists an $n \times m$ matrix X such that $AXA = A$. An $n \times n$ matrix A is *idempotent* if $A^2 = A$. In [8] Proposition 3.2, we know that in a regular $\mathscr{D}$-class each $\mathscr{R}$-class and each $\mathscr{L}$-class contains at least one idempotent. Let E be an $n \times n$ idempotent matrix. If $B \in D_E$, then by Lemma 3.2 we have

$$B = Q \begin{bmatrix} C & CH \\ WC & WCH \end{bmatrix} P$$

where $C \in D_{\bar{E}}$, P and Q are permutation matrices, and H and W are matrices of appropriate sizes. Hence, the regular $\mathscr{D}$-class D_E is determined by $D_{\bar{E}}$. We will study the nonsingular regular $\mathscr{D}$-classes in the next section.

4. Nonsingular regular $\mathscr{D}$-classes

In this section, we discuss the nonsingular regular $\mathscr{D}$-classes of the semigroup $(M_n(\mathbb{T}), \otimes)$. The $\mathscr{R}$-class ($\mathscr{L}$-class, $\mathscr{H}$-class, and $\mathscr{D}$-class resp.) in the semigroup $(M_n(\mathbb{T}), \otimes)$ is the restriction of the corresponding class in $\cup_{m=1}^{\infty} \cup_{n=1}^{\infty} M_{m\times n}(\mathbb{T})$. If a matrix of a regular $\mathscr{D}$-class is nonsingular, then by Corollary 3.3 and Proposition 3.1 in [8] we have that the matrices of this $\mathscr{D}$-class are all nonsingular regular matrices. We call the $\mathscr{D}$-class nonsingular regular $\mathscr{D}$-classes.

Next, recall the partial order [3] $\leq$ on $M_{m\times n}(\mathbb{T})$ by

$$A \leq B \Longleftrightarrow A \oplus B = B.$$

Lemma 4.1 ([3]) *Let A, B be $m \times n$ matrices, let X be an $n \times p$ matrix, and let Y be a $p \times m$ matrix. Then the following statements hold.*

(i) $A \leq A \oplus B$;

(i) If $A \leq B$, then $AX \leq BX$ and $YA \leq YB$.

Lemma 4.2 *If $E = (e_{ij})$ is an $n \times n$ idempotent matrix, then $e_{ii} \leq 0$ for all $1 \leq i \leq n$.*

Proof Let $E = (e_{ij})_{n\times n}$ be an idempotent matrix. Then for any $1 \leq i \leq n$,

$$e_{ii} \otimes e_{ii} \leq (e_{i1} \otimes e_{1i}) \oplus \cdots \oplus (e_{ii} \otimes e_{ii}) \oplus \cdots \oplus (e_{in} \otimes e_{ni}) = e_{ii}.$$

This implies that $e_{ii} \leq 0$. □

Lemma 4.3 *Let $E = (e_{ij})$ be an $n \times n$ idempotent matrix. If $e_{ii} < 0$ for some $i \in \{1, 2, \ldots, n\}$, then the ith column (row, resp.) of E is a linear combination of the remaining columns (rows, resp.). Furthermore, the matrix obtained from E by deleting the ith column and the ith row is an $(n-1) \times (n-1)$ idempotent matrix.*

Proof Let $E = (e_{ij})_{n\times n}$ be an idempotent matrix. Suppose that $e_{ii} < 0$ for some $1 \leq i \leq n$. Without loss of generality, we assume that $e_{11} < 0$. Partition E as $\begin{bmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$. Then we have

$$E^2 = \begin{bmatrix} e_{11} \otimes e_{11} \oplus E_{12}E_{21} & e_{11}E_{12} \oplus E_{12}E_{22} \\ E_{21}e_{11} \oplus E_{22}E_{21} & E_{21}E_{12} \oplus E_{22}^2 \end{bmatrix} = \begin{bmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} E_{12}E_{21} & E_{12}E_{22} \\ E_{22}E_{21} & E_{21}E_{12} \oplus E_{22}^2 \end{bmatrix} = \begin{bmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

since $e_{11} < 0$. Thus, it follows that

$$\begin{bmatrix} e_{11} \\ E_{21} \end{bmatrix} = \begin{bmatrix} E_{12}E_{21} \\ E_{22}E_{21} \end{bmatrix} = \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} E_{21}, \tag{4.1}$$

$$\begin{bmatrix} e_{11} & E_{12} \end{bmatrix} = \begin{bmatrix} E_{12}E_{21} & E_{12}E_{22} \end{bmatrix} = E_{12} \begin{bmatrix} E_{21} & E_{22} \end{bmatrix}, \tag{4.2}$$

$$E_{21}E_{12} \oplus E_{22}^2 = E_{22}. \tag{4.3}$$

Equation (4.1) ((4.2), resp.) tells us that the 1st column (the 1st row, resp.) of E is a linear combination of the remaining columns (rows, resp.). By Lemma 4.1 and (4.3), we have

$$E_{22}^2 \leq E_{22} \text{ and } E_{21}E_{12} \leq E_{22}. \tag{4.4}$$

Thus, it follows by (4.4) and Lemma 4.1 that $E_{21}E_{12} = E_{21}E_{12}E_{22} \leq E_{22}^2$, since $E_{12}E_{22} = E_{12}$. We therefore have

$$E_{22} = E_{21}E_{12} \oplus E_{22}^2 \leq E_{22}^2 \oplus E_{22}^2 = E_{22}^2 \tag{4.5}$$

by Lemma 4.1. Thus, (4.4) and (4.5) tell us that $E_{22}^2 = E_{22}$. □

The above lemma tells us that if $E = (e_{ij})$ is an $n \times n$ idempotent matrix and $e_{ii} < 0$ for some $1 \leq i \leq n$, then $c(E) < n$ and $r(E) < n$. Thus, by Lemmas 4.2 and 4.3, we immediately have the following result, which was obtained previously in [9] by a different method.

Corollary 4.4 *All main diagonal entries of a nonsingular idempotent matrix are* 0.

Proposition 4.5 *Let* E *be a nonsingular idempotent matrix. If there exists a monomial matrix* M*, such that* $EME = E$*, then* $M = I_n$.

Proof Suppose that $E = (e_{ij})$ is an $n \times n$ nonsingular idempotent matrix and that there exists a matrix M, such that $EME = E$. It follows that EM is idempotent and $E \mathscr{R} EM$. Thus, by Corollary 3.3 we can see that EM is a nonsingular idempotent matrix. It follows by Corollary 4.4 that the main diagonal entries of E and EM are all 0. Since $EME = E$, $M_{ij} \leq (EME)_{ij} = E_{ij}$, and so $M \leq E$. Hence,

$$EM \leq E^2 = E \tag{4.6}$$

by Lemma 4.1 (ii). It follows by $(EM)E(EM) = EM$ that $E_{ij} \leq ((EM)E(EM))_{ij} = (EM)_{ij}$. Thus,

$$E \leq EM. \tag{4.7}$$

(4.6) and (4.7) tell us that $EM = E$.

Finally, assume that M is monomial. If M is not diagonal, it follows from $EM = E$ that there exist two distinct indices j and l such that $\boldsymbol{e}_{*j} = a\boldsymbol{e}_{*l}$ for some real number a, a contradiction, since E is nonsingular. Then M is diagonal and hence $M = I_n$. □

Proposition 4.6 *Let E be a nonsingular idempotent matrix. If F is an idempotent matrix in D_E, then there exists a monomial matrix M such that $F = MEM^{-1}$.*

Proof Suppose that E and F are $n \times n$ nonsingular idempotent matrices. If $F \in D_E$, then, by Lemma 3.2, we can show that $F = MEN$ for some $M,\ N \in GL_n(\mathbb{T})$. Thus, it follows that $MEN = F = F^2 = MENMEN$. This implies that $E = EMNE$. Hence, we have by Proposition 4.5 that $MN = I_n$ and so $F = MEM^{-1}$. This completes the proof. □

The following result is a corollary of Theorem 5.7 in [9]. We note that our result is obtained by elementary matrix techniques.

Proposition 4.7 *Any nonsingular regular $\mathscr{R}$-class ($\mathscr{L}$-class, resp.) contains a unique idempotent.*

Proof Suppose that R_A is a nonsingular regular $\mathscr{R}$-class. Then by Proposition 3.2 in [8] there exists a nonsingular idempotent matrix E such that $R_A = R_E$. If F is an idempotent matrix in R_E, then by Lemma 3.1 we can show that $F = EM$ for some monomial matrix M. Thus, $EM = F = F^2 = EMEM$ and so $E = EME$. It follows by Proposition 4.5 that $M = I_n$. Hence, $F = E$. A similar argument establishes that there exists a unique idempotent in each nonsingular regular $\mathscr{L}$-class. □

Lemma 4.8 *Let E be a nonsingular idempotent matrix. If there exist monomial matrices M_1 and M_2 such that $EM_1 = M_2E$, then $M_1 = M_2$.*

Proof Let E be an $n \times n$ nonsingular idempotent matrix. Suppose that there exist monomial matrices M_1 and M_2 such that $EM_1 = M_2E$. Then we have

$$\begin{aligned} EM_1 = M_2E &\Longrightarrow E = M_2EM_1^{-1} \\ &\Longrightarrow M_2EM_1^{-1}M_2EM_1^{-1} = M_2EM_1^{-1} \\ &\Longrightarrow EM_1^{-1}M_2E = E \\ &\Longrightarrow M_1^{-1}M_2 = I_n \qquad\qquad (\textit{by Proposition } 4.5) \\ &\Longrightarrow M_1 = M_2. \end{aligned}$$

□

Lemma 4.9 *If E is a nonsingular idempotent, then the set*

$$C_E(GL_n(\mathbb{T})) = \{M \in GL_n(\mathbb{T}) \mid EM = ME\}$$

is a subgroup of the group $GL_n(\mathbb{T})$.

Proof Suppose that E is a nonsingular idempotent. Since $EI_n = I_nE = E$ we have that $I_n \in C_E(GL_n(\mathbb{T}))$. If $M_1, M_2 \in C_E(GL_n(\mathbb{T}))$, then $EM_1 = M_1E, EM_2 = M_2E$, and it follows that

$$EM_1M_2 = M_1EM_2 = M_1M_2E,$$

and so $M_1M_2 \in C_E(GL_n(\mathbb{T}))$. If $M \in C_E(GL_n(\mathbb{T}))$, then $EM = ME$, and so

$$M^{-1}E = EM^{-1}.$$

Thus, $M^{-1} \in C_E(GL_n(\mathbb{T}))$. Hence, $C_E(GL_n(\mathbb{T}))$ is a subgroup of $GL_n(\mathbb{T})$. □

Proposition 4.10 *Let E be an $n \times n$ nonsingular idempotent matrix. Then*

$$H_E = \{A \mid (\exists M \in C_E(GL_n(\mathbb{T}))A = ME\}.$$

Proof Suppose that E is an $n\times n$ nonsingular idempotent matrix. Then $H_E = \{EM \mid M \in GL_n(\mathbb{T})\} \cap \{ME \mid M \in GL_n(\mathbb{T})\}$ and so $H_E = \{A \mid (\exists M,\ N \in GL_n(\mathbb{T}))A = NE = EM\}$. It follows by Lemma 4.8 that $H_E = \{A \mid (\exists M \in C_E(GL_n(\mathbb{T})))A = ME = EM\}$. □

Proposition 4.11 *Let E be a nonsingular idempotent matrix and F be an idempotent matrix in D_E. Then there exists a monomial matrix M such that*

$$H_F = \{MBM^{-1} \mid B \in H_E\}.$$

Proof Suppose that E is an $n \times n$ nonsingular idempotent matrix and F is an idempotent $n \times n$ matrix in D_E. Then by Proposition 4.6 we have that there exists a monomial matrix M such that $F = MEM^{-1}$. It follows by Proposition 4.10 that

$$H_F = \{A \mid (\exists M \in GL_n(\mathbb{T}))A = MF = FM\}$$

and that

$$H_E = \{A \mid (\exists M \in GL_n(\mathbb{T}))A = ME = EM\}.$$

If $A \in H_F$, then there exists a monomial matrix M_1 such that $A = FM_1 = M_1F$. Then

$$\begin{aligned} FM_1 = M_1F &\Longleftrightarrow MEM^{-1}M_1 = M_1MEM^{-1} \\ &\Longleftrightarrow EM^{-1}M_1M = M^{-1}M_1ME \\ &\Longleftrightarrow EM^{-1}M_1M \in H_E. \end{aligned}$$

It follows that $A = MEM^{-1}M_1 = M(EM^{-1}M_1M)M^{-1}$ and so $A \in \{MBM^{-1} \mid B \in H_E\}$. Thus, we can see that $H_F \subseteq \{MBM^{-1} \mid B \in H_E\}$. A similar argument establishes that $\{MBM^{-1} \mid B \in H_E\} \subseteq H_F$. □

We define a relation ϱ on the set $GL_n(\mathbb{T}) \times GL_n(\mathbb{T})$ as follows:

$$(M_1, N_1)\varrho(M_2, N_2) \Longleftrightarrow {M_1}^{-1}M_2, N_1{N_2}^{-1} \in C_E(GL_n(\mathbb{T})).$$

Lemma 4.12 *If E is a nonsingular idempotent, then ϱ is a equivalence relation on the set $GL_n(\mathbb{T}) \times GL_n(\mathbb{T})$.*

Proof Suppose that E is a nonsingular idempotent. If $(M, N) \in GL_n(\mathbb{T}) \times GL_n(\mathbb{T})$, then by Lemma 4.9, we have that

$$M^{-1}M = NN^{-1} = I_n \in C_E(GL_n(\mathbb{T})),$$

and so $(M, N)\varrho(M, N)$.

If $(M_1, N_1), (M_2, N_2) \in GL_n(\mathbb{T}) \times GL_n(\mathbb{T})$ and $(M_1, N_1)\varrho(M_2, N_2)$, then

$${M_1}^{-1}M_2, N_1{N_2}^{-1} \in C_E(GL_n(\mathbb{T})).$$

It follows by Lemma 4.9 that

$$M_2{}^{-1}M_1, N_2N_1{}^{-1} \in C_E(GL_n(\mathbb{T})).$$

This implies that $(M_2, N_2)\varrho(M_1, N_1)$.

Finally, if $(M_1, N_1)\varrho(M_2, N_2)$ and $(M_2, N_2)\varrho(M_3, N_3)$, then

$$M_1{}^{-1}M_2, N_1N_2{}^{-1}, M_2{}^{-1}M_3, N_2N_3{}^{-1} \in C_E(GL_n(\mathbb{T})),$$

and so

$$M_1{}^{-1}M_3 = M_1{}^{-1}M_2M_2{}^{-1}M_3 \in C_E(GL_n(\mathbb{T})),$$

$$N_1N_3{}^{-1} = N_1N_2{}^{-1}N_2N_3{}^{-1} \in C_E(GL_n(\mathbb{T})).$$

Hence, $(M_1, N_1)\varrho(M_3, N_3)$.

We have therefore proved that ϱ is an equivalence relation on the set $GL_n(\mathbb{T}) \times GL_n(\mathbb{T})$. □

Lemma 4.13 *Let E be a nonsingular idempotent matrix. If $A, B \in D_E$, then there exist monomial matrices M_1, M_2, N_1, N_2 such that $A = M_1EN_1$, $B = M_2EN_2$. Further,*

$$H_A = H_B \iff (M_1, N_1)\varrho(M_2, N_2).$$

Proof If E is a nonsingular idempotent matrix and $A, B \in D_E$, then by Lemma 3.2 we can see that

$$A = M_1EN_1, B = M_2EN_2$$

for some monomial matrices M_1, N_1, M_2, and N_2. Then

$$\begin{aligned}
H_A = H_B &\iff H_{M_1EN_1} = H_{M_2EN_2} \\
&\iff M_1EN_1\mathscr{H}M_2EN_2 \\
&\iff M_1EN_1\mathscr{L}M_2EN_2, M_1EN_1\mathscr{R}M_2EN_2 \\
&\iff (\exists S, T \in GL_n(\mathbb{T}))M_1EN_1 = SM_2EN_2, M_1EN_1 = M_2EN_2T \quad (\textit{by Lemma } 3.1) \\
&\iff (\exists S, T \in GL_n(\mathbb{T}))EN_1N_2{}^{-1} = M_1{}^{-1}SM_2E, EN_1T^{-1}N_2{}^{-1} = M_1{}^{-1}M_2E \\
&\iff (\exists S, T \in GL_n(\mathbb{T}))M_1{}^{-1}SM_2 = N_1N_2{}^{-1} \in C_E(GL_n(\mathbb{T})), \\
&\qquad N_1T^{-1}N_2{}^{-1} = M_1{}^{-1}M_2 \in C_E(GL_n(\mathbb{T})) \quad (\textit{by Lemma } 4.8) \\
&\Longrightarrow M_1{}^{-1}M_2, N_1N_2{}^{-1} \in C_E(GL_n(\mathbb{T})) \qquad (\textit{by Lemma } 4.9) \\
&\Longrightarrow (M_1, N_1)\varrho(M_2, N_2).
\end{aligned}$$

Conversely, if $(M_1, N_1)\varrho(M_2, N_2)$, then $M_1{}^{-1}M_2, N_1N_2{}^{-1} \in C_E(GL_n(\mathbb{T}))$, and so

$$M_1EN_1 = M_1EN_1N_2{}^{-1}N_2 = M_1N_1N_2{}^{-1}EN_2 = (M_1N_1N_2{}^{-1}M_2{}^{-1})M_2EN_2,$$

$$M_1EN_1 = M_2M_2{}^{-1}M_1EN_1 = M_2EM_2{}^{-1}M_1N_1 = M_2EN_2(N_2{}^{-1}M_2{}^{-1}M_1N_1).$$

Thus, $M_1EN_1\mathscr{H}M_2EN_2$.

Hence, we have therefore proved that $H_A = H_B$ if and only if $(M_1, N_1)\varrho(M_2, N_2)$. □

By Lemma 3.2 and Lemma 4.13, we now have the following result:

Theorem 4.14 *Let E be a nonsingular idempotent matrix. Then*

$$D_E = \bigcup \{H_{MEN} \mid (M,N)\varrho \in (GL_n(\mathbb{T}) \times GL_n(\mathbb{T}))/\varrho\}.$$

H_E	H_{EN_1}	H_{EN_2}	$\cdots$
$H_{N_1{}^{-1}E}$	$H_{N_1{}^{-1}EN_1}$	$H_{N_1{}^{-1}EN_2}$	$\cdots$
$H_{N_2{}^{-1}E}$	$H_{N_2{}^{-1}EN_1}$	$H_{N_2{}^{-1}EN_2}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\ddots$

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