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Quantitative Voronovskaya- and Grüss–Voronovskaya-type theorems by the blending variant of Szász operators including Brenke-type polynomials

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Abstract: The present paper aims to investigate a class of linear positive operators by combining Szász–Jain operators and Brenke polynomials and studies their approximation properties. We also prove quantitative Voronovskaya-type results and establish Grüss–Voronovskaja-type theorem. Furthermore, we show the rate of convergence for Szász–Jain–Brenke operators to functions having derivative of bounded variation and not having derivative of bounded variation by illustrative graphics using MATLAB.

Key words: Brenke polynomials, Szász operators, Voronovskaya-type, Grüss–Voronovskaya, bounded variation.

1. Introduction

Jain [16] proposed the Szász–Mirakyan-type operators and studied the following class of positive linear operators:

$$\mathfrak{J}_n^{[\vartheta]}(f;x)=\sum_{j=0}^{\infty}\varkappa_{n,j}^{(\vartheta)}(x)f\left(\frac{j}{n}\right),\ x\in[0,\infty), \tag{1.1}$$

where $\vartheta\in[0,1)$ and $\varkappa_{n,j}^{(\vartheta)}(x)$ is a modification of Poisson-type distribution given by

$$\varkappa_{n,j}^{(\vartheta)}(x)=\frac{nx(nx+j\vartheta)^{j-1}e^{-(nx+j\vartheta)}}{j!}.$$

In the particular case for $\vartheta=0$, the operator $\mathfrak{J}_n^{[\vartheta]}(f;x)$ turns out to be the well known Szász–Mirakyan type [20, 22]. Research work related to generalization and properties of the operators (1.1) can be found in [10, 12, 14, 18, 19, 21]. The Durrmeyer modification of operators defined by (1.1) and their direct results are introduced by Gupta and Greubel [13]. Later, Jakimovski and Leviatan [17] studied a new type of Szász operators by means of the Appell polynomials while Varma et al. [25] generalized Szász operators by means of the Brenke-type polynomials. Suppose that

$$E(z)=\sum_{j=0}^{\infty}e_jz^j, E(0)\neq 0, F(z)=\sum_{j=0}^{\infty}f_jz^j, F(0)\neq 0. \tag{1.2}$$

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be analytic functions in the disk $|z| < R, (R > 1)$. The Brenke-type polynomials $p_j(x)$ are defined with the help of the following generating relation:

$$E(t)F(tx) = \sum_{j=0}^{\infty} p_j(x)t^j. \tag{1.3}$$

An explicit formula for $p_j(x)$ is given by

$$p_j(x) = \sum_{l=0}^{j} e_{j-l} f_l x^l, j = 0, 1, 2, ... \tag{1.4}$$

Varma et al. in [25] introduced the following positive linear operators involving the Brenke-type polynomials:

$$\mathfrak{S}_n(f; x) = \sum_{j=0}^{\infty} f\left(\frac{j}{n}\right) \frac{p_j(nx)}{E(1)F(nx)}, \tag{1.5}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

Let $C_E[0, \infty]$ be the set of all continuous functions with the property $|f(t)| \leq Ae^{Bt}(t \geq 0)$ for some finite constants $A, B > 0$.

We introduce the Brenke analogue by combining the operators of Szász–Jain [16] as follows:

$$\begin{aligned} \mathfrak{B}_n^{[\vartheta]}(f; x) &= \sum_{j=0}^{\infty} \frac{p_j(nx)}{E(1)F(nx)} \frac{\int_0^{\infty} \varkappa_{n,j}^{(\vartheta)}(t) f(t)dt}{\int_0^{\infty} \varkappa_{n,j}^{(\vartheta)}(t)dt} \\ &= \sum_{j=0}^{\infty} \frac{p_j(nx)}{E(1)F(nx)} \frac{< \varkappa_{n,j}^{(\vartheta)}(t), f(t) >}{< \varkappa_{n,j}^{(\vartheta)}(t), 1 >}, \end{aligned} \tag{1.6}$$

where

$$< f, g >= \int_0^{\infty} f(t)g(t)dt.$$

The operator defined by (1.6) may be rewritten as

$$\mathfrak{B}_n^{[\vartheta]}(f; x) = \int_0^{\infty} \mathfrak{L}_n^{[\vartheta]}(x; u) f(u)du,$$

where $\mathfrak{L}_n^{[\vartheta]}(x; u) = \sum_{j=0}^{\infty} \frac{p_j(nx)\varkappa_{n,j}^{(\vartheta)}(u)}{E(1)F(nx)\int_0^{\infty} \varkappa_{n,j}^{(\vartheta)}(t)dt}$.

During the rest of paper we assume that

$$\lim_{y\to\infty} \frac{F^{(r)}(y)}{F(y)} = 1, for\ r \in \{1, 2, ..., j\}. \tag{1.7}$$

We shall restrict ourselves to the operators given by (1.6) satisfying:

(i) $E(1) \neq 0,\ \frac{e_{j-l} f_l}{E(1)} \geq 0, j \in \mathbb{N} \cup \{0\}$

(ii) $F : \mathbb{R}_0^+ \to \mathbb{R}^+$,

(iii) (1.3) converges for $\mid t \mid < R, (R > 1)$.

2. Auxiliary results

Lemma 2.1 *[13] For $0 \leq v < 1$, we have*

$$\frac{< \varkappa_{n,j}^{(\vartheta)}(t), t^s >}{< \varkappa_{n,j}^{(\vartheta)}(t), 1 >} = g_s(j, \vartheta),$$

where

$$< f, g >= \int_0^\infty f(t)g(t)dt,$$

and $g_s(j,\vartheta)$ is a polynomial of degree s in the variable j.

In particular,

(i) $g_0(j, \vartheta) = 1$,

(ii) $g_1(j, \vartheta) = \frac{1}{n}\left[(1-\vartheta)j + \frac{1}{1-\vartheta}\right]$,

(iii) $g_2(j, \vartheta) = \frac{1}{n^2}\left[(1-\vartheta)^2 j^2 + 3j + \frac{2!}{1-\vartheta}\right]$,

(iv) $g_3(j, \vartheta) = \frac{1}{n^3}\left[(1-\vartheta)^3 j^3 + 6(1-\vartheta)j^2 + \frac{(11-8\vartheta)j}{1-\vartheta} + \frac{3!}{1-\vartheta}\right]$,

(v) $g_4(j, \vartheta) = \frac{1}{n^4}\left[(1-\vartheta)^4 j^4 + 10(1-\vartheta)^2 j^3 + 5(7-4\vartheta)j^2 + \frac{10(5-3\vartheta)j}{1-\vartheta} + \frac{4!}{1-\vartheta}\right]$.

Now let $\phi_{s,x}(u) = (u-x)^s, s = 0, 1, 2, 3, 4$.

Lemma 2.2 *For the operators $\mathfrak{B}_n^{[\vartheta]}(\phi_{s,x}(u); x)$, one has*

(i) $\mathfrak{B}_n^{[\vartheta]}(\phi_{0,0}(u); x) = 1$;

(ii) $\mathfrak{B}_n^{[\vartheta]}(\phi_{1,0}(u); x) = (1-\vartheta)\frac{F'(nx)}{F(nx)}x + \frac{(1-\vartheta)^2 E'(1) + E(1)}{n(1-\vartheta)E(1)}$;

(iii)
$$\begin{aligned}\mathfrak{B}_n^{[\vartheta]}(\phi_{2,0}(u); x) =& (1-\vartheta)^2 \frac{F''(nx)}{F(nx)}x^2 + \frac{x(1-\vartheta)^2}{n}\frac{F'(nx)}{F(nx)}\left(\frac{2E'(1)+E(1)}{E(1)} + \frac{3}{(1-\vartheta)^2}\right) \\ &+ \frac{(1-\vartheta)^2}{n^2}\left(\frac{E''(1)+E'(1)}{E(1)} + \frac{3E'(1)}{(1-\vartheta)^2 E(1)} + \frac{2}{(1-\vartheta)^3}\right);\end{aligned}$$

(iv) $$\mathfrak{B}_n^{[\vartheta]}(\phi_{3,0}(u);x) = \frac{(1-\vartheta)^3F'''(nx)}{F(nx)}x^3 + \frac{3(1-\vartheta)^3x^2F''(nx)}{nF(nx)}\left(\frac{E'(1)+E(1)}{E(1)} + \frac{2}{(1-\vartheta)^2}\right)$$
$$+ \frac{(1-\vartheta)^3xF'(nx)}{n^2E(1)F(nx)}\left(3E''(1)+6E'(1)+E(1)+\frac{6(2E'(1)+E(1))}{(1-\vartheta)^2} + \frac{(11-8\vartheta)E(1)}{(1-\vartheta)^4}\right)$$
$$+ \frac{(1-\vartheta)^3}{n^3}\left(\frac{E'''(1)+3E''(1)+E'(1)}{E(1)} + \frac{6(E''(1)+E'(1))}{E(1)(1-\vartheta)^2} + \frac{11-8\vartheta}{(1-\vartheta)^4}\frac{E'(1)}{E(1)} + \frac{3!}{(1-\vartheta)^4}\right);$$

(v) $$\mathfrak{B}_n^{[\vartheta]}(\phi_{4,0}(u);x) = \frac{(1-\vartheta)^4F^{(4)}(nx)x^4}{F(nx)} + \frac{2(1-\vartheta)^4B'''(nx)x^3}{nE(1)B(nx)}\left(2A'(1)+3E(1)+\frac{5E(1)}{(1-\vartheta)^2}\right)$$
$$+ \frac{(1-\vartheta)^4E''(nx)x^2}{n^2E(1)F(nx)}\left(6E''(1)+18E'(1)+7E(1)+\frac{30(E'(1)+E(1))}{(1-\vartheta)^2)}\right.$$
$$\left.+ \frac{5(7-4\vartheta)E(1)}{(1-\vartheta)^4}\right) + \frac{(1-\beta)^4F'(nx)x}{n^3E(1)F(nx)}\left(4E'''(1)+18E''(1)+14E'(1)+E(1)\right.$$
$$\left.+ \frac{10(3E''(1)+6E'(1)+E(1))}{(1-\vartheta)^2} + \frac{5(7-4\vartheta)(2E'(1)+E(1))}{(1-\vartheta)^4} + \frac{10(5-3\vartheta)E(1)}{(1-\vartheta)^5}\right)$$
$$+ \frac{(1-\vartheta)^4}{n^4E(1)}\left(E^{(4)}(1)+6E'''(1)+7E''(1)+E'(1)+\frac{10(E'''(1)+3E''(1)+E'(1))}{(1-\vartheta)^2}\right.$$
$$\left.+ \frac{5(7-4\vartheta)(E''(1)+E'(1))}{(1-\vartheta)^4} + \frac{10(5-3\vartheta)E'(1)+4!E(1)}{(1-\vartheta)^5}\right).$$

Proof

(i) Using the generating relation (1.3), we have

$$\begin{aligned}\mathfrak{B}_n^{[\vartheta]}(\phi_{0,0}(u);x) &= \mathfrak{B}_n^{[\vartheta]}(1;x)\\ &= \sum_{j=0}^{\infty}\frac{p_j(nx)}{E(1)F(nx)} = 1.\end{aligned}$$

(ii) Using Lemma (2.1) (ii), we have

$$\begin{aligned}\mathfrak{B}_n^{[\vartheta]}(\phi_{1,0}(u);x) &= \mathfrak{B}_n^{[\vartheta]}(u;x)\\ &= \sum_{j=0}^{\infty}\frac{p_j(nx)\int_0^\infty \varkappa_{n,j}^{(\vartheta)}(u)udu}{E(1)F(nx)\int_0^\infty \varkappa_{n,j}^{(\vartheta)}(u)du}\\ &= \sum_{j=0}^{\infty}\frac{p_j(nx)g_1(j,\vartheta)}{E(1)F(nx)}\\ &= \frac{1}{n}\left[(1-\vartheta)\left(\sum_{j=0}^{\infty}\frac{p_j(nx)j}{E(1)F(nx)}\right) + \frac{1}{(1-\vartheta)}\right].\end{aligned}$$

Differentiating (1.3) with respect to t and then putting t=1 and replacing x by nx, we get

$$\sum_{j=0}^{\infty} \frac{p_j(nx)j}{E(1)F(nx)} = \frac{E'(1)}{E(1)} + \frac{nxF'(nx)}{F(nx)},$$

hence,

$$\mathfrak{B}_n^{[\vartheta]}(\phi_{1,0}(u);x) = \frac{(1-\vartheta)F'(nx)x}{F(nx)} + \frac{(1-\vartheta)^2E'(1)+E(1)}{n(1-\vartheta)E(1)}.$$

(iii) Using Lemma (2.1) (iii)

$$\begin{aligned}\mathfrak{B}_n^{[\vartheta]}(\phi_{2,0}(u);x) &= \mathfrak{B}_n^{[\vartheta]}(u^2;x)\\ &= \sum_{j=0}^{\infty} \frac{p_j(nx)g_2(j,\vartheta)}{E(1)F(nx)},\\ &= \sum_{j=0}^{\infty} \frac{p_j(nx)}{E(1)F(nx)}\frac{1}{n^2}\left((1-\vartheta)^2j^2+3j+\frac{2!}{(1-\vartheta)}\right).\end{aligned}$$

Differentiating (1.3) with respect to t twice and then putting t=1 and replacing x by nx, we get

$$\sum_{j=0}^{\infty} p_j(nx)j^2 = E''(1)F(nx) + 2nxF'(nx)E'(1) + E'(1)F(nx) + (nx)^2F''(nx)E(1) + nxF'(nx)E(1).$$

Hence

$$\begin{aligned}\mathfrak{B}_n^{[\vartheta]}(\phi_{2,0}(u);x) &= \frac{(1-\vartheta)^2F''(nx)x^2}{F(nx)} + \frac{F'(nx)}{F(nx)}\frac{x(1-\vartheta)^2}{n}\left(\frac{(2E'(1)+E(1))}{E(1)} + \frac{3}{(1-\vartheta)^2}\right) + \\ &\quad \frac{(1-\vartheta)^2}{n^2}\left(\frac{E''(1)+E'(1)}{E(1)} + \frac{3E'(1)}{E(1)(1-\vartheta)^2} + \frac{2}{(1-\vartheta)^3}\right).\end{aligned}$$

Proceeding similarly, the equalities (iv) and (v) can be proved.

□

In what follows, we assume that $\vartheta = \vartheta(n) \to 0$, as $n \to \infty$, $\lim_{n\to\infty} n\vartheta = l \in \mathbb{R}$ and $\lim_{y\to\infty} y\frac{F'(y)-F(y)}{F(y)} = 0$,

$$\lim_{y\to\infty} y\frac{F''(y)-2F'(y)+F(y)}{F(y)} = 0, \quad \lim_{y\to\infty} y^2\frac{F^{(4)}(y)-4F'''(y)+6F''(y)-4F'(y)+F(y)}{F(y)} = 0,$$

$$\lim_{y\to\infty} y\frac{F^{(4)}(y)-3F'''(y)+3F''(y)-F'(y)}{F(y)} = 0, \quad \lim_{y\to\infty} y\frac{F'''(y)-3F''(y)+3F'(y)-F(y)}{F(y)} = 0,$$

$$\lim_{y\to\infty} y\frac{F'''(y)-2F''(y)+F'(y)}{F(y)} = 0, \text{ and } \lim_{y\to\infty} y\frac{5F'''(y)-12F''(y)+9F'(y)-2F(y)}{F(y)} = 0$$

As a consequence of the condition (1.7) and the above assumptions, we have the following lemma:

Lemma 2.3 *If we denote the central moment as* $\eta_{n,s}^{[\vartheta]}(x) = \mathfrak{B}(\phi_{s,x}(u); x)$, *then*

(i) $\lim_{n\to\infty} n\eta_{n,1}^{[\vartheta]}(x) = -lx + 1 + \frac{E'(1)}{E(1)}$;

(ii) $\lim_{n\to\infty} n\eta_{n,2}^{[\vartheta]}(x) = 3x$;

(iii) $\lim_{n\to\infty} n^2\eta_{n,4}^{[\vartheta]}(x) = 44lx^3 + 56x^2$.

From Lemma 2.3, for each $x \in [0,\infty)$ there exists a positive constant $C_1(l)$ depending on l such that

$$\eta_{n,2}^{[\vartheta]}(x) \leq \frac{C_1(l)(1+x^2)}{n}. \tag{2.1}$$

Consequently, using the Cauchy–Schwarz inequality

$$\mathfrak{B}(|\phi_{1,x}(u)|; x) \leq \{\eta_{n,2}^{[\vartheta]}(x))\}^{\frac{1}{2}} \leq \sqrt{\frac{C_1(l)(1+x^2)}{n}}. \tag{2.2}$$

3. Main results

In what follows, let $\mathfrak{F}(f)(x) = \mathfrak{B}_n^{[\vartheta]}(f;x) - f(x)$.

Theorem 3.1 *Let* $f \in C_E[0,\infty)$, *and assume that condition (1.7) holds for* $r = 1, 2$. *Then*

$$\lim_{n\to\infty} \mathfrak{B}_n^{[\vartheta]}(f;x) = f(x)$$

uniformly in $[a,b] \subset [0,\infty)$.

Proof By the assumptions in the theorem and using Lemma 2.2, we find that

$$\lim_{n\to\infty} \mathfrak{B}_n^{[\vartheta]}(\phi_{s,0}(u);x) = x^s, \; s = 0,1,2$$

uniformly on every compact subset of $[0,\infty)$. Hence applying Korovkin theorem presented by [7], the desired result is obtained.

Let $C_B[0,\infty)$ be the space of all bounded and uniformly continuous functions on $[0,\infty)$. The norm on $C_B[0,\infty)$ is defined as $\| f \| = \sup_{x\in[0,\infty)} |f(x)|$. Further, let $C_B^2[0,\infty) = \{f : [0,\infty) \to R | f^{(i)} \in C_B[0,\infty),\, i = 1,2\}$. For $f \in C_B[0,\infty)$ and $\delta > 0$, the Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \{\| f - g \| + \delta \| g'' \|\},$$

and the mth modulus of continuity is defined as

$$\omega_m(f;\sqrt{\delta}) = \sup_{0<|h|<\delta} \sup_{x\in[0,\infty)} |\sum_{i=0}^{m} (-1)^{m-i} f(x+ih)|.$$

It is well known [6] that the K-functional $K(f;\delta)$ and the second order modulus of continuity $\omega_2(f;\sqrt{\delta})$ are related by

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{3.1}$$

for some constant $C>0$ and $\delta>0$. □

Theorem 3.2 *If $f \in C_B[0,\infty)$, then for any $x \in [0,\infty]$, the following statement holds:*

$$|\mathfrak{F}(f)| \leq 2\omega(f;\sqrt{\eta_{n,2}^{[\vartheta]}(x)}).$$

Proof By linearity of $\mathfrak{B}_n^{[\vartheta]}$, and using the property of $\omega(f;\delta)$ we get the following:

$$\begin{aligned} |\mathfrak{F}(f)| &\leq \left\{\sum_{j=0}^{\infty} \frac{p_j(nx)}{E(1)F(nx)} \frac{< \varkappa_{n,j}^{(\vartheta)}(u), |f(u)-f(x)| >}{< \varkappa_{n,j}^{(\vartheta)}(t), 1 >}\right\} \\ &\leq \omega(f;\delta)\left\{1+\frac{1}{\delta}\sum_{j=0}^{\infty} \frac{p_j(nx)}{E(1)F(nx)} \frac{< \varkappa_{n,j}^{(\vartheta)}(u), |\phi_{1,x}(u)| >}{< \varkappa_{n,j}^{(\vartheta)}(u), 1 >}\right\}, \ \delta>0. \end{aligned}$$

By the Cauchy–Schwarz inequality, we obtain

$$|\mathfrak{F}(f)| \leq \left\{1+\left(\eta_{n,2}^{[\vartheta]}(x)\right)^{\frac{1}{2}}\delta^{-1}\right\}\omega(f;\delta). \tag{3.2}$$

Taking $\delta = \sqrt{\eta_{n,2}^{[\vartheta]}(x)}$ in (3.2), we get the required result. □

Theorem 3.3 *If $f \in C_B^1[0,\infty)$, then for any $x \in [0,\infty)$, we have*

$$\begin{aligned} |\mathfrak{F}(f)| &\leq |f'(x)|\left|\frac{((1-\vartheta)F'(nx)-F(nx))\,x}{F(nx)} + \frac{(1-\vartheta)^2E'(1)+E(1)}{n(1-\vartheta)E(1)}\right| \\ &+2\omega(f';\sqrt{\eta_{n,2}^{\vartheta}(x)})\sqrt{\eta_{n,2}^{[\vartheta]}(x)}. \end{aligned}$$

Proof Let $f \in C_B^1[0,\infty)$. For any $u, x \in [0,\infty)$, we have

$$f(u)-f(x) = f'(x)\phi_{1,x}(u) + \int_x^u (f'(u)-f'(x))du.$$

Applying $\mathfrak{B}_n^{[\vartheta]}$ on both sides of the above relation, we get

$$\mathfrak{F}(f) = f'(x)\eta_{n,1}^{[\vartheta]}(x) + \mathfrak{B}_n^{[\vartheta]}(\int_x^u (f'(u)-f'(x))du;x).$$

By using the property of modulus of continuity it follows that

$$\left|\int_x^u |f'(u)-f'(x)|\right| \leq |\phi_{1,x}(u)|\omega(f';|\phi_{1,x}(u)|) \leq \omega(f';\delta)\left(\frac{\phi_{2,x}(u)}{\delta}+|\phi_{1,x}(u)|\right).$$

Thus,

$$|\mathfrak{F}(f)| \leq |f'(x)| \left|\eta_{n,1}^{[\vartheta]}(x)\right| + \omega(f';\delta) \left(\frac{\eta_{n,2}^{[\vartheta]}(x)}{\delta} + \mathfrak{B}_n^{[\vartheta]}(|\phi_{1,x}(u)|;x)\right).$$

Using the Cauchy–Schwarz inequality, we have

$$|\mathfrak{F}(f)| \leq |f'(x)| \left|\eta_{n,1}^{[\vartheta]}(x)\right| + \omega(f';\delta) \left(\frac{1}{\delta}\sqrt{\eta_{n,2}^{[\vartheta]}(x)} + 1\right) \sqrt{\eta_{n,2}^{[\vartheta]}(x)}.$$

Choosing $\delta = \sqrt{\eta_{n,2}^{[\vartheta]}(x)}$, the required result follows. □

Theorem 3.4 *Let* $0 < r \leq 1$, *and* $f \in C_B[0,\infty)$. *If* $f \in Lip_M^*(r)$, *i.e. the condition*

$$|f(u) - f(x)| \leq M_f \frac{|\phi_{1,x}(u)|^r}{(u+x)^{\frac{r}{2}}}; u \in [0,\infty), x > 0$$

holds, then for each $x \in (0,\infty)$ *we have*

$$|\mathfrak{F}(f)| \leq M_f \left(\frac{\eta_{n,2}^{[\vartheta]}(x)}{x}\right)^{r/2},$$

where $M_f > 0$ *is a constant.*

Proof By using the Hölder inequality we find

$$\begin{aligned}
|\mathfrak{F}(f)| &\leq \left\{\frac{1}{E(1)F(nx)} \sum_{j=0}^{\infty} p_j(nx) \frac{< \varkappa_{n,j}^{(\vartheta)}(u), |f(u)-f(x)|^{\frac{2}{r}} >}{< \varkappa_{n,j}^{(\vartheta)}(u), 1 >}\right\}^{\frac{r}{2}} \\
&\leq M_f \left\{\frac{1}{E(1)F(nx)} \sum_{j=0}^{\infty} p_j(nx) \frac{< \varkappa_{n,j}^{(\vartheta)}(u), \frac{\phi_{2,x}(u)}{u+x} >}{< \varkappa_{n,j}^{(\vartheta)}(u), 1 >}\right\}^{\frac{r}{2}} \\
&\leq \frac{M_f}{x^{\frac{r}{2}}} \left\{\frac{1}{E(1)F(nx)} \sum_{j=0}^{\infty} p_j(nx) \frac{< \varkappa_{n,j}^{(\vartheta)}(u), \phi_{2,x}(u) >}{< \varkappa_{n,j}^{(\vartheta)}(u), 1 >}\right\}^{\frac{r}{2}} \\
&\leq M_f \left(\frac{\eta_{n,2}^{[\vartheta]}(x)}{x}\right)^{r/2}.
\end{aligned}$$

Hence the proof is completed. □

Theorem 3.5 *Let* $f \in C_B[0,\infty)$. *Then for any* $x \geq 0$, *the following inequality holds:*

$$|\mathfrak{F}(f)| \leq 5\omega(f;(\eta_{n,2}^{[\vartheta]})^{1/2}) + \frac{13}{2}\omega_2(f;\eta_{n,2}^{[\vartheta]}).$$

Proof Let f_h be the second order Steklov mean of $f \in C_B[0,\infty)$, i.e.

$$f_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\Big(2f(x+u+v) - f(x+2u+2v)\Big)dudv, h > 0.$$

In [26], it is proved that

$$\|f_h'\| \le \frac{5}{h}\omega(f;h); \|f_h''\| \le \frac{9}{h^2}\omega_2(f;h) \ and \ \|f_h - f\| \le \omega_2(f;h). \tag{3.3}$$

By virtue of the equality $\mathfrak{B}_n^{[\vartheta]}(1;x) = 1$, we can write

$$|\mathfrak{F}(f)| \le |\mathfrak{B}_n^{[\vartheta]}((f-f_h)(u);x)| + |f_h(x) - f(x)| + |\mathfrak{B}_n^{[\vartheta]}(f_h(u) - f_h(x);x)|. \tag{3.4}$$

Applying Lemma 2.2(i)

$$|\mathfrak{B}_n^{[\vartheta]}(f;x)| \quad \le \quad \sum_{j=0}^{\infty}\frac{p_j(nx)}{A(1)B(nx)}\frac{<\varkappa_{n,j}^{(\beta)}(t), |f(t)|>}{<\varkappa_{n,j}^{(\vartheta)}(t), 1>} \le \|f\|,$$

hence using relation (3.3), we have

$$\mathfrak{B}_n^{[\vartheta]}(|(f-f_h)(u)|;x) \le \|f - f_h\| \le \omega_2(f;h), \tag{3.5}$$

and

$$|f_h(x) - f(x)| \le \| f - f_h \| \le \omega_2(f,\delta). \tag{3.6}$$

By the Taylor's expansion and the Cauchy–Schwarz inequality, we have

$$|\mathfrak{B}_n^{[\vartheta]}(f_h;x) - f_h(x)| \le \|f_h'\|\sqrt{\eta_{n,2}^{[\vartheta]}(x)} + \frac{1}{2}\|f_h''\|\eta_{n,2}^{[\vartheta]}(x).$$

Using the relation (3.3), we obtain

$$|\mathfrak{B}_n^{[\vartheta]}(f_h;x) - f_h(x)| \le \frac{5}{h}\omega(f;h)\sqrt{\eta_{n,2}^{[\vartheta]}(x)} + \frac{9}{2h^2}\omega_2(f;h)\eta_{n,2}^{[\vartheta]}(x). \tag{3.7}$$

Now taking $h = \sqrt{\eta_{n,2}^{[\vartheta]}(x)}$ in the above inequality (3.7) and then substituing the inequalities (3.5)–(3.7) in (3.4), the proof is completed. □

Theorem 3.6 *Let $f \in C_B^2[0,\infty)$; then*

$$\lim_{n\to\infty} n(\mathfrak{F}(f)) = (-lx + \frac{E'(1)+E(1)}{E(1)})f'(x) + \frac{3x}{2}f''(x), \tag{3.8}$$

uniformly with respect to $x \in [a,b]$, where $0 \le a < b < \infty$.

Proof By Taylor's expansion of f, we obtain

$$f(t)=\sum_{k=0}^{2}\frac{f^{(k)}(x)(t-x)^k}{k!}+\xi(t,x)(t-x)^2,$$

where the function $\xi(t,x)$ is the Peano form of the remainder and $\lim_{t\to x}\xi(t,x)=0$.
Applying the operator $\mathfrak{B}_n^{[\vartheta]}$ on both sides of the above equation we get

$$\begin{aligned} n(\mathfrak{B}_n^{[\vartheta]}(f;x)-f(x)) &= n\eta_{n,1}^{[\vartheta]}(x)f'(x)+\frac{1}{2}n\eta_{n,2}^{[\vartheta]}(x)f''(x)\\ &\quad +n\mathfrak{B}_n^{[\vartheta]}((\xi(t,x)(t-x)^2;x). \end{aligned}$$

Hence using Lemma 2.3, we have

$$\begin{aligned} \lim_{n\to\infty} n(\mathfrak{B}_n^{[\vartheta]}(f;x)-f(x)) &= (-lx+\frac{E'(1)+E(1)}{E(1)})f'(x)+\frac{3x}{2}f''(x)\\ &\quad +\lim_{n\to\infty} n\mathfrak{B}_n^{[\vartheta]}((\xi(t,x)(t-x)^2;x). \end{aligned} \tag{3.9}$$

For the last term of the right-hand side in (3.9), using the Cauchy–Schwarz inequality, we are led to

$$n\mathfrak{B}_n^{[\vartheta]}((\xi(t,x)(t-x)^2;x)\le n\sqrt{\mathfrak{B}_n^{[\vartheta]}((\xi^2(t,x);x)}\sqrt{\mathfrak{B}_n^{[\vartheta]}(((t-x)^4;x)}.$$

We observe that $\xi^2(x,x)=0$, and $\xi^2(,x)\in C_B[0,\infty)$. Hence applying Theorem 3.1

$$\lim_{n\to\infty}\mathfrak{B}_n^{[\vartheta]}(\xi^2(t,x);x)=\xi^2(x,x)=0,$$

uniformly in $x\in[a,b]$. Hence using Lemma 2.3, we obtain

$$\lim_{n\to\infty} n\mathfrak{B}_n^{[\vartheta]}(\xi(t,x)(t-x)^2;x)=0, \tag{3.10}$$

uniformly in $x\in[a,b]$. Combining the equations (3.9) and (3.10) we have

$$\lim_{n\to\infty} n(\mathfrak{F}(f))=(-lx+\frac{E'(1)+E(1)}{E(1)})f'(x)+\frac{3x}{2}f''(x),$$

uniformly in $x\in[a,b]$. Thus, the proof is completed. □

Theorem 3.7 *For every function $f\in C_B^2[0,\infty)$, and $x\in[0,\infty)$ the following statement holds*

$$|\mathfrak{F}(f)|\le\lambda_n(x)\|f''\|,$$

where

$$\lambda_n(x)=\eta_{n,2}^{[\vartheta]}(x)+(\eta_{n,1}^{[\vartheta]}(x))^2.$$

Proof Let us define the auxiliary operator

$$\tilde{\mathfrak{B}}_n^{[\vartheta]}(f;x)=\mathfrak{B}_n^{[\vartheta]}(f;x)-f(\mathfrak{B}_n^{[\vartheta]}(u;x))+f(x).$$

Then using Lemma (2.2), we have

$$\tilde{\mathfrak{B}}_n^{[\vartheta]}(1;x)=1 \text{ and } \tilde{\mathfrak{B}}_n^{[\vartheta]}((u-x);x)=0.$$

Let $f\in C_B^2[0,\infty)$; by Taylor's formula we have

$$f(u)=f(x)+(u-w)f'(x)+\int_x^u(u-w)f''(w)dw.$$

Hence,

$$\begin{aligned}|\tilde{\mathfrak{B}}_n^{[\vartheta]}(f;x)-f(x)| &= |\tilde{\mathfrak{B}}_n^{[\vartheta]}(\int_x^u(u-w)f''(w)dw;x)|\\ &= \Bigg|\mathfrak{B}_n^{[\vartheta]}\Bigg(\int_x^u(u-w)f''(w)dw;x\Bigg)\\ &\quad -\Bigg(\int_x^{\mathfrak{B}_n^{[\vartheta]}(u;x)}\Bigg(\frac{(1-\vartheta)F'(nx)x}{F(nx)}+\frac{(1-\vartheta)^2E'(1)+E(1)}{n(1-\vartheta)E(1)}-w\Bigg)f''(w)dw,x\Bigg)\Bigg|\end{aligned}$$

$$\begin{aligned}|\tilde{\mathfrak{B}}_n^{[\vartheta]}(f;x)-f(x)| &\le \parallel f''\parallel\Bigg(\mathfrak{B}_n^{[\vartheta]}((u-x)^2;x)+(\mathcal{B}_n^{\vartheta}(u;x)-x)^2\\ &\le \|\| f''\parallel(\eta_{n,2}^{\vartheta}(x)+(\mathfrak{B}_n^{[\vartheta]}(u;x)-x)^2)\\ &= \parallel f''\parallel(\eta_{n,2}^{\vartheta}(x)+(\eta_{n,1}^{\vartheta}(x))^2).\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.8 *For every function $f\in C_B[0,\infty)$, and $x\in[0,\infty)$ the following statement holds:*

$$|\mathfrak{F}(f)|\le C\omega_2(f;\sqrt{\frac{\lambda_n(x)}{3}}),$$

where $\lambda_n(x)$ is given as in Theorem 3.7.

Proof For $f\in C_B[0,\infty)$, from Lemma 2.2(i)

$$\begin{aligned}|\mathfrak{B}_n^{[\vartheta]}(f;x)| &\le |\mathfrak{B}_n^{[\vartheta]}(f;x)|+|f(\mathfrak{B}_n^{[\vartheta]}(u;x)|+|f(x)|\\ &\le 3||f||. \end{aligned} \tag{3.11}$$

Hence, for any $g\in C_B^2[0,\infty)$

$$\begin{aligned}\mathfrak{B}_n^{[\vartheta]}(f(u);x)-f(x) &= \tilde{\mathfrak{B}}_n^{[\vartheta]}(f(u);x)-f(x)+f(\mathfrak{B}_n^{[\vartheta]}(u;x))-f(x)\\ &= \tilde{\mathfrak{B}}_n^{[\vartheta]}(f(u)-g(u);x)+\tilde{\mathfrak{B}}_n^{[\vartheta]}(g(u);x)-g(x)+g(x)-f(x)+f(\mathfrak{B}_n^{[\vartheta]}(u;x))-f(x)\end{aligned}$$

Using (3.11) and applying Theorem 3.7

$$|\mathfrak{B}_n^{[\vartheta]}(f;x) - f(x)| \leq 3||f-g|| + ||g''||\lambda_n(x).$$

Now, taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and then using the relation (3.1), we have

$$\begin{aligned} |\mathfrak{B}_n^{\vartheta}(f;x) - f(x)| &\leq 3K_2\left(f; \sqrt{\frac{\lambda_n(x)}{3}}\right) \\ &\leq C\omega_2\left(f; \sqrt{\frac{\lambda_n(x)}{3}}\right), \end{aligned}$$

which completes the proof.

4. Quantitative Voronovskaya-type results

Let $\xi(x) = 1 + x^2$ be the weighted function and M_f be a positive constant depending only on f. Then we define $B_\xi[0,\infty) = \left\{ f : [0,\infty) \to R : |f(x)| \leq M_f \xi(x) \right\}$. By $C[0,\infty)$, we denote the subspace of all continuous functions belonging to $B_\xi[0,\infty)$. Moreover, $C_\xi^*[0,\infty) = \left\{ f \in C_\xi[0,\infty) : \lim\limits_{x\to\infty} \frac{|f(x)|}{1+x^2} = k_f < \infty \right\}$. The norm on $C_\xi^*[0,\infty)$ is $\|f\|_\xi = \sup\limits_{x\geq 0} \frac{|f(x)|}{1+x^2}$.

For any function $f \in C_\xi^*[0,\infty)$, Ispir and Atakut [15] defined the weighted modulus of continuity of the function f as follows:

$$\Omega(f;\delta) = \sup_{0\leq|h|<\delta, x\in[0,\infty)} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}. \tag{4.1}$$

In the following lemma we collect some properties of $\Omega(f,\delta)$. □

Lemma 4.1 *[15]. For the function $\Omega(f,\delta)$, we have*

(i) $\Omega(f,\delta)$ is a monotonically increasing function of δ,

(ii) $lim_{\delta\to 0^+}\Omega(f,\delta) = 0$;

(iii) for any $\lambda \in [0,\infty)$ and $\delta > 0, \Omega(f,\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(f,\delta)$.

In what follows, let $\rho(x,\delta) = (1+x^2)(1+\delta^2)$.

Theorem 4.2 *Let $f \in C_\xi^*[0,\infty)$, $\vartheta = \vartheta(n) \to 0, n\vartheta(n) \to l \in \mathbb{R}$ as $n \to \infty$, then there exists a constant $C' = C'(l) > 0$, such that*

$$\sup_{x\in[0,\infty)} \frac{|\mathfrak{F}(f)|}{(1+x^2)^{\frac{5}{2}}} \leq C'\Omega(f; n^{\frac{-1}{2}}). \tag{4.2}$$

Proof For $f \in C^*_\xi[0,\infty)$, considering the definition (4.1) and applying Lemma 4.1 (iii), we have

$$|f(u)-f(x)| \quad \leq \quad 2\Big(1+|\phi_{1,x}(u)|\delta^{-1}\Big)\rho(x,\delta)(1+\phi_{2,x}(u))\Omega(f;\delta),\ u,x\in[0,\infty),$$

where $\delta > 0$.

Since the operator $\mathfrak{B}_n^{[\vartheta]}$ is linear and positive, we get

$$\begin{aligned}|\mathfrak{F}(f)| \quad &\leq 2\Big[\mathfrak{B}_n^{[\vartheta]}\Big(\Big(1+|\phi_{1,x}(u)|\delta^{-1}\Big)(1+\phi_{2,x}(u));x\Big)\Big]\rho(x,\delta)\Omega(f;\delta)\\ &\leq 2\rho(x,\delta)\Omega(f;\delta)\Big\{\mathfrak{B}_n^{[\vartheta]}(1;x)+\eta_{n,2}^{[\vartheta]}(x)\\ &+\frac{1}{\delta}\mathfrak{B}_n^{[\vartheta]}(|\phi_{1,x}(u)|;x)+\frac{1}{\delta}\mathfrak{B}_n^{[\vartheta]}(|\phi_{1,x}(u)|\phi_{2,x}(u);x)\Big\}.\end{aligned}$$

Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\mathfrak{F}(f)| \quad &\leq 2\rho(x,\delta)\Omega(f;\delta)\Big\{1+\eta_{n,2}^{[\vartheta]}(x)\\ &+\frac{1}{\delta}\sqrt{\eta_{n,2}^{[\vartheta]}(x)}+\frac{1}{\delta}\sqrt{\eta_{n,2}^{[\vartheta]}(x)}\sqrt{\eta_{n,4}^{[\vartheta]}(x)}\Big\}.\end{aligned}$$

From Lemma 2.3, there exists a positive constant $C_2(l)$ depending on l such that

$$\eta_{n,4}^{[\vartheta]}(x) \leq \frac{C_2(l)(1+x^2)^2}{n^2}. \tag{4.3}$$

Hence, using (2.1) and (4.3) we have

$$\begin{aligned}|\mathfrak{F}(f)| \quad &\leq 2\rho(x,\delta)\Omega(f;\delta)\Big\{1+\frac{C_1(l)(1+x^2)}{n}\\ &+\frac{\sqrt{C_1(l)}}{\delta\sqrt{n}}(1+x^2)^{\frac{1}{2}}+\sqrt{C_1(l)C_2(l)}\frac{(1+x^2)^{\frac{3}{2}}}{\delta n\sqrt{n}}\Big\}.\end{aligned}$$

Taking $C' = 4(1+C_1(l)+\sqrt{C_1(l)}+\sqrt{C_1(l)C_2(l)})$, $\delta = \frac{1}{\sqrt{n}}$, we get the required result. □

In our next theorem we prove quantitative Voronovskaya-type theorem.

Theorem 4.3 *Let $f \in C^*_\xi[0,\infty)$ such that $f'(x), f''(x) \in C^*_\xi[0,\infty)$. Further, let $\vartheta = \vartheta_n$ be a sequence such that $\vartheta_n \to 0$, as $n\to\infty$ and $\lim_{n\to\infty} n\vartheta = l$ $(l\in\mathbb{R})$; then for sufficiently large n and each $x\in[0,\infty)$, the* *equality*

$$\Bigg|n[\mathfrak{F}(f)] - n\bigg(\frac{((1-\vartheta)F'(nx) - F(nx))x}{F(nx)} + \frac{(1-\vartheta)^2E'(1) + E(1)}{n(1-\vartheta)E(1)}\bigg)f'(x) - n\frac{f''(x)}{2!}\Bigg[\bigg(\frac{(1-\vartheta)^2F''(nx)}{F(nx)} - 2\frac{(1-\vartheta)F'(nx)}{F(nx)} + 1\bigg)x^2 + \bigg(\frac{(1-\vartheta)^2F'(nx)}{nF(nx)}\bigg(\frac{2E'(1)+E(1)}{E(1)} + \frac{3}{(1-\vartheta)^2}\bigg) - \frac{2\{(1-\vartheta)^2E'(1)+E(1)\}}{n(1-\vartheta)E(1)}\bigg)x + \frac{(1-\vartheta)^2}{n^2}\bigg(\frac{E''(1)+E'(1)}{E(1)} + \frac{3E'(1)}{(1-\vartheta)^2E(1)} + \frac{2}{(1-\vartheta)^3}\bigg)\Bigg]\Bigg| = O(1)\Omega(f''; n^{-1/2}),\ \ as\ \ n\to\infty$$

holds true.

Proof By Taylor's theorem, we have

$$\begin{aligned} f(u) &= f(x) + f'(x)\phi_{1,x}(u) + \frac{f''(\xi)}{2!}\phi_{2,x}(u) \\ &= f(x) + f'(x)\phi_{1,x}(u) + \frac{f''(x)}{2!}\phi_{2,x}(u) + h_2(u,x) \end{aligned} \tag{4.4}$$

where ξ is a number between u and x and $h_2(u,x) = \frac{f''(\xi)-f''(x)}{2!}\phi_{2,x}(u)$. Applying the operator $\mathfrak{B}_n^{[\vartheta]}$ to both sides of (4.4) and using Lemma 2.2, we have

$$\bigg|n\mathfrak{F}(f) - f'(x)n\eta_{n,1}^{[\vartheta]}(x) - n\frac{f''(x)}{2!}\eta_{n,2}^{[\vartheta]}(x)\bigg| = \Bigg|n[\mathfrak{F}(f)] - n\bigg(\frac{((1-\vartheta)F'(nx) - F(nx))x}{F(nx)} + \frac{(1-\vartheta)^2E'(1)+A(1)}{n(1-\vartheta)E(1)}\bigg)f'(x) - n\frac{f''(x)}{2!}\Bigg[\bigg(\frac{(1-\vartheta)^2F''(nx)}{F(nx)} - 2\frac{(1-\vartheta)F'(nx)}{F(nx)} + 1\bigg)x^2 + \bigg(\frac{(1-\vartheta)^2B'(nx)}{nF(nx)}\bigg(\frac{2E'(1)+E(1)}{E(1)} + \frac{3}{(1-\vartheta)^2}\bigg) - \frac{2\{(1-\vartheta)^2E'(1)+E(1)\}}{n(1-\vartheta)E(1)}\bigg)x + \frac{(1-\vartheta)^2}{n^2}\bigg(\frac{E''(1)+E'(1)}{E(1)} + \frac{3E'(1)}{(1-\vartheta)^2E(1)} + \frac{2}{(1-\vartheta)^3}\bigg)\Bigg]\Bigg| \leq n\mathfrak{B}_n^{[\vartheta]}(|h_2(u,x)|;x).$$

Using the definition (4.1) of the weighted modulus of continuity, we obtain

$$\begin{aligned} \left|\frac{f''(\xi)-f''(x)}{2}\right| &\leq \frac{1}{2}\Omega(f''; |\xi - x|)(1+(\xi-x)^2)(1+x^2) \\ &\leq \frac{1}{2}\Omega(f''; |\phi_{1,x}(u)|)(1+\phi_{2,x}(u))(1+x^2) \\ &\leq \bigg(1 + \frac{|\phi_{1,x}(u)|}{\delta}\bigg)\rho(x,\delta)\Omega(f'',\delta)(1+\phi_{2,x}(u)). \end{aligned}$$

Hence,

$$\left|\frac{f''(\xi)-f''(x)}{2}\right| \leq \begin{cases} 2\rho(x,\delta)\Omega(f'';\delta), & |\phi_{1,x}(u)| < \delta, \\ 2\rho(x,\delta)\frac{\phi_{4,x}(u)}{\delta^4}\Omega(f'';\delta), & |\phi_{1,x}(u)| \geq \delta. \end{cases}$$

Now choosing $0 < \delta < 1$, we have

$$\begin{aligned}\left|\frac{f''(\xi)-f''(x)}{2}\right| &\leq 2(1+\delta^2)^2(1+x^2)\Omega(f'';\delta)\left(1+\frac{\phi_{4,x}(u)}{\delta^4}\right)\\ &\leq 8(1+x^2)\Omega(f'',\delta)\left(1+\frac{\phi_{4,x}(u)}{\delta^4}\right).\end{aligned}$$

Thus,

$$|h_2(u;,x)| \leq 8(1+x^2)\Omega(f'',\delta)\left(\phi_{2,x}(u)+\frac{\phi_{6,x}(u)}{\delta^4}\right).$$

Hence applying Lemma 2.3, for sufficiently large n and each $x \in [0,\infty)$,

$$\begin{aligned}\mathfrak{B}_n^{[\vartheta]}(|h_2(u;x)| &\leq 8(1+x^2)\Omega(f'';\delta)\{\eta_{n,2}^{[\vartheta]}(x)+\frac{1}{\delta^4}\eta_{n,6}^{[\vartheta]}(x)\}\\ &= 8(1+x^2)\Omega(f'',\delta)\{O\left(\frac{1}{n}\right)+\frac{1}{\delta^4}O\left(\frac{1}{n^3}\right)\}.\end{aligned}$$

Choosing $\delta = n^{-1/2}$, we get $\mathfrak{B}_n^{[\vartheta]}(h_2(u,x);x) = 8(1+x^2)\Omega\left(f'';n^{-1/2}\right)O\left(\frac{1}{n}\right)$, *as* $n \to \infty$. Consequently, for sufficiently large n and each $x \in [0,\infty)$, $n\mathfrak{B}_n^{[\vartheta]}(h_2(u,x)|;x) = O(1)\Omega\left(f'';n^{-1/2}\right)$, which completes the proof.

□

In our next result, we establish Gr$\ddot{u}$ss–Voronovskaya-type theorem that shows the nonmultiplicativity of the operator $\mathfrak{B}_n^{[\vartheta]}(.;x)$. The Gr$\ddot{u}$ss inequality [11] is an estimate of the difference of integral of the two functions with the product of the integral of the two functions. Many researchers have made significant contributions in this direction (cf. [1–4, 8, 9, 23, 24] etc.)

Theorem 4.4 *Let* $f,\ g \in C_\xi^*[0,\infty)$ *such that* $f',\ g',\ f'',\ g'',\ (fg)'' \in C_\xi^*([0,\infty))$ *and* $\vartheta = \vartheta_n \to 0,$ *as* $n \to \infty$ *and* $\lim_{n\to\infty} n\vartheta = l$ $(l \in \mathbb{R})$*; then the following equality holds:*

$$\lim_{n\to\infty} n\{\mathfrak{B}_n^{[\vartheta]}(fg)(x) - \mathfrak{B}_n^{[\vartheta]}(f)(x)\mathfrak{B}_n^{[\vartheta]}(g)(x)\} = 3xf'(x)g'(x).$$

Proof Using the equality

$$(fg)''(x) = (f(x)g(x))'' + g'(x)f'(x) + g''(x)f(x) + g'(x)f'(x)$$

by simple calculations, we get

$$n\{\mathfrak{B}(fg)(x)-\mathfrak{B}(f)(x)\mathfrak{B}(g)(x)\}=n\Bigg(\mathfrak{B}_n^{[\vartheta]}(fg)(x)-f(x)g(x)-(fg)'(x)\eta_{n,1}^{[\vartheta]}(x) \tag{4.5}$$

$$-\frac{\eta_{n,2}^{[\vartheta]}(x)(fg)''(x)}{2}-g(x)\Bigg(\mathfrak{B}_n^{[\vartheta]}(f;x)-f(x)-f'(x)\eta_{n,1}^{[\vartheta]}(x)-\frac{\eta_{n,2}^{[\vartheta]}(x))}{2}f''(x)\Bigg) \tag{4.6}$$

$$-\mathfrak{B}_n^{[\vartheta]}(f;x)\Bigg(\mathfrak{B}_n^{[\vartheta]}(g;x)-g(x)-g'(x)\eta_{n,1}^{[\vartheta]}(x)-\frac{\eta_{n,2}^{[\vartheta]}(x)}{2}g''(x)\Bigg)+\eta_{n,2}^{[\vartheta]}(x)f'(x)g'(x) \tag{4.7}$$

$$+g''(x)\frac{\eta_{n,2}^{[\vartheta]}(x)}{2!}(f(x)-\mathfrak{B}_n^{[\vartheta]}(f;x))+g'(x)\eta_{n,1}^{[\vartheta]}(x)(f(x)-\mathfrak{B}_n^{[\vartheta]}(f;x))\Bigg). \tag{4.8}$$

In view of Theorem 3.1, for each $f\in C_\xi^*[0,\infty)$ it follows that $\mathfrak{B}_n^{[\vartheta]}(f;x)\to f(x)$, as $n\to\infty$. Also by quantitative Theorem 4.3, for $f''\in C_\xi^*[0,\infty)$ and each $x\in[0,\infty)$ we have $n\Bigg(\mathfrak{B}_n^{[\vartheta]}(f)(x)-f(x)-f'(x)\eta_{n,1}^{[\vartheta]}(x)-\frac{\eta_{n,2}^{[\vartheta]}(x)f''(x)}{2}\Bigg)\to 0$, *as* $n\to\infty$. Hence, using Lemma 2.3, we have $\lim_{n\to\infty}n\{\mathfrak{B}_n^{[\vartheta]}(fg)(x)-\mathfrak{B}_n^{[\vartheta]}(f)(x)\mathfrak{B}_n^{[\vartheta]}(g)(x)\}=3xf'(x)g'(x)$. Thus, we reach the desired result. □

Example 4.5 *Let us consider* $A(t)=t,B(t)=e^t$, $\vartheta=\frac{n^2}{n^3+1}$, *and the function*

$$f(x)=\begin{cases} x^{\frac{3}{2}}\cdot cos\left(\frac{\pi}{\sqrt{x}}\right) & if \quad 0<x\leq 1\\ 0 & if \quad x=0.\end{cases}$$

The function f *has a bounded derivative on* $[0,1]$ *and we have that* f *is of bounded variation. The convergence of* $\mathfrak{B}_n^{[\vartheta]}(f;x)$ *to* $f(x)$ *for* $n=50,100,$ *and* 500 *is illustrated in Figure 1.*

Example 4.6 *Let us consider* $A(t)=t,B(t)=e^t$, $\vartheta=\frac{n^2}{n^3+1}$, *and the function*

$$f(x)=\begin{cases} x\cdot cos\left(\frac{\pi}{x}\right) & if \quad x\neq 0\\ 0 & if \quad x=0.\end{cases}$$

This function f *is continuous, but is not of bounded variation because it wobbles too much near* $x=0$. *The convergence of* $\mathfrak{B}_n^{[\vartheta]}(f;x)$ *to* $f(x)$ *for* $n=50,100,$ *and* 500 *is illustrated in Figure 2.*

5. Rate of approximation

In this section, we shall estimate the approximation function that belongs to the class of derivatives of bounded variation. First we shall need the following result:

Lemma 5.1 *For n sufficiently large,* $x>0$ *and* $\vartheta=\vartheta(n)\to 0, n\vartheta(n)\to l\in\mathbb{R}$ *as* $n\to\infty$, *we have*

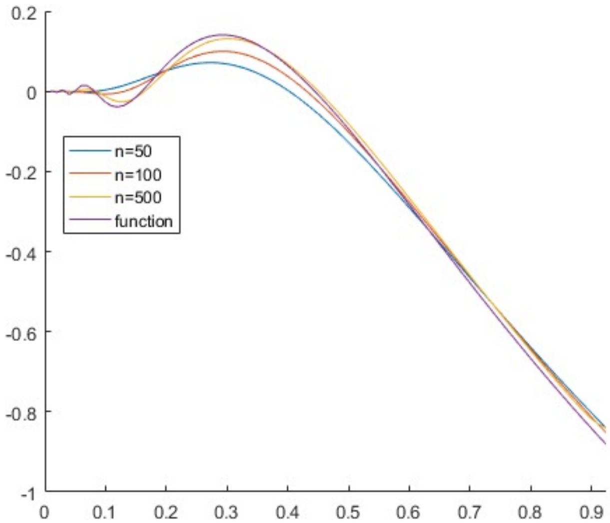

Figure 1. The convergence of $\mathfrak{B}_n^{[\vartheta]}(f;x)$ *to* $f(x)$.

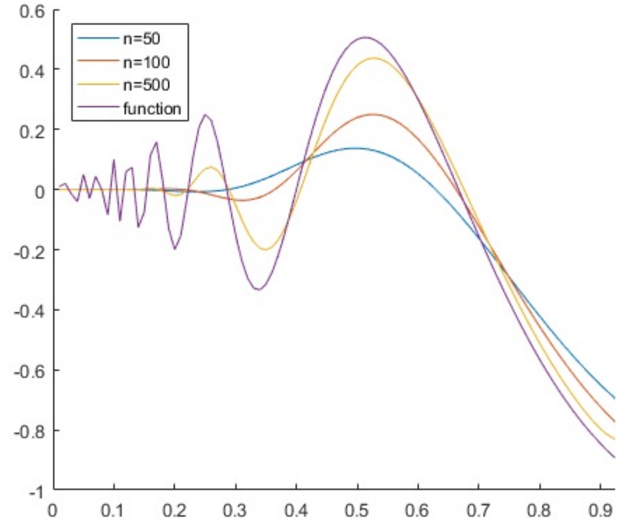

Figure 2. The convergence of $\mathfrak{B}_n^{[\vartheta]}(f;x)$ *to* $f(x)$.

1. *For* $0 \leq y < x$, *we have* $\varrho_n^{[\vartheta]}(x,y) = \int_0^y \mathfrak{L}_n^{[\vartheta]}(x,u)du \leq \frac{C_1(l)(1+x^2)}{n(x-y)^2}$,

2. *For* $x \leq z < \infty$, *we have* $1 - \varrho_n^{[\vartheta]}(x,z) = \int_0^\infty \mathfrak{L}_n^{[\vartheta]}(x,u)du \leq \frac{C_1(l)(1+x^2)^2}{n^2(z-x)^2}$,

where $C_1(l)$ *is a constant given by (2.1).*

Proof From (2.1), for sufficiently large n

$$\begin{aligned}\varrho_n^{[\vartheta]}(x,y) &= \int_0^y \mathfrak{L}_n^{[\vartheta]}(x,u)du \leq \int_0^y \left(\frac{x-u}{x-y}\right)^2 \mathfrak{L}_n^{[\vartheta]}(x,u)du \\ &\leq \frac{\eta_{n,2}^{[\vartheta]}(x)}{(x-y)^2} \leq \frac{C_1(l)(1+x^2)}{n(x-y)^2}.\end{aligned}$$

The inequality (2) can be proved in a similar manner and hence we omit the details.

Let $f \in DBV[0,\infty)$ be the class of all continuous functions in $[0,\infty)$ that have a derivative of bounded variation on every finite subinterval of $[0,\infty)$ and $f(t) = O(t^{2s})$, as $t \to \infty$. It can be observed that all functions $f \in DBV[0,\infty)$ possess the equation

$$f(x) = \int_0^x g(t)dt + f(0).$$

We also use the following notations:

$$\Delta_1(x) = \frac{f'(x+) + f'(x-)}{2} \quad \text{and} \quad \Delta_2(x) = \frac{f'(x+) - f'(x-)}{2}.$$

Now, as usual, we define the auxiliary function f_x, by

$$f_x'(t) = \begin{cases} f'(t) - f'(x+), & x < t \leq \infty \\ 0\ , & t = x \\ f'(t) - f'(x-), & 0 \leq t < x. \end{cases}$$

Then we mainly obtain the following result. □

Theorem 5.2 *Let* $f \in DBV[0,\infty)$, $\vartheta = \vartheta(n) \to 0, n\vartheta(n) \to l \in \mathbb{R}$ *as* $n \to \infty$. *If* $x > 0$ *and* $s > 1$ *are given and* $f(t) = O(t^{2s}), t \to \infty$, *for sufficiently large* n *we have*

$$|\mathfrak{F}(f)| \leq \frac{C_1(x)}{n}\left|\Delta_1(x)\right|\left| - lx + \frac{E'(1)+E(1)}{E(1)}\right| + \left(\left|\Delta_2(x)\right| + \left|f(x^+)\right|\right)\sqrt{\frac{C_2 x}{n}} + O(n^{-s})$$
$$+ \frac{C}{n}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x+\frac{x}{k}} f'_x\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} f'_x\right) + \frac{C}{nx}\left(\left|f(x)\right| + \left|f(2x) - f(x) - xf'(x^+)\right|\right).$$

Proof For $f \in DBV[0,\infty)$, and $x \in (0,\infty)$, we can write

$$f'(t) \quad = \quad \Delta_1(x) + f'_x(t) + \Delta_2(x)sgn(t-x) + \delta_x(t)\Big(f'(t) - \Delta_1(x)\Big), \tag{5.1}$$

where

$$\delta_x(t) = \begin{cases} 1 \ , & t = x \\ 0 \ , & t \neq x. \end{cases}$$

Now using the equality (5.1) and by the fact that $\mathfrak{B}_n^{[\vartheta]}$ is a linear operator, we have

$$\begin{aligned} |\mathfrak{F}(f)| \quad &\leq \quad \left|\mathfrak{B}_n^{[\vartheta]}(f'(u)du;x)\right| \leq \left|\Delta_1(x)\right|\left|\eta_{n,1}^{[\beta]}(x)\right| + \left|\Delta_2(x)\right|\mathfrak{B}_n^{[\vartheta]}|\phi_{1,x}(t)|;x) \\ &+\left|\mathfrak{B}_n^{[\vartheta]}(\Big(\int_t^x f'_x(u)du\Big);x)\right| + \left|f'(t) - \Delta_1(x)\right|\left|\mathfrak{B}_n^{[\vartheta]}(\int_t^x \delta_x(u)du;x)\right| \end{aligned} \tag{5.2}$$

Since $\int_t^x \delta_x(u)du = 0$, we conclude that

$$\begin{aligned} |\mathfrak{F}(f)| \quad &\leq \quad \left|\Delta_1(x)\right|\left|\eta_{n,1}^{[\vartheta]}(x)\right| + \left|\Delta_2(x)\right|\mathfrak{B}_n^{[\vartheta]}(|\phi_{1,x}(t)|;x) \\ &+\left|P_n^{[\vartheta]}(f,x)\right| + \left|Q_n^{[\vartheta]}(f,x)\right|, \end{aligned} \tag{5.3}$$

where $P_n^{[\vartheta]}(f,x) = \int_0^x \mathfrak{L}_n^{[\vartheta]}(x,t)\Big(\int_t^x f'_x(u)du\Big)dt$, and $Q_n^{[\vartheta]}(f,x) = \int_x^\infty \mathfrak{L}_n^{[\vartheta]}(x,t)\Big(\int_x^t f'_x(u)du\Big)dt$.

Then it follows from relation (2.1) that

$$|\mathfrak{F}(f)| \quad \leq \quad \left|\Delta_1(x)\right|\left|\eta_{n,1}^{[\vartheta]}(x)\right| + \left|\Delta_2(x)\right|\sqrt{\frac{Cx}{n}} + \left|P_n^{[\vartheta]}(f,x)\right| + \left|Q_n^{[\vartheta]}(f,x)\right|. \tag{5.4}$$

Thus, to complete the proof of the theorem, it is sufficient to estimate the terms $P_n^{[\vartheta]}(f,x)$ and $Q_n^{[\vartheta]}(f,x)$. Using Lemma 5.1, and applying integration by parts, we can write

$$P_n^{[\vartheta]}(f,x) = \int_0^x \Big(\int_t^x f'_x(u)du\Big)d_t(\varrho_n^{[\vartheta]}(x,t)) = \int_0^x f'_x(t)\varrho_n^{[\vartheta]}(x,t)dt.$$

Thus,

$$|P_n^{[\vartheta]}(f,x)| \leq (\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^x)|f'_x(t)|\varrho_n^{[\vartheta]}(x,t)dt.$$

Since $|f'_x(t)| \leq \bigvee_t^x f'_x$ and $\varrho_n^{[\vartheta]}(x,t) \leq 1$, we get

$$\int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(t)|\varrho_n^{[\vartheta]}(x,t)dt \quad \leq \quad \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right).$$

Using Lemma 5.1 and considering $t = x - \frac{x}{u}$, we have

$$\int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)|\varrho_n^{[\vartheta]}(x,t)dt \leq \frac{Cx}{n}\int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)|\frac{dt}{\phi_{2,x}(t)} \leq \frac{Cx}{n}\int_0^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_t^x f'_x\right)\frac{dt}{\phi_{2,x}(t)}$$

$$= \frac{C}{n}\int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{u}}^{x} f'_x\right)du \leq \frac{C}{n}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right).$$

Therefore,

$$|P_n^{[\vartheta]}(f,x)| \leq \frac{C}{n}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right). \tag{5.5}$$

Now to estimate $Q_n^{[\vartheta]}(f,x)$ we proceed in a manner similar to Theorem 3.8 in [19] and deduce:

$$|Q_n^{[\vartheta]}(f,x)| \leq O(n^{-s}) + C\frac{\big|f(x)\big|}{nx} + |f(x^+)|\sqrt{\frac{Cx}{n}} + \frac{C}{nx}\Big|f(2x) - f(x) - xf'(x^+)\Big|$$

$$+ \frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x\right) + \frac{C}{n}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{k}} f'_x\right). \tag{5.6}$$

Finally combining the estimates of (5.4), (5.5), and (5.6), we obtain the desired result. □

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