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On radial solutions for Monge–Ampère equations

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Abstract: In this paper,we obtain some new existence, uniqueness, and multiplicity results of radial solutions of an elliptic system coupled by Monge–Ampère equations using the fixed point theorem.

Key words: Monge–Ampère equations, radial solution, uniqueness and multiplicity, fixed point theorem

1. Introduction

Monge–Ampère equations are fully nonlinear second order PDEs that have many important applications in geometry and other scientific fields. Much attention has been focused on the study of problems with a single equation (see [1, 3, 5–10, 12, 17]) like

$$\begin{cases} \det(D^2u) = f(u), & in\ B, \\ u(x) = 0, & on\ \partial B, \end{cases} \tag{1.1}$$

or equations (see [2, 11, 13–16]) like

$$\begin{cases} \det(D^2u_1) = f_1(-u_1, \cdots, -u_n), & in\ B, \\ \cdots \\ \det(D^2u_n) = f_n(-u_1, \cdots, -u_n), & in\ B, \\ u_i(x) = 0, & on\ \partial B, \end{cases} \tag{1.2}$$

where $f_i (i = 1, 2 \cdots n)$ are continuous, $D^2u(x) = (\frac{\partial^2 u}{\partial x_j \partial x_k})$, $j, k = 1, 2 \cdots n$, is the Hessian matrix of $u(x)$, $B = \{x \in R^N : |x| < 1\}$. One of the important research directions is the existence of nontrivial solutions of (1.1) or (1.2), which has been studied by many researchers in the nonsingular case [2, 6–8, 11–15, 17] as well as the singular case [3, 10, 16].

On one hand, authors [11] investigated the existence, uniqueness, and nonexistence results of radial solutions of the following system of equations:

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$$\begin{cases} \det D^2 u_1 = (-u_2)^{\alpha}, & in\ B, \\ \det D^2 u_2 = (-u_1)^{\beta}, & in\ B, \\ u_1 = u_2 = 0, & on\ \partial B, \end{cases} \tag{1.3}$$

which are expressed as follows:

Theorem A *[11] Assume that $\alpha, \beta > 0$. Then we have*

(I) (existence) If $\alpha\beta \neq N^2$, then (1.3) has at least a radial convex solution.

(II) (uniqueness) If $\alpha\beta < N^2$, then (1.3) has a unique radial convex solution.

(III) (nonexistence) If $\alpha\beta = N^2$, then (1.3) has no radial convex solution.

Furthermore, the authors handle more general ones, i.e.

$$\begin{cases} \det D^2 u = f_1(-v), & in\ B, \\ \det D^2 v = f_2(-u), & in\ B, \\ u = v = 0, & on\ \partial B. \end{cases} \tag{1.4}$$

Similar arguments go through and the authors get the following conclusion: If $f_1, f_2 : [0,\infty) \to [0,\infty)$ are continuous, both nondecreasing, then (1.4) admits a solution if one of the following cases is satisfied:

(i) $\lim_{x\to 0^+} \frac{f_1^{\frac{1}{N}}(f_2^{\frac{1}{N}}(x))}{x} = 0$ *and* $\lim_{x\to\infty} \frac{f_1^{\frac{1}{N}}(f_2^{\frac{1}{N}}(x))}{x} = \infty$;

(ii) $\lim_{x\to\infty} \frac{f_1^{\frac{1}{N}}(f_2^{\frac{1}{N}}(x))}{x} = 0$ *and* $\lim_{x\to 0^+} \frac{f_1^{\frac{1}{N}}(f_2^{\frac{1}{N}}(x))}{x} = \infty$.

However, the uniqueness and multiplicity results of radial solutions of (1.4) are not considered. Thus, one of the important results of this paper is concerned with the existence, uniqueness, and multiplicity of the positive radial solutions of a more general system, i.e.

$$\begin{cases} \det D^2 u = f_1(x, -v), & in\ B, \\ \det D^2 v = f_2(x, -u), & in\ B, \\ u = v = 0, & on\ \partial B. \end{cases} \tag{1.5}$$

The results we are going to present reveal how the behavior of the function $f_i (i = 1, 2)$ at zero and infinity, where $f_i : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous, and they are nondecreasing on the second variable,

$$\underline{\varphi}_i^0 = \liminf_{c\to 0^+} \min_{t\in[0,1]} \frac{f_i(t,c)}{c^{\alpha_i}},\ \overline{\psi}_i^\infty = \limsup_{c\to\infty} \max_{t\in[0,1]} \frac{f_i(t,c)}{c^{\beta_i}},$$

$$\overline{\varphi}_i^0 = \limsup_{c\to 0^+} \max_{t\in[0,1]} \frac{f_i(t,c)}{c^{\alpha_i}},\ \underline{\psi}_i^\infty = \liminf_{c\to\infty} \min_{t\in[0,1]} \frac{f_i(t,c)}{c^{\beta_i}},$$

has a profound effect on the number of nontrivial solutions of problem (1.5). In particular, we assume that $f_2(x, 0) = 0$ in Section 2.

On the other hand, in [2, 14], the authors mainly consider the existence of nontrivial solutions of (1.2) under the notations

(N) there exist nonnegative constants f_i^0, f_i^∞ defined as

$$f_i^0 = \lim_{\sum|u_i|\to 0} \frac{f_i(u)}{(\sum|u_i|)^N},\ and\ \ f_i^\infty = \lim_{\sum|u_i|\to\infty} \frac{f_i(u)}{(\sum|u_i|)^N}.$$

Another important result of this paper is concerned with the existence of positive radial solutions of problem (1.2) under the following assumption:

(H) there exist two pairs of nonnegative continuous functions F_{i*}, F_i^* $(i=1,\ldots,n)$ such that

$$\begin{aligned} &F_{i*}(-u_{i_0}) \le f_i(-u_1,\cdots,-u_n) \le F_i^*(-u_{i_0}),\\ &For\ some\ -u_{i_0} \in \{-u_j\},\ u_{i_0} \ne u_{k_0}, if\ i \ne k. \end{aligned}$$

For convenience, we give the following notations:

$$\underline{F_{i*}^0} = \liminf_{c\to 0^+} \frac{F_{i*}(c)}{c^{\alpha_i}},\ \ \overline{F_i^*}^\infty = \limsup_{c\to+\infty} \frac{F_i^*(c)}{c^{\beta_i}},$$

$$\underline{F_{i*}^\infty} = \liminf_{c\to+\infty} \frac{F_{i*}(c)}{c^{\alpha_i}},\ \ \overline{F_i^*}^0 = \limsup_{c\to 0^+} \frac{F_i^*(c)}{c^{\beta_i}}.$$

At the end of Section 3, we also give examples to illustrate that $f_i(u)$ satisfies the assumption (H).

Finally, for radial solution $u(r)$ with $r=\sqrt{\sum_1^N x_i^2}$, the Monge–Ampère operator simply becomes

$$\det(D^2u) = \frac{(u')^{N-1}u''}{r^{N-1}} = \frac{1}{Nr^{N-1}}((u')^N)'.$$

Inspired by [11, 14], in the rest of this paper, we mainly pay more attention to the boundary value problems, respectively,

$$\begin{cases} ((-u'(t))^N)' = Nt^{N-1}f_1(t,v), \quad t\in(0,1),\\ ((-v'(t))^N)' = Nt^{N-1}f_2(t,u), \quad t\in(0,1),\\ u(t)>0, v(t)>0, \quad t\in(0,1),\\ u'(0)=u(1)=0, v'(0)=v(1)=0 \end{cases} \tag{1.6}$$

and

$$\begin{cases} ((-v_1'(t))^N)' = Nt^{N-1}f_1(v_1,v_2,\ldots,v_n),\\ ((-v_2'(t))^N)' = Nt^{N-1}f_2(v_1,v_2,\ldots,v_n),\\ \cdots\\ ((-v_n'(t))^N)' = Nt^{N-1}f_n(v_1,v_2,\ldots,v_n),\\ v_i'(0)=v_i(1)=0, \quad i=1,2,\ldots,n. \end{cases} \tag{1.7}$$

The discussion is based on the following lemmas.

Lemma 1.1 *[4] Let E be a Banach space and $K\subset E$ be a cone in E. Assume Ω_1, Ω_2 are bounded, open subsets of E with $0\in\Omega_1, \overline{\Omega}_1\subset\Omega_2$, and let $A: K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be a completely continuous operator such that either* *(i)* $\|Au\| \leq \|u\|, \ \ u \in K \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|, \ \ u \in K \cap \partial\Omega_2$ *; or*

(ii) $\|Au\| \geq \|u\|, \ \ u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|, \ \ u \in K \cap \partial\Omega_2$ *.*

Then A *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ *.*

Definition 1.2 *[4] Let* K *be a cone in a real Banach space* E*. With some* $u_0 \in K$ *positive,* $A : K \to K$ *is called* u_0*-sublinear if*
(a) *for any* $u > 0$*,there exists* $\theta_1 > 0, \theta_2 > 0$ *such that* $\theta_1 u_0 \leq Au \leq \theta_2 u_0$*;*
(b) *for any* $\theta_1 u_0 \leq u \leq \theta_2 u_0$ *and* $t \in (0,1)$*, there always exists some* $\eta = \eta(u,t) > 0$ *such that* $A(tu) \geq (1+\eta)tAu$*.*

Lemma 1.3 *[4] An increasing and* u_0*-sublinear operator* A *has at most one positive fixed point.*

2. Results of (1.6)

Theorem 2.1 *Assume that* $\alpha_i, \ \beta_i > 0 \ (i = 1, 2)$ *with*

$$\alpha_1\alpha_2 \leq N^2, \ \beta_1\beta_2 \leq N^2.$$

If $\underline{\varphi}_1^0 > 0, \ \underline{\varphi}_2^0 = +\infty, \ \overline{\psi}_1^\infty < +\infty, \ \overline{\psi}_2^\infty = 0,$ *then (1.6) has at least a solution.*

Proof Define a mapping $A : C[0,1] \to C[0,1]$ by

$$A(u)(t) = A_1 \circ A_2(u)(t),$$

where

$$A_1(v)(t) = \int_t^1 (\int_0^s N\tau^{N-1} f_1(\tau, v(\tau))d\tau)^{\frac{1}{N}} ds, \quad t \in [0,1],$$

$$A_2(u)(t) = \int_t^1 (\int_0^s N\tau^{N-1} f_2(\tau, u(\tau))d\tau)^{\frac{1}{N}} ds, \quad t \in [0,1].$$

Now our main goal is to look for nontrivial fixed points of A in a subcone $K \subset C[0,1]$ defined by

$$K = \{u \in C[0,1] : u(t) \geq 0, \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq \frac{1}{4}\|u\|\},$$

where $\|u(t)\| = \max_{t\in[0,1]} |u(t)|$. From the standard process (see [11]), it follows that $A : K \to K$ is completely continuous.

For any $\gamma > 0$, in the following paragraphs, we set

$$\Omega_\gamma = \{u \in C[0,1] : \|u\| < \gamma\},$$

and

$$\partial\Omega_\gamma = \{u \in C[0,1] : \|u\| = \gamma\}.$$

On one hand, by the definitions of $\underline{\varphi}_1^0 > 0, \ \underline{\varphi}_2^0 = +\infty$, there exist $C_1 > 0$ and $r_1 \in (0,1)$ such that

$$f_1(t,v) \geq (\underline{\varphi}_1^0 - \epsilon)v^{\alpha_1}, \ for \ t \in [0,1], \ 0 \leq v \leq r_1,$$

$$f_2(t,u) \geq C_1 u^{\alpha_2},\ for\ t\in[0,1],\ 0\leq u\leq r_1,$$

where C_1 satisfies

$$\frac{1}{4^{\frac{\alpha_1 N+\alpha_1\alpha_2}{N^2}}} C_1^{\frac{\alpha_1}{N^2}} \Gamma^{\frac{\alpha_1+N}{N}} (\underline{\varphi}_1^0-\epsilon)^{\frac{1}{N}} \geq 1, \Gamma = \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\int_{\frac{1}{4}}^{s} N\tau^{N-1}d\tau\right)^{\frac{1}{N}} ds.$$

Since f_2 is continuous and $f_2(t,0)=0$, there exists an $r_2\in(0,r_1)$ such that

$$f_2(t,u)\leq r_1^N,\ \forall\, t\in[0,1],\ u\in[0,r_2].$$

For any $u\in K\cap\partial\Omega_{r_2}$, we have

$$\begin{aligned} v(t)=A_2(u)(t) &= \int_t^1 (\int_0^s N\tau^{N-1} f_2(\tau,u(\tau))d\tau)^{\frac{1}{N}} ds \\ &\leq \int_0^1 (\int_0^s N\tau^{N-1} r_1^N d\tau)^{\frac{1}{N}} ds \leq r_1. \end{aligned}$$

Furthermore, we can get

$$\begin{aligned} A_1(v)(\frac{1}{4}) &= \int_{\frac{1}{4}}^1 (\int_0^s N\tau^{N-1} f_1(\tau,v(\tau))d\tau)^{\frac{1}{N}} ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^s N\tau^{N-1} (\underline{\varphi}_1^0-\epsilon)(\frac{1}{4}\|v\|)^{\alpha_1} d\tau)^{\frac{1}{N}} ds \\ &\geq (\underline{\varphi}_1^0-\epsilon)^{\frac{1}{N}} (\frac{1}{4})^{\frac{\alpha_1}{N}} \Gamma \|v\|^{\frac{\alpha_1}{N}} \end{aligned}$$

and

$$\begin{aligned} v(\frac{1}{4})=A_2(u)(\frac{1}{4}) &= \int_{\frac{1}{4}}^1 (\int_0^s N\tau^{N-1} f_2(\tau,u(\tau))d\tau)^{\frac{1}{N}} ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^s N\tau^{N-1} C_1(\frac{1}{4}\|u\|)^{\alpha_2} d\tau)^{\frac{1}{N}} ds \\ &\geq (C_1)^{\frac{1}{N}} (\frac{1}{4})^{\frac{\alpha_2}{N}} \Gamma \|u\|^{\frac{\alpha_2}{N}}. \end{aligned}$$

Thus for any $u\in K\cap\partial\Omega_{r_2}$, from the above inequalities it follows that

$$\begin{aligned} \|A(u)(t)\| &\geq A_1\circ A_2(u)(\frac{1}{4}) \\ &\geq \frac{1}{4^{\frac{\alpha_1 N+\alpha_1\alpha_2}{N^2}}} C_1^{\frac{\alpha_1}{N^2}} \Gamma^{\frac{\alpha_1+N}{N}} (\underline{\varphi}_1^0-\epsilon)^{\frac{1}{N}} \|u\|^{\frac{\alpha_1\alpha_2}{N^2}} \geq \|u\|. \end{aligned}$$

On the other hand, by the definitions of $\overline{\psi}_1^\infty<+\infty$, $\overline{\psi}_2^\infty=0$, there exist $\epsilon_1>0$ and $R_1>0$ such that

$$f_1(t,v)\leq(\overline{\psi}_1^\infty+\varepsilon)v^{\beta_1},\ for\ t\in[0,1],\ v\geq R_1,$$

$$f_2(t,u) \le \epsilon_1 u^{\beta_2},\ for\ t \in [0,1],\ u \ge R_1,$$

where ϵ_1 satisfies

$$\epsilon_1^{\frac{\beta_1}{N^2}} (\overline{\psi}_1^{\infty} + \varepsilon)^{\frac{1}{N}} < 1.$$

Since f_i is continuous, let

$$M_1 = \max\{f_1(t,v) : 0 \le t \le 1,\ 0 \le v \le R_1\},$$

$$M_2 = \max\{f_2(t,u) : 0 \le t \le 1, 0 \le u \le R_1\}.$$

Then we have

$$f_1(t,v) \le (\overline{\psi}_1^{\infty} + \varepsilon) v^{\beta_1} + M_1,$$

$$f_2(t,u) \le \ \epsilon_1 u^{\beta_2} + M_2.$$

Furthermore, we have

$$\begin{aligned} & A(u)(t) = A_1 \circ A_2(u)(t) \\ = \ & \int_t^1 (\int_0^s N\tau^{N-1} f_1(\tau, A_2(u)(\tau)) d\tau)^{\frac{1}{N}} ds \\ \le \ & [\int_0^1 N\tau^{N-1} ((\overline{\psi}_1^{\infty} + \varepsilon)(A_2(u)(\tau))^{\beta_1} + M_1) d\tau]^{\frac{1}{N}} \\ \le \ & (\int_0^1 N\tau^{N-1} d\tau)^{\frac{1}{N}} [(\overline{\psi}_1^{\infty} + \varepsilon) \|A_2(u)\|^{\beta_1} + M_1]^{\frac{1}{N}} \\ = \ & [(\overline{\psi}_1^{\infty} + \varepsilon) \|A_2(u)\|^{\beta_1} + M_1]^{\frac{1}{N}} \end{aligned}$$

and

$$\begin{aligned} A_2(u)(t) &= \int_t^1 (\int_0^s N\tau^{N-1} f_2(\tau, u(\tau)) d\tau)^{\frac{1}{N}} ds \\ &\le [\int_0^1 N\tau^{N-1} (\epsilon_1 u(\tau)^{\beta_2} + M_2) d\tau]^{\frac{1}{N}} \\ &\le (\int_0^1 N\tau^{N-1} d\tau)^{\frac{1}{N}} [\epsilon_1 \|u\|^{\beta_2} + M_2]^{\frac{1}{N}} \\ &= [\epsilon_1 \|u\|^{\beta_2} + M_2]^{\frac{1}{N}}. \end{aligned}$$

From the above inequalities, it is clear that the term with the highest index is

$$\epsilon_1^{\frac{\beta_1}{N^2}} (\overline{\psi}_1^{\infty} + \varepsilon)^{\frac{1}{N}} u^{\frac{\beta_1 \beta_2}{N^2}} < u^{\frac{\beta_1 \beta_2}{N^2}}.$$

Hence we can choose a sufficiently large $R_2 > 0$ such that

$$\|A(u)(t)\| \le \|u\|,\ \ u \in K \cap \partial\Omega_{R_2}.$$

Therefore, by Lemma 1.1, the operator A has at least one fixed point in $K \cap (\overline{\Omega}_{R_2} \setminus \Omega_{r_2})$. □

Corollary 2.2 *Assume that $f_i(t,c)$ satisfies the following conditions:*

(i) $f_i(t,c)$ *is continuous and nondecreasing on* c*;*

(ii) there exist positive constants $\alpha, \beta, k_i, l_i (i=1,2)$*, such that*

$$k_1 v^{\alpha} \leq f_1(t,v) \leq k_2 v^{\alpha}, l_1 u^{\beta} \leq f_2(t,u) \leq l_2 u^{\beta}.$$

If $\alpha\beta < N^2$*, then (1.6) has a unique positive solution.*

Proof The existence result can be obtained from Theorem 2.1. Now we just give the proof of uniqueness of solutions.

Let $P = \{u \in C[0,1], u(t) \geq 0, t \in [0,1]\}$. Now we show that A has at most one fixed point in P. Since $f_i(t,c)$ is nondecreasing on c, the operator $A = A_1 \circ A_2$ is nondecreasing. By Lemma 2.2, we only need to verify that A is u_0-sublinear for $u_0 = 1-t$.

Let $M = \max_{t\in[0,1]}\{f_1(t,v(t))\}$. Then

$$\begin{aligned} A(u)(t) &= \int_t^1 (\int_0^s N\tau^{N-1} f_1(\tau, A_2(u)(\tau)) d\tau)^{\frac{1}{N}} ds \\ &\leq \int_t^1 (\int_0^s N\tau^{N-1} M d\tau)^{\frac{1}{N}} ds \\ &\leq M^{\frac{1}{N}}(1-t). \end{aligned}$$

Therefore, we take $\theta_2 = M^{\frac{1}{N}}$.

Choose any $c \in (0,1)$ and set

$$m = \left(\int_0^c N\tau^{N-1} f_1(v)\, d\tau\right)^{\frac{1}{N}}.$$

Since $(Au)(t)$ is strictly decreasing in t and vanishes at $t=1$, we have

$$(Au)(t) \geq (Au)(c) \geq m(1-c),\ for\ all\ t \in [0,c],$$

$$(Au)(t) \geq \int_t^1 m ds = m(1-t),\ for\ any\ t \in [c,1].$$

It is clear that $(Au)(t) \geq m(1-c)(1-t)$, for all $t \in [0,1]$. Thus we choose $\theta_1 = m(1-c)$.

Finally, for $u \in [\theta_1 u_0, \theta_2 u_0]$, $\xi \in (0,1)$, we have

$$\begin{aligned} A_2(\xi u) &= \int_t^1 (\int_0^s N\tau^{N-1} f_2(\tau, \xi u) d\tau)^{\frac{1}{N}} ds \\ &\geq \int_t^1 (\int_0^s N\tau^{N-1} l_1 (\xi u(\tau))^{\beta} d\tau)^{\frac{1}{N}} ds \\ &\geq (\frac{l_1}{l_2})^{\frac{1}{N}} \xi^{\frac{\beta}{N}} A_2(u) \end{aligned}$$

and

$$\begin{aligned} A(\xi u) &= A_1 \circ A_2(\xi u) \\ &\geq \int_t^1 (\int_0^s N\tau^{N-1} k_1 A_2^{\alpha}(\xi u) d\tau)^{\frac{1}{N}} ds \\ &\geq (\frac{k_1}{k_2})^{\frac{1}{N}} (\frac{l_1}{l_2})^{\frac{\alpha}{N^2}} \xi^{\frac{\alpha\beta}{N^2}} A(u). \end{aligned}$$

Further, since $\alpha\beta < N^2$, for any $\xi \in (0,1)$, there always exists some $\eta > 0$ such that $A(\xi u) \geq (1+\eta)\xi Au$. Therefore, by Lemma 1.1 and 1.3, (1.6) has a unique positive solution. $\square$

Theorem 2.3 *Assume that* $\alpha_i,\ \beta_i > 0\ (i = 1,2)$ *with*

$$\alpha_1\alpha_2 \geq N^2,\ \beta_1\beta_2 \geq N^2.$$

If $\overline{\varphi}_1^0 < +\infty,\ \overline{\varphi}_2^0 = 0,\ \underline{\psi}_1^\infty > 0,\ \underline{\psi}_2^\infty = +\infty,$ *then (1.6) has at least a solution.*

Proof On one hand, from the definitions of $\overline{\varphi}_1^0 < +\infty,\ \overline{\varphi}_2^0 = 0$, there exist $\epsilon_2 > 0$ and $r_3 \in (0,1)$ such that

$$f_1(t,v) \leq (\overline{\varphi}_1^0 + \varepsilon) v^{\alpha_1},\ for\ t \in [0,1],\ v \in [0,1],$$

$$f_2(t,u) \leq \epsilon_2 u^{\alpha_2},\ for\ t \in [0,1],\ u \in [0,r_3],$$

where ϵ_2 satisfies

$$\frac{1}{2}\epsilon_2^{\frac{1}{N}} \leq 1\ and\ \frac{1}{2^{\frac{\alpha_1+N}{N}}} \epsilon_2^{\frac{\alpha_1}{N^2}} (\overline{\varphi}_1^0 + \varepsilon)^{\frac{1}{N}} \leq 1.$$

Then for any $u \in K \cap \partial\Omega_{r_3}$, we have

$$\begin{aligned} v(t) = A_2(u)(t) &= \int_t^1 (\int_0^s N\tau^{N-1} f_2(\tau, u(\tau)) d\tau)^{\frac{1}{N}} ds \\ &\leq \int_0^1 (\int_0^s N\tau^{N-1} \epsilon_2 u^{\alpha_2} d\tau)^{\frac{1}{N}} ds \\ &\leq \int_0^1 (\int_0^s N\tau^{N-1} \epsilon_2 \|u\|^{\alpha_2} d\tau)^{\frac{1}{N}} ds \\ &\leq \frac{1}{2}\epsilon_2{}^{\frac{1}{N}} \|u\|^{\frac{\alpha_2}{N}} \leq 1. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} A_1(v)(t) &= \int_t^1 (\int_0^s N\tau^{N-1} f_1(\tau, v(\tau))d\tau)^{\frac{1}{N}} ds \\ &\le \int_t^1 (\int_0^s N\tau^{N-1} (\overline{\varphi}_1^0 + \varepsilon) v^{\alpha_1}(\tau) d\tau)^{\frac{1}{N}} ds \\ &\le \int_0^1 (\int_0^s N\tau^{N-1} (\overline{\varphi}_1^0 + \varepsilon) \|v\|^{\alpha_1} d\tau)^{\frac{1}{N}} ds \\ &\le \frac{1}{2} (\overline{\varphi}_1^0 + \varepsilon)^{\frac{1}{N}} \|v\|^{\frac{\alpha_1}{N}} \\ &\le \frac{1}{2^{\frac{\alpha_1 + N}{N}}} \epsilon_2^{\frac{\alpha_1}{N^2}} (\overline{\varphi}_1^0 + \varepsilon)^{\frac{1}{N}} \|u\|^{\frac{\alpha_1 \alpha_2}{N^2}} \\ &\le \|u\|^{\frac{\alpha_1 \alpha_2}{N^2}} \le \|u\|, \end{aligned}$$

namely, $\|A(u)(t)\| \le \|u\|, \ \ u \in K \cap \partial\Omega_{r_3}$.

On the other hand, from the definitions of $\underline{\psi}_1^\infty > 0, \ \underline{\psi}_2^\infty = +\infty$, it follows that there exist $C_2, \ C_3 > 0$ and $R_3 > 1$ such that

$$f_1(t, v) \ge C_2 v^{\beta_1}, \ for \ t \in [0, 1], \ v \ge R_3,$$

$$f_2(t, u) \ge C_3 u^{\beta_2}, \ for \ t \in [0, 1], \ u \ge R_3,$$

where C_2, C_3 satisfy

$$C_3^{\frac{1}{N}} \frac{1}{4^{\frac{\beta_2}{N}}} \Gamma \ge 4,$$

$$C_3^{\frac{\beta_1}{N^2}} \Gamma^{\frac{\beta_1 + N}{N}} C_2^{\frac{1}{N}} \frac{1}{4^{\frac{\beta_1 \beta_2 + \beta_1 N}{N^2}}} \ge 1.$$

Set $R_4 = \max\{4R_3, R_3^{\frac{N}{\beta_2}}\}$. Then for any $u \in K \cap \partial\Omega_{R_4}$, we have

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \ge \frac{1}{4} R_4 \ge R_3$$

and

$$\begin{aligned} \|v\| \ge v(\frac{1}{4}) = A_2(u)(\frac{1}{4}) &= \int_{\frac{1}{4}}^1 (\int_0^s N\tau^{N-1} f_2(\tau, u(\tau)) d\tau)^{\frac{1}{N}} ds \\ &\ge \int_{\frac{1}{4}}^1 (\int_0^s N\tau^{N-1} C_3 u^{\beta_2}(\tau) d\tau)^{\frac{1}{N}} ds \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^s N\tau^{N-1} C_3 \frac{1}{4^{\beta_2}} \|u\|^{\beta_2} d\tau)^{\frac{1}{N}} ds \\ &\ge C_3^{\frac{1}{N}} \frac{1}{4^{\frac{\beta_2}{N}}} \Gamma \|u\|^{\frac{\beta_2}{N}} \ge 4\|u\|^{\frac{\beta_2}{N}} \ge 4R_3. \end{aligned}$$

Furthermore, we also get

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} v(t) \geq \frac{1}{4}\|v\| \geq R_3$$

and

$$\begin{aligned} A_1(v)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^s N\tau^{N-1}f_1(\tau,v(\tau))d\tau)^{\frac{1}{N}}ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^{s} N\tau^{N-1}C_2v^{\beta_1}(\tau)d\tau)^{\frac{1}{N}}ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^{s} N\tau^{N-1}C_2\frac{1}{4^{\beta_1}}\|v\|^{\beta_1}d\tau)^{\frac{1}{N}}ds \\ &\geq C_2^{\frac{1}{N}}\frac{1}{4^{\frac{\beta_1}{N}}}\Gamma\|v\|^{\frac{\beta_1}{N}}. \end{aligned}$$

From the above inequalities, we have

$$\begin{aligned} A(u)(\frac{1}{4}) &\geq C_3^{\frac{\beta_1}{N^2}}\Gamma^{\frac{\beta_1+N}{N}}C_2^{\frac{1}{N}}\frac{1}{4^{\frac{\beta_1\beta_2+\beta_1N}{N^2}}}\|u\|^{\frac{\beta_1\beta_2}{N^2}} \\ &\geq \|u\|^{\frac{\beta_1\beta_2}{N^2}} \geq \|u\|. \end{aligned}$$

Hence, we have $\|A(u)(t)\| \geq \|u\|,\ \ u \in K\cap\partial\Omega_{R_4}$.

Therefore, by Lemma 1.1, the operator A has at least one fixed point in $K\cap(\overline{\Omega}_{R_4}\setminus\Omega_{r_3})$. □

Theorem 2.4 *Assume that* $\alpha_i,\ \beta_i > 0\ (i=1,2)$ *with*

$$\alpha_1\alpha_2 \leq N^2,\ \beta_1\beta_2 \geq N^2.$$

In addition, the following conditions hold:

(i) $\underline{\varphi}_1^0 > 0,\ \underline{\varphi}_2^0 = +\infty,\ \underline{\psi}_1^\infty > 0,\ \underline{\psi}_2^\infty = +\infty;$

(ii) there exists an $\widetilde{R}$ *such that*

$$\frac{1}{2}(N_{\widetilde{R}}^1)^{\frac{1}{N}} \leq \widetilde{R},$$

where

$$N_{\widetilde{R}}^1 = \max\{f_1(t,v): 0\leq t\leq 1, 0\leq v\leq \frac{1}{2}(N_{\widetilde{R}}^2)^{\frac{1}{N}}\},$$

$$N_{\widetilde{R}}^2 = \max\{f_2(t,u): 0\leq t\leq 1, 0\leq u\leq \widetilde{R}\}.$$

Then (1.6) has at least two solutions.

Proof For any $u \in K \cap \partial\Omega_{\widetilde{R}}$, we have

$$\begin{aligned} v(t) = A_2(u)(t) &= \int_t^1 \left(\int_0^s N\tau^{N-1} f_2(\tau, u(\tau)) d\tau\right)^{\frac{1}{N}} ds \\ &\le \int_0^1 \left(\int_0^s N\tau^{N-1} N^2_{\widetilde{R}} d\tau\right)^{\frac{1}{N}} ds \\ &\le \frac{1}{2}(N^2_{\widetilde{R}})^{\frac{1}{N}}. \end{aligned}$$

Furthermore, we can get

$$\begin{aligned} & A(u)(t) = A_1 \circ A_2(u)(t) \\ = & \int_t^1 \left(\int_0^s N\tau^{N-1} f_1(\tau, A_2(u)(\tau)) d\tau\right)^{\frac{1}{N}} ds \\ \le & \int_0^1 \left(\int_0^s N\tau^{N-1} N^1_{\widetilde{R}} d\tau\right)^{\frac{1}{N}} ds \\ \le & \frac{1}{2}(N^1_{\widetilde{R}})^{\frac{1}{N}} \le \widetilde{R}, \end{aligned}$$

namely, $\|A(u)(t)\| \le \|u\|, \ u \in K \cap \partial\Omega_{\widetilde{R}}$.

Since $\alpha_1\alpha_2 \le N^2, \ \beta_1\beta_2 \ge N^2, \ \underline{\varphi}_1^0 > 0, \ \underline{\varphi}_2^0 = +\infty, \ \underline{\psi}_1^\infty > 0, \ \underline{\psi}_2^\infty = +\infty$, from the proof of Theorem 2.1 and Theorem 2.3, there exist $r_2 > 0$ (sufficiently small) and $R_4 > 0$ (sufficiently large) such that $\|A(u)(t)\| \ge \|u\|, \ u \in K \cap \partial\Omega_{r_2}$ and $\|A(u)(t)\| \ge \|u\|, \ u \in K \cap \partial\Omega_{R_4}$. Therefore, by Lemma 1.1, the operator A has at least two fixed points in $K \cap (\overline{\Omega}_{\widetilde{R}} \setminus \Omega_{r_2})$ and $K \cap (\overline{\Omega}_{R_4} \setminus \Omega_{\widetilde{R}})$. □

Theorem 2.5 *Assume that* $\alpha_i, \ \beta_i > 0 \ (i = 1, 2)$ *with*

$$\alpha_1\alpha_2 \ge N^2, \ \beta_1\beta_2 \le N^2.$$

In addition, the following conditions hold:

(i) $\overline{\varphi}_1^0 < +\infty, \ \overline{\varphi}_2^0 = 0, \ \overline{\psi}_1^\infty < +\infty, \ \overline{\psi}_2^\infty = 0;$

(ii) there exists an $\widehat{R}$ *such that*

$$\Gamma(K^1_{\widehat{R}})^{\frac{1}{N}} \ge \widehat{R},$$

where

$$K^1_{\widehat{R}} = \min\{f_1(t, v) : \frac{1}{4} \le t \le \frac{3}{4}, \frac{1}{4}\Gamma(K^2_{\widehat{R}})^{\frac{1}{N}} \le v \le \frac{1}{2}(N^2_{\widehat{R}})^{\frac{1}{N}}\},$$

$$N^2_{\widehat{R}} = \max\{f_2(t, u) : 0 \le t \le 1, 0 \le u \le \widehat{R}\},$$

$$K^2_{\widehat{R}} = \min\{f_2(t, u) : \frac{1}{4} \le t \le \frac{3}{4}, \frac{1}{4}\widehat{R} \le u \le \widehat{R}\}.$$

Then (1.6) has at least two solutions.

Proof For any $u \in K \cap \partial\Omega_{\widehat{R}}$, we have

$$\begin{aligned} v(t) = A_2(u)(t) &= \int_t^1 (\int_0^s N\tau^{N-1} f_2(\tau, u(\tau)) d\tau)^{\frac{1}{N}} ds \\ &\leq \int_0^1 (\int_0^s N\tau^{N-1} N^2_{\widehat{R}} d\tau)^{\frac{1}{N}} ds \\ &\leq \frac{1}{2}(N^2_{\widehat{R}})^{\frac{1}{N}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \min_{t\in[\frac{1}{4},\frac{3}{4}]} v(t) \geq \frac{1}{4}\|v\| &\geq \frac{1}{4}\int_0^1 (\int_0^s N\tau^{N-1} f_2(\tau, u(\tau)) d\tau)^{\frac{1}{N}} ds \\ &\geq \frac{1}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^s N\tau^{N-1} K^2_{\widehat{R}} d\tau)^{\frac{1}{N}} ds \\ &\geq \frac{1}{4}\Gamma(K^2_{\widehat{R}})^{\frac{1}{N}}. \end{aligned}$$

For $t \in [\frac{1}{4}, \frac{3}{4}]$, we have the estimates

$$\frac{1}{4}\Gamma(K^2_{\widehat{R}})^{\frac{1}{N}} \leq v(t) \leq \frac{1}{2}(N^2_{\widehat{R}})^{\frac{1}{N}}$$

and

$$\begin{aligned} & A(u)(\frac{1}{4}) = A_1 \circ A_2(u)(\frac{1}{4}) \\ = & \int_{\frac{1}{4}}^1 (\int_0^s N\tau^{N-1} f_1(\tau, A_2(u)(\tau)) d\tau)^{\frac{1}{N}} ds \\ \geq & \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^s N\tau^{N-1} K^1_{\widehat{R}} d\tau)^{\frac{1}{N}} ds \\ \geq & \Gamma(K^1_{\widehat{R}})^{\frac{1}{N}}, \end{aligned}$$

namely, $\|A(u)(t)\| \geq \|u\|, \ u \in K \cap \partial\Omega_{\widehat{R}}$.

Since $\alpha_1\alpha_2 \geq N^2$, $\beta_1\beta_2 \leq N^2$, $\overline{\varphi}_1^0 < +\infty$, $\overline{\varphi}_2^0 = 0$, $\overline{\psi}_1^\infty < +\infty$, $\overline{\psi}_2^\infty = 0$, from the proof of Theorem 2.1 and Theorem 2.3, there exist $R_2 > 0$ (sufficiently large) and $r_3 > 0$ (sufficiently small) such that $\|A(u)(t)\| \leq \|u\|, \ u \in K \cap \partial\Omega_{R_2}$ and $\|A(u)(t)\| \leq \|u\|, \ u \in K \cap \partial\Omega_{r_3}$. Therefore, by Lemma 1.1, the operator A has at least two fixed points in $K \cap (\overline{\Omega}_{\widehat{R}} \setminus \Omega_{r_3})$ and $K \cap (\overline{\Omega}_{R_2} \setminus \Omega_{\widehat{R}})$. □

Now we give some examples to illustrate our main results.

Example 2.6 *If $N = 3$, then (1.6) is related to the second-order system*

$$\begin{cases} ((-u'(t))^3)' = 3t^2 f_1(t,v), \quad t \in (0,1), \\ ((-v'(t))^3)' = 3t^2 f_2(t,u), \quad t \in (0,1), \\ u(t) > 0, v(t) > 0, \quad t \in (0,1), \\ u'(0) = u(1) = 0, v'(0) = v(1) = 0, \end{cases} \tag{2.1}$$

where $f_1(t,v) = (\sin v)^2 + t^2, f_2(t,u) = u + t$. Choosing $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 4, \beta_2 = 2$, it is easy to verify that

$$\underline{\varphi}_1^0 = \liminf_{v \to 0+} \min_{t \in [0,1]} \frac{f_1(t,v)}{v^{\alpha_1}} = \liminf_{v \to 0+} \frac{(\sin v)^2}{v^2} = 1 > 0,$$

$$\underline{\varphi}_2^0 = \liminf_{u \to 0+} \min_{t \in [0,1]} \frac{f_2(t,u)}{u^{\alpha_2}} = \liminf_{u \to 0+} \frac{u}{u^3} = +\infty,$$

$$\overline{\psi}_1^\infty = \limsup_{v \to \infty} \max_{t \in [0,1]} \frac{f_1(t,v)}{v^{\beta_1}} = \limsup_{v \to \infty} \frac{(\sin v)^2 + 1}{v^4} = 0 < +\infty,$$

$$\overline{\psi}_2^\infty = \limsup_{u \to \infty} \max_{t \in [0,1]} \frac{f_2(t,u)}{u^{\beta_2}} = \limsup_{u \to \infty} \frac{u+1}{u^2} = 0,$$

which implies that the assumptions of Theorem 2.1 hold. Therefore, the problem (2.1) has at least one positive solution. $\diamond$

Example 2.7 *If $N = 3$, then (1.6) is related to the second-order system*

$$\begin{cases} ((-u'(t))^3)' = 3t^2 f_1(t,v), \quad t \in (0,1), \\ ((-v'(t))^3)' = 3t^2 f_2(t,u), \quad t \in (0,1), \\ u(t) > 0, v(t) > 0, \quad t \in (0,1), \\ u'(0) = u(1) = 0, v'(0) = v(1) = 0, \end{cases} \tag{2.2}$$

where $f_1(t,v) = (v^3 + v^5)\sin(t + \frac{\pi}{2} - 1), f_2(t,u) = u^5(t^2 + t + 1)$. Choosing $\alpha_1 = 3, \alpha_2 = 4, \beta_1 = 5, \beta_2 = 3$, it is easy to verify that

$$\overline{\varphi}_1^0 = \limsup_{v \to 0+} \max_{t \in [0,1]} \frac{f_1(t,v)}{v^{\alpha_1}} = \limsup_{v \to 0+} \frac{v^3 + v^5}{v^3} = 1 < +\infty,$$

$$\overline{\varphi}_2^0 = \limsup_{u \to 0+} \max_{t \in [0,1]} \frac{f_2(t,u)}{u^{\alpha_2}} = \limsup_{u \to 0+} \frac{3u^5}{u^4} = 0,$$

$$\underline{\psi}_1^\infty = \liminf_{v \to \infty} \min_{t \in [0,1]} \frac{f_1(t,v)}{v^{\beta_1}} = \liminf_{v \to \infty} \frac{(v^3 + v^5)\sin(\frac{\pi}{2} - 1)}{v^5} = \sin(\frac{\pi}{2} - 1) > 0,$$

$$\underline{\psi}_2^\infty = \liminf_{u \to \infty} \min_{t \in [0,1]} \frac{f_2(t,u)}{u^{\beta_2}} = \liminf_{u \to \infty} \frac{u^5}{u^3} = +\infty,$$

which implies that the assumptions of Theorem 2.3 hold. Therefore, the problem (2.2) has at least one positive solution. $\diamond$

Example 2.8 *If $N=3$, then (1.6) is related to the second-order system*

$$\begin{cases} ((-u'(t))^3)' = 3t^2 f_1(t,v), \quad t\in(0,1),\\ ((-v'(t))^3)' = 3t^2 f_2(t,u), \quad t\in(0,1),\\ u(t)>0, v(t)>0, \quad t\in(0,1),\\ u'(0)=u(1)=0, v'(0)=v(1)=0, \end{cases} \tag{2.3}$$

where

$$f_1(t,v) = \begin{cases} v^2, & 0\le v\le 1,\\ v^4, & 1<v. \end{cases} \quad \textit{and} \quad f_2(t,u) = \begin{cases} u^2, & 0\le u\le 1,\\ u^6, & 1<u. \end{cases}$$

Choosing $\alpha_1=2, \alpha_2=3, \beta_1=4, \beta_2=5$, it is easy to verify that

$$\underline{\varphi}_1^0 = \liminf_{v\to 0+} \min_{t\in[0,1]} \frac{f_1(t,v)}{v^{\alpha_1}} = \liminf_{v\to 0+} \frac{v^2}{v^2} = 1 > 0,$$

$$\underline{\varphi}_2^0 = \liminf_{u\to 0+} \min_{t\in[0,1]} \frac{f_2(t,u)}{u^{\alpha_2}} = \liminf_{u\to 0+} \frac{u^2}{u^3} = +\infty,$$

$$\underline{\psi}_1^\infty = \liminf_{v\to \infty} \min_{t\in[0,1]} \frac{f_1(t,v)}{v^{\beta_1}} = \liminf_{v\to \infty} \frac{v^4}{v^4} = 1 > 0,$$

$$\underline{\psi}_2^\infty = \liminf_{u\to \infty} \min_{t\in[0,1]} \frac{f_2(t,u)}{u^{\beta_2}} = \liminf_{u\to \infty} \frac{u^6}{u^5} = +\infty,$$

which implies that (i) of Theorem 2.4 holds.

Choosing $\widetilde{R}=\frac{1}{2}$, via some computations we can get

$$N_{\widetilde{R}}^2 = \max\{f_2(t,u) : 0\le t\le 1, 0\le u\le \widetilde{R}\} = \widetilde{R}^2,$$

$$N_{\widetilde{R}}^1 = \max\{f_1(t,v) : 0\le t\le 1, 0\le v\le \frac{1}{2}(N_{\widetilde{R}}^2)^{\frac{1}{N}}\} = \widetilde{R}^{\frac{10}{3}},$$

$$\frac{1}{2}(N_{\widetilde{R}}^1)^{\frac{1}{N}} = \widetilde{R}^{\frac{19}{9}} \le \widetilde{R},$$

which yields that (ii) of Theorem 2.4 holds. Therefore, the problem (2.3) has at least two positive solutions. ⋄

Example 2.9 *If $N=3$, then (1.6) is related to the second-order system*

$$\begin{cases} ((-u'(t))^3)' = 3t^2 f_1(t,v), \quad t\in(0,1),\\ ((-v'(t))^3)' = 3t^2 f_2(t,u), \quad t\in(0,1),\\ u(t)>0, v(t)>0, \quad t\in(0,1),\\ u'(0)=u(1)=0, v'(0)=v(1)=0, \end{cases} \tag{2.4}$$

where

$$f_1(t,v)=\begin{cases} v^3, & 0\le v\le \frac{2048}{\Gamma^4},\\ (\frac{2048}{\Gamma^4})^3, & \frac{2048}{\Gamma^4}<v\le(\frac{2048}{\Gamma^4})^{\frac{3}{2}},\\ v^2, & (\frac{2048}{\Gamma^4})^{\frac{3}{2}}<v.\end{cases}$$

$$f_2(t,u)=\begin{cases} u^6, & 0\le u\le \frac{64}{\Gamma^2}\\ (\frac{64}{\Gamma^2})^6, & \frac{64}{\Gamma^2}<u\le(\frac{64}{\Gamma^2})^2,\\ u^3, & (\frac{64}{\Gamma^2})^2<u.\end{cases}$$

Choosing $\alpha_1=3,\alpha_2=5,\beta_1=2,\beta_2=4$, *then it is easy to verify that*

$$\overline{\varphi}_1^0=\limsup_{v\to0+}\max_{t\in[0,1]}\frac{f_1(t,v)}{v^{\alpha_1}}=\limsup_{v\to0+}\frac{v^3}{v^3}=1<+\infty,$$

$$\overline{\varphi}_2^0=\limsup_{u\to0+}\max_{t\in[0,1]}\frac{f_2(t,u)}{u^{\alpha_2}}=\limsup_{u\to0+}\frac{u^6}{u^5}=0,$$

$$\overline{\psi}_1^\infty=\limsup_{v\to\infty}\max_{t\in[0,1]}\frac{f_1(t,v)}{v^{\beta_1}}=\limsup_{v\to\infty}\frac{v^2}{v^2}=1<+\infty,$$

$$\overline{\psi}_2^\infty=\limsup_{u\to\infty}\max_{t\in[0,1]}\frac{f_2(t,u)}{u^{\beta_2}}=\limsup_{u\to\infty}\frac{u^3}{u^4}=0,$$

which implies that (i) of Theorem 1.5 holds.

Choosing $\widehat{R}=\frac{64}{\Gamma^2}$, *via some computations we can get*

$$N_{\widehat{R}}^2=\max\{f_2(t,u):0\le t\le1,0\le u\le\widehat{R}\}=\widehat{R}^6,$$

$$K_{\widehat{R}}^2=\min\{f_2(t,u):\frac14\le t\le\frac34,\frac14\widehat{R}\le u\le\widehat{R}\}=(\frac{\widehat{R}}{4})^6,$$

$$K_{\widehat{R}}^1=\min\{f_1(t,v):\frac14\le t\le\frac34,\frac{\Gamma}{4}(K_{\widehat{R}}^2)^{\frac1N}\le v\le\frac12(N_{\widehat{R}}^2)^{\frac1N}\}=(\frac{\Gamma}{4})^3(\frac{\widehat{R}}{4})^6,$$

$$\Gamma(K_{\widehat{R}}^1)^{\frac1N}=\widehat{R},$$

which yields that (ii) of Theorem 2.5 holds. Therefore, the problem (2.4) has at least two positive solutions. $\diamond$

3. Main results of (1.7)

Theorem 3.1 *Assume that (H) holds. In addition,* $0<\alpha_i,\ \beta_i\le N\ (i=1,\ldots,n)$. *If* $\underline{F_{i*}^0}=+\infty$, $\overline{F_i^*}^\infty=0$, *then (1.7) has at least one positive solution.*

Proof Let E denote the Banach space $\overbrace{C[0,1]\times\cdots\times C[0,1]}^{n}$ with the norm $\|\vec{v}\| = \max_{1\le i\le n}\{|v_i|_1\}$, where $|v_i|_1 = \max_{t\in[0,1]}|v_i(t)|$. Define a mapping $A: E\to E$ by

$$A(\vec{v})(t) = (A_1(\vec{v})(t),\ldots,A_n(\vec{v})(t)),$$

where

$$A_i(\vec{v})(t) = \int_t^1 (\int_0^s N\tau^{N-1} f_i(v_1(\tau),\ldots,v_n(\tau))d\tau)^{\frac{1}{N}}ds, \quad t\in[0,1].$$

Define a subcone $K\subset E$ by $K = K_1\times\cdots\times K_n$, where $K_i = \{v_i(t) : \min_{\frac{1}{4}\le t\le\frac{3}{4}} v_i(t) \ge \frac{1}{4}|v_i|_1\}$. From the standard process (see [11]) and the Arzelà–Ascoli theorem, it follows that $A: K\to K$ is completely continuous.

On one hand, from the definition of $\overline{F_i^*}^{\infty} = 0$, there exist a sufficiently small $\epsilon > 0$ and $\overline{R} > 0$ such that $F_i^*(v_{i_0}) \le \epsilon v_{i_0}^{\beta_i},\ for\ v_{i_0}\ge\overline{R}$. For the given $\overline{R}$, let

$$M_i = \max_{0\le v_{i_0}\le\overline{R}} F_i^*(v_{i_0}).$$

Then we have

$$f_i(\vec{v}) \le F_i^*(v_{i_0}) \le \epsilon v_{i_0}^{\beta_i} + M_i.$$

Furthermore, we have the estimates

$$\begin{aligned}
A_i(\vec{v})(t) &= \int_t^1 (\int_0^s N\tau^{N-1} f_i(v_1(\tau),\ldots,v_n(\tau))d\tau)^{\frac{1}{N}}ds \\
&\le \int_0^1 (\int_0^s N\tau^{N-1}[\epsilon v_{i_0}^{\beta_i} + M_i]d\tau)^{\frac{1}{N}}ds \\
&\le \frac{1}{2}(\epsilon|v_{i_0}|_1^{\beta_i} + M_i)^{\frac{1}{N}}.
\end{aligned}$$

Therefore, combining with the assumption $\beta_i\le N$, there exists a sufficiently large $R>0$ such that, for any $\vec{v}\in\partial\Omega_R\cap K$,

$$\begin{aligned}
\|A(\vec{v})\| &\le \max_{1\le i\le n}\{\frac{1}{2}(\epsilon|v_{i_0}|_1^{\beta_i} + M_i)^{\frac{1}{N}}\} \\
&\le \max_{1\le i\le n}\{\frac{1}{2}(\epsilon R^{\beta_i} + M_i)^{\frac{1}{N}}\} \le R = \|\vec{v}\|.
\end{aligned}$$

On the other hand, since $\underline{F_{i*}^0} = +\infty$, there exist $M>0$ and $r<1$ such that

$$F_{i*}(v_{i_0}) \ge M v_{i_0}^{\alpha_i},\ for\ 0\le v_{i_0}\le r,$$

where M satisfies

$$\min_{1\le i\le n}\{M^{\frac{1}{N}}(\frac{1}{4})^{\frac{\alpha_i}{N}}\Gamma\} \ge 1.$$

Then for any $\vec{v} \in \partial\Omega_r \cap K$, we have

$$\begin{aligned}
A_i(\vec{v})(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_{0}^{s} N\tau^{N-1} f_i(v_1(\tau),\ldots,v_n(\tau))d\tau)^{\frac{1}{N}} ds \\
&\geq \int_{\frac{1}{4}}^{1}(\int_{0}^{s} N\tau^{N-1} F_{i*}(v_{i_0})d\tau)^{\frac{1}{N}} ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^{s} N\tau^{N-1} F_{i*}(v_{i_0})d\tau)^{\frac{1}{N}} ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^{s} N\tau^{N-1} M v_{i_0}^{\alpha_i})d\tau)^{\frac{1}{N}} ds \\
&\geq M^{\frac{1}{N}}(\frac{1}{4})^{\frac{\alpha_i}{N}}\Gamma|v_{i_0}|_1^{\frac{\alpha_i}{N}}.
\end{aligned}$$

Furthermore, there exists an index i_0 such that

$$\|A(\vec{v})\| \;>\; \max_{1\leq i\leq n}\{|v_{i_0}|_1^{\frac{\alpha_i}{N}}\} = r^{\frac{\alpha_i}{N}} \geq r.$$

Therefore, for any $\vec{v} \in \partial\Omega_r \cap K$, we have $\|A(\vec{v})\| > \|\vec{v}\|$.

Therefore, by Lemma 1.1, the operator A has at least one fixed point in $K \cap (\overline{\Omega}_R \setminus \Omega_r)$. $\square$

Theorem 3.2 *Assume that (H) holds. In addition,* $\alpha_i,\ \beta_i \geq N$. *If* $\underline{F_{i*}^{\infty}} = +\infty$, $\overline{F_i^{*}}^{0} = 0$, *then (1.7) has at least one positive solution.*

Proof On one hand, from the definition of $\overline{F_i^{*}}^{0} = 0$, there exist a sufficiently small $\epsilon > 0$ with $\frac{1}{2}\epsilon^{\frac{1}{N}} \leq 1$ and $r < 1$ such that

$$F_i^{*}(v_{i_0}) \leq \epsilon v_{i_0}^{\beta_i},\ for\ 0 \leq v_{i_0} \leq r.$$

Then, for any $\vec{v} \in \partial\Omega_r \cap K$, we have

$$\begin{aligned}
A_i(\vec{v})(t) &= \int_{t}^{1}(\int_{0}^{s} N\tau^{N-1} f_i(v_1(\tau),\ldots,v_n(\tau))d\tau)^{\frac{1}{N}} ds \\
&\leq \int_{0}^{1}(\int_{0}^{s} N\tau^{N-1} \epsilon v_{i_0}^{\beta_i} d\tau)^{\frac{1}{N}} ds \\
&\leq \frac{1}{2}(\epsilon|v_{i_0}|_1^{\beta_i})^{\frac{1}{N}}.
\end{aligned}$$

Therefore, combining with the assumption $\beta_i \geq N$, we have

$$\|A(\vec{v})\| < \max_{1\leq i\leq n}\{|v_{i_0}|_1^{\frac{\beta_i}{N}}\} \leq r.$$

On the other hand, from the definitions of $\underline{F_{i*}^{\infty}} = +\infty$, there exist $M > 0$ and $\overline{R} > r$ such that

$$F_{i*}(v_{i_0}) \geq M v_{i_0}^{\alpha_i},\ for\ v_{i_0} \geq \overline{R},$$

where M satisfies

$$\min_{1\le i\le n}\{M^{\frac{1}{N}}(\frac{1}{4})^{\frac{\alpha_i}{N}}\Gamma\}\ge 1.$$

Set $R=4\overline{R}+1$. Let

$$D_i=\min_{0\le v_{i_0}\le R}F_{i*}(v_{i_0}).$$

Then for any $\vec{v}\in\partial\Omega_R\cap K$, if $\|\vec{v}\|=|v_{i_0}|_1=R$, then

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]}v_{i_0}(t)\ge\frac{1}{4}|v_{i_0}|_1>\overline{R}.$$

Further, we have

$$\begin{aligned}
A_i(\vec{v})(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^s N\tau^{N-1}f_i(v_1(\tau),\dots,v_n(\tau))d\tau)^{\frac{1}{N}}ds\\
&\ge \int_{\frac{1}{4}}^{1}(\int_0^s N\tau^{N-1}F_{i*}(v_{i_0})d\tau)^{\frac{1}{N}}ds\\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^s N\tau^{N-1}F_{i*}(v_{i_0})d\tau)^{\frac{1}{N}}ds\\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^s N\tau^{N-1}Mv_{i_0}^{\alpha_i})d\tau)^{\frac{1}{N}}ds\\
&\ge M^{\frac{1}{N}}(\frac{1}{4})^{\frac{\alpha_i}{N}}\Gamma|v_{i_0}|_1^{\frac{\alpha_i}{N}}\ge|v_{i_0}|_1^{\frac{\alpha_i}{N}}
\end{aligned}$$

and for $j\neq i$,

$$\begin{aligned}
A_j(\vec{v})(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^s N\tau^{N-1}f_j(v_1(\tau),\dots,v_n(\tau))d\tau)^{\frac{1}{N}}ds\\
&\ge \int_{\frac{1}{4}}^{1}(\int_0^s N\tau^{N-1}F_{j*}(v_{j_0})d\tau)^{\frac{1}{N}}ds\\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^s N\tau^{N-1}D_jd\tau)^{\frac{1}{N}}ds=D_j^{\frac{1}{N}}\Gamma.
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
\max_{1\le i\le n}\{A_i(\vec{v})(\frac{1}{4})\} &> \max\{|v_{i_0}|_1^{\frac{\alpha_i}{N}},D_j^{\frac{1}{N}}\Gamma\}\\
&\ge |v_{i_0}|_1^{\frac{\alpha_i}{N}}=R^{\frac{\alpha_i}{N}}\ge R.
\end{aligned}$$

Hence, for any $\vec{v}\in\partial\Omega_R\cap K$, we have $\|A(\vec{v})\|>\|\vec{v}\|$.

Therefore, by Lemma 1.1, the operator A has at least one fixed point in $K\cap(\overline{\Omega}_R\setminus\Omega_r)$. □

Example 3.3 *Assume that $\alpha, \beta > 0$, $N = 4$. Then for the problem*

$$\begin{cases} ((-u'(t))^N)' = Nt^{N-1}(\sin(u+v)+2)v^{\alpha}(t), & t \in (0,1), \\ ((-v'(t))^N)' = Nt^{N-1}(e^{-v}+\arctan(u+1))u^{\beta}(t), & t \in (0,1), \\ u(t) > 0, v(t) > 0, \quad t \in (0,1), \\ u'(0) = u(1) = 0, v'(0) = v(1) = 0, \end{cases} \tag{3.1}$$

It is obvious that

$$F_{1*}(v) = v^{\alpha}(t) \leq f_1(u,v) = (\sin(u+v)+2)v^{\alpha}(t) \leq F_1^*(v) = 3v^{\alpha}(t),$$

$$F_{2*}(u) = \frac{\pi}{4}u^{\beta}(t) \leq f_2(u,v) = (e^{-v}+\arctan(u+1))u^{\beta}(t) \leq F_2^*(u) = (1+\frac{\pi}{2})u^{\beta}(t).$$

Choosing $\alpha_1 = 4$, $\alpha_2 = \frac{\beta+4}{2}$, $\beta_1 = \frac{3\alpha+4}{4}$, $\beta_2 = \frac{4\beta+4}{5}$.

Case I. If α, $\beta < 4$, then it is easy to verify that

$$\underline{F_{1*}^0} = \liminf_{c\to 0^+} \frac{F_{1*}(c)}{c^{\alpha_1}} = \liminf_{v\to 0^+} \frac{v^{\alpha}}{v^4} = +\infty,$$

$$\underline{F_{2*}^0} = \liminf_{c\to 0^+} \frac{F_{2*}(c)}{c^{\alpha_2}} = \liminf_{u\to 0^+} \frac{u^{\beta}}{u^{\frac{\beta+4}{2}}} = +\infty,$$

$$\overline{F_1^*}^{\infty} = \limsup_{c\to +\infty} \frac{F_1^*(c)}{c^{\beta_1}} = \limsup_{v\to +\infty} \frac{v^{\alpha}}{v^{\frac{3\alpha+4}{4}}} = 0,$$

$$\overline{F_2^*}^{\infty} = \limsup_{c\to +\infty} \frac{F_2^*(c)}{c^{\beta_2}} = \limsup_{u\to +\infty} \frac{u^{\beta}}{u^{\frac{4\beta+4}{5}}} = 0.$$

Thus, by Theorem 3.1, the problem (3.1) has at least a positive solution.

Case II. If α, $\beta > 4$, then it is easy to verify that

$$\underline{F_{1*}^{\infty}} = \liminf_{c\to +\infty} \frac{F_{1*}(c)}{c^{\alpha_1}} = \liminf_{v\to +\infty} \frac{v^{\alpha}}{v^4} = +\infty,$$

$$\underline{F_{2*}^{\infty}} = \liminf_{c\to +\infty} \frac{F_{2*}(c)}{c^{\alpha_2}} = \liminf_{u\to +\infty} \frac{u^{\beta}}{u^{\frac{\beta+4}{2}}} = +\infty,$$

$$\overline{F_1^*}^{0} = \limsup_{c\to 0^+} \frac{F_1^*(c)}{c^{\beta_1}} = \limsup_{v\to 0^+} \frac{v^{\alpha}}{v^{\frac{3\alpha+4}{4}}} = 0,$$

$$\overline{F_2^*}^{0} = \limsup_{c\to 0^+} \frac{F_2^*(c)}{c^{\beta_2}} = \limsup_{u\to 0^+} \frac{u^{\beta}}{u^{\frac{4\beta+4}{5}}} = 0.$$

Thus, by Theorem 3.2, the problem (3.1) has at least a positive solution.

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