

Bound states and spectral singularities of an impulsive Schrödinger equation

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Abstract: In this paper, we study the analytical properties of the Jost function of an impulsive Schrödinger equation. We also investigate the bound states and spectral singularities of this equation. We present some conditions on the potential function that guarantee that the impulsive Schrödinger equation has a finite number of bound states and spectral singularities with finite multiplicities.

Key words: Spectral analysis, spectral singularities, bound states, Schrödinger operators, point interaction

1. Introduction

Impulsive differential equations are a basic tool to study dynamics of processes that are subjected to abrupt changes in their states. Many chemical, physical, and biological phenomena involving thresholds, bursting rhythm models in medicine, pharmacokinetics, and frequency-modulated systems and mathematical models in economics do exhibit impulsive effects [15,18,22]. For the mathematical theory of impulsive differential equations, we refer to the monographs [6,7]. In the literature impulsive conditions are referred to by different names. Some of these are ‘jump condition’, ‘interface condition’, ‘point interaction condition’, and ‘transmissions condition’. In particular, impulsive dissipative boundary value problems have been studied in detail in [12,13,25,26]. The spectral analysis of a nonself-adjoint Schrödinger operator was investigated by Naimark [21]. He proved that the spectrum of this operator is composed of a continuous spectrum, bound states, and spectral singularities. Schwartz studied the spectral singularities of a certain class of abstract operators in a Hilbert space [24]. Lyance investigated the effects of spectral singularities in spectral expansion in terms of the principal functions of Schrödinger operators [19]. Some problems of spectral theory of differential and some other types of operators with spectral singularities were studied by others in [1–5,8–11,16,17].

Let us consider the Schrödinger equation

$$-\psi''(x) + q(x)\psi(x) = \lambda^2\psi(x), \quad x \in \mathbb{R} \setminus \{0\} \quad (1.1)$$

with the impulsive condition

$$\begin{pmatrix} \psi(0^+) \\ \psi'(0^+) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi(0^-) \\ \psi'(0^-) \end{pmatrix}, \quad (1.2)$$

where $a, b, c, d \in \mathbb{C}$, q is a complex valued function and λ is a spectral parameter. The bounded solutions of

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(1.1) satisfying the conditions

$$\lim_{x \rightarrow \pm\infty} \psi(x)e^{\pm i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ = \{\lambda: \lambda \in \mathbb{C}, \text{Im}\lambda \geq 0\} \tag{1.3}$$

will be denoted by $e_{\pm}(x, \lambda)$. The solutions $e_{\pm}(x, \lambda)$ are called Jost solutions of (1.1). Under the condition

$$\int_{-\infty}^{\infty} (1 + |x|) |q(x)| dx < \infty, \tag{1.4}$$

the solutions $e_{\pm}(x, \lambda)$ have the representations

$$e_-(x, \lambda) = e^{-i\lambda x} + \int_{-\infty}^x K^-(x, t) e^{-i\lambda t} dt, \tag{1.5}$$

$$e_+(x, \lambda) = e^{i\lambda x} + \int_x^{\infty} K^+(x, t) e^{i\lambda t} dt$$

for every $\lambda \in \overline{\mathbb{C}}_+$ and the kernel functions $K^+(x, t)$ and $K^-(x, t)$ satisfy

$$K^+(x, t) = \frac{1}{2} \int_{\frac{x+t}{2}}^{\infty} q(s) ds + \frac{1}{2} \int_x^{\frac{x+t}{2}} \int_{x+t-s}^{t-x+s} q(s) K^+(s, r) ds dr$$

$$+ \frac{1}{2} \int_{\frac{x+t}{2}}^{\infty} \int_s^{t-x+s} q(s) K^+(s, r) ds dr, \tag{1.6}$$

$$K^-(x, t) = \frac{1}{2} \int_{-\infty}^{\frac{x+t}{2}} q(s) ds + \frac{1}{2} \int_{\frac{x+t}{2}}^x \int_{x+t-s}^{x+t-s} q(s) K^-(s, r) ds dr$$

$$+ \frac{1}{2} \int_{-\infty}^{\frac{x+t}{2}} \int_s^{t-x+s} q(s) K^-(s, r) ds dr.$$

Furthermore, $K^{\pm}(x, t)$ are continuously differentiable with respect to their arguments and satisfy the following inequalities:

$$|K^{\pm}(x, t)| \leq c\sigma^{\pm} \left(\frac{x+t}{2} \right)$$

$$\left| K_x^{\pm}(x, t) \pm \frac{1}{4} \left| q \left(\frac{x+t}{2} \right) \right| \right| \leq c\sigma^{\pm} \left(\frac{x+t}{2} \right) \tag{1.7}$$

$$\left| K_t^{\pm}(x, t) \pm \frac{1}{4} \left| q \left(\frac{x+t}{2} \right) \right| \right| \leq c\sigma^{\pm} \left(\frac{x+t}{2} \right),$$

where

$$\sigma^+(t) = \int_x^\infty |q(t)| dt, \quad \sigma^-(t) = \int_{-\infty}^x |q(t)| dt$$

and $c > 0$ is a constant [17].

This paper is organized as follows. In the next section, we investigate the effect of the impulsive condition on the Schrödinger equation with complex potential, which is different from those in literature, and we obtain a different form of the Jost function of this equation. In Section 3, we find asymptotic behavior of the Jost function and get some properties of bound states and spectral singularities of (1.1)–(1.2) by using this asymptotic equation. In the fourth section, we prove that bound states and spectral singularities of (1.1)–(1.2) and their multiplicities are finite under the Naimark condition.

2. Jost Function of (1.1) – (1.2)

Let ψ be a solution of Schrödinger equation (1.1) and ψ_-, ψ_+ be the restrictions of ψ on positive and negative semiaxes, respectively, given by

$$\begin{aligned} \psi_+(x) &= A_+e_+(x, \lambda) + B_+e_+(x, -\lambda), \quad 0 < x < \infty, \quad \lambda \in \mathbb{R} \setminus \{0\} \\ \psi_-(x) &= A_-e_-(x, \lambda) + B_-e_-(x, -\lambda), \quad -\infty < x < 0, \quad \lambda \in \mathbb{R} \setminus \{0\}. \end{aligned} \tag{2.1}$$

We will define the matrix \mathbf{M} as follows:

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \mathbf{M} \begin{pmatrix} A_- \\ B_- \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{2.2}$$

By using (1.2), (1.5), and (2.1), we get

$$\begin{pmatrix} A_+e_+(0, \lambda) + B_+e_+(0, -\lambda) \\ A_+e'_+(0, \lambda) + B_+e'_+(0, -\lambda) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_-e_-(0, \lambda) + B_-e_-(0, -\lambda) \\ A_-e'_-(0, \lambda) + B_-e'_-(0, -\lambda) \end{pmatrix}$$

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \frac{1}{c} \begin{pmatrix} e'_+(0, -\lambda) & -e_+(0, -\lambda) \\ -e'_+(0, \lambda) & e_+(0, \lambda) \end{pmatrix} \begin{pmatrix} A_-e_-(0, \lambda) + B_-e_-(0, -\lambda) \\ A_-e'_-(0, \lambda) + B_-e'_-(0, -\lambda) \end{pmatrix},$$

where

$$\begin{aligned} C &= W[e_+(0, \lambda), e_+(0, -\lambda)] = -2i\lambda, \quad \lambda \in \mathbb{R} \setminus \{0\}, \\ M_{11} &= ae'_+(0, -\lambda)e_-(0, \lambda) + be'_+(0, -\lambda)e'_-(0, \lambda) - ce_+(0, -\lambda)e_-(0, \lambda) \\ &\quad - de_+(0, -\lambda)e'_-(0, \lambda), \\ M_{12} &= ae'_+(0, -\lambda)e_-(0, -\lambda) + be'_+(0, -\lambda)e'_-(0, -\lambda) - ce_+(0, -\lambda)e_-(0, -\lambda) \\ &\quad - de_+(0, -\lambda)e'_-(0, -\lambda), \\ M_{21} &= -ae'_+(0, \lambda)e_-(0, \lambda) - be'_+(0, \lambda)e'_-(0, \lambda) + ce_+(0, \lambda)e_-(0, \lambda) \\ &\quad + de_+(0, \lambda)e'_-(0, \lambda), \\ M_{22} &= -ae'_+(0, \lambda)e_-(0, -\lambda) - be'_+(0, \lambda)e'_-(0, -\lambda) + ce_+(0, \lambda)e_-(0, -\lambda) \\ &\quad + de_+(0, \lambda)e'_-(0, -\lambda). \end{aligned} \tag{2.3}$$

Both bound states and spectral singularities are related to the zeros of M_{22} , which is a component of the transfer matrix M . The function M_{22} is called the Jost function of (1.1)–(1.2).

Theorem 2.1 *Under the assumptions*

$$q \in AC(-\infty, \infty), \quad \lim_{x \rightarrow \pm\infty} q(x) = 0, \quad \int_{-\infty}^{\infty} x^3 |q'(x)| dx < \infty, \quad (2.4)$$

M_{22} has the representation

$$M_{22}(\lambda) = -b\lambda^2 + m\lambda + n + \int_0^{\infty} f(t) e^{i\lambda t} dt, \quad \lambda \in \mathbb{R} \setminus \{0\}, \quad (2.5)$$

where

$$\begin{aligned} m &= -ia - biK^-(0, 0) - biK^+(0, 0) - id, \\ n &= aK^+(0, 0) + bK^+(0, 0)K^-(0, 0) + dK^-(0, 0) + c \\ &\quad + bK_x^-(0, 0) - bK_x^+(0, 0) + dK^+(0, 0) + aK^-(0, 0), \\ f(t) &= aK_x^-(0, -t) - aK^+(0, 0)K^-(0, -t) + dK^+(0, t) - bK_{xt}^+(0, t) \\ &\quad - dK_x^-(0, -t) - aK_x^+(0, t) - K_{xt}^-(0, -t) - cK^-(0, -t) \\ &\quad - cK^+(0, t) + dK^-(0, 0)K^+(0, t) - bK^-(0, 0)K_x^+(0, t) \\ &\quad + b(K_x^+(0, t) * K_x^+(0, -t)) - c(K^+(0, t) * K^+(0, -t)) \\ &\quad - d(K^+(0, t) * K_x^+(0, -t)) - a(K_x^+(0, t) * K^+(0, -t)) \\ &\quad - bK^+(0, 0)K_x^-(0, -t) \end{aligned} \quad (2.6)$$

and then $m, n \in \mathbb{C}$ and $f \in L_1(0, \infty)$.

Proof In view of (1.5) and (2.3), we find

$$\begin{aligned} M_{22}(\lambda) &= -b\lambda^2 + (-ia - biK^+(0, 0) - id)\lambda + aK^+(0, 0) + bK^+(0, 0) \\ &\quad K^-(0, 0) + c - i\lambda a \int_{-\infty}^0 K^-(0, t) e^{-i\lambda t} dt - bi\lambda \int_{-\infty}^0 K_x^-(0, t) e^{-i\lambda t} dt \\ &\quad - a \int_0^{\infty} K_x^+(0, t) e^{i\lambda t} dt + aK^+(0, 0) \int_{-\infty}^0 K^-(0, t) e^{-i\lambda t} dt \\ &\quad + bK^+(0, 0) \int_{-\infty}^0 K_x^-(0, t) e^{-i\lambda t} dt + dK^-(0, 0) \int_0^{\infty} K^+(0, t) e^{i\lambda t} dt \end{aligned}$$

$$\begin{aligned}
 & -bK^-(0,0) \int_0^\infty K_x^+(0,t) e^{i\lambda t} dt + c \int_{-\infty}^0 K^-(0,t) e^{-i\lambda t} dt + dK^-(0,0) \\
 & + bi\lambda \int_0^\infty K_x^+(0,t) e^{i\lambda t} dt + c \int_0^\infty K^+(0,t) e^{i\lambda t} dt + d \int_{-\infty}^0 K_x^-(0,t) e^{-i\lambda t} dt \\
 & - di\lambda \int_0^\infty K^+(0,t) e^{i\lambda t} dt - a \int_0^\infty K_x^+(0,t) e^{i\lambda t} dt \int_{-\infty}^0 K^-(0,t) e^{-i\lambda t} dt \\
 & - b \int_0^\infty K_x^+(0,t) e^{i\lambda t} dt \int_{-\infty}^0 K_x^-(0,t) e^{-i\lambda t} dt + c \int_0^\infty K^+(0,t) e^{i\lambda t} dt \\
 & \int_{-\infty}^0 K^-(0,t) e^{-i\lambda t} dt + d \int_0^\infty K^+(0,t) e^{i\lambda t} dt \int_{-\infty}^0 K_x^-(0,t) e^{-i\lambda t} dt,
 \end{aligned}$$

where $(g_1 * g_2)(x) := \int_0^x g_1(t) g_2(x-t) dt$ denotes the convolution of the functions $g_1, g_2 \in L_1(0, \infty)$. It is also known from [20] that

$$\begin{aligned}
 K^+(0,t), K_x^+(0,t), K_t^+(0,t) & \in L_1(0, \infty), \\
 K^-(0,t), K_x^-(0,t), K_t^-(0,t) & \in L_1(-\infty, 0)
 \end{aligned}$$

and by (1.6)

$$\begin{aligned}
 K_{xt}^+(x,t) & = -\frac{1}{8}q'\left(\frac{x+t}{2}\right) + \frac{1}{4}K^+\left(\frac{x+t}{2}, \frac{x+t}{2}\right)q\left(\frac{x+t}{2}\right) \\
 & \quad - \frac{1}{2} \int_x^{\frac{x+t}{2}} [K_t^+(s, t+x-s) + K_t^+(s, t-x+s)] q(s) ds \\
 & \quad - \frac{1}{2} \int_{\frac{x+t}{2}}^\infty K_t^+(s, t-x+s) q(s) ds.
 \end{aligned}$$

In a similar way, we can obtain $K_{xt}^-(x,t)$. Using (2.1) and (2.5), we find that

$$|K_{xt}^+(0,t)| \leq c \left[\left|q'\left(\frac{t}{2}\right)\right| + \left|q\left(\frac{t}{2}\right)\right| + \int_{\frac{t}{2}}^\infty |q(s)| ds \right], \tag{2.7}$$

where $c > 0$ is a constant. In this case

$$\begin{aligned}
 K_{xt}^+(0,t) & \in L_1(0, \infty) \\
 K_{xt}^-(0,t) & \in L_1(-\infty, 0)
 \end{aligned}$$

and it follows from (2.6) that

$$f \in L_1(0, \infty). \tag{2.8}$$

□

Therefore, we can understand that M_{22} has an analytic continuation to $\mathbb{C}_+ := \{\lambda: \lambda \in \mathbb{C}, \text{Im}\lambda > 0\}$ and is continuous up to the real axis. In the next sections, we can give some theorems and conditions by using this property of M_{22} , and then we find some properties of bound states and spectral singularities of Schrödinger equation (1.1)–(1.2).

3. Bound states and spectral singularities of (1.1) – (1.2)

We will denote the sets of bound states and spectral singularities of (1.1)–(1.2) by σ_d and σ_{ss} , respectively. By the definition of the sets of bound states and spectral singularities of (1.1)–(1.2), we write

$$\begin{aligned} \sigma_d &:= \{\mu: \mu = \lambda^2, \lambda \in \mathbb{C}_+, M_{22}(\lambda) = 0\}, \\ \sigma_{ss} &:= \{\mu: \mu = \lambda^2, \lambda \in \mathbb{R} \setminus \{0\}, M_{22}(\lambda) = 0\}. \end{aligned} \tag{3.1}$$

It is obvious from (2.5) that to study the structure of the bound states and the spectral singularities of (1.1)–(1.2), we need to investigate the structure of the zeros of M_{22} in $\overline{\mathbb{C}_+}$. Let us define

$$\begin{aligned} T_1 &= \{\lambda: \lambda \in \mathbb{C}_+, M_{22}(\lambda) = 0\}, \\ T_2 &= \{\lambda: \lambda \in \mathbb{R} \setminus \{0\}, M_{22}(\lambda) = 0\}. \end{aligned}$$

The following can be easily seen:

$$\sigma_d = \{\mu: \mu = \lambda^2, \lambda \in T_1\}, \quad \sigma_{ss} = \{\mu: \mu = \lambda^2, \lambda \in T_2\}. \tag{3.2}$$

In the following, we will give a lemma that is essential for us to investigate the properties of bound states and spectral singularities of (1.1)–(1.2).

Lemma 3.1 *Under the condition (2.4),*

i) the set of T_1 is bounded and has at most a countable number of elements, and its limit points can lie only on a bounded subinterval of the real axis;

ii) the set of T_2 is bounded and its linear Lebesgue measure is zero.

Proof From (1.5) and (2.3), we get that

$$M_{22}(\lambda) = -b\lambda^2 + m\lambda + n + o(1), \quad \lambda \in \overline{\mathbb{C}_+}, \quad |\lambda| \rightarrow \infty. \tag{3.3}$$

Equation (3.3) indicates the boundedness of the sets T_1 and T_2 . From the analyticity of function M_{22} in \mathbb{C}_+ , we obtain that T_1 has at most a countable number of elements, and its limit points can lie only on a bounded subinterval of the real axis. Using the uniqueness theorem of analytic functions, we get that the set T_2 is closed and its linear Lebesgue measure is zero [14]. □

From (3.3) and Lemma 2.2, we obtain the following theorem.

Theorem 3.2 Under the condition (2.4),

- i) the set of bound states of (1.1)–(1.2) is bounded and has at most a countable number of elements, and its limit points can lie only on a bounded subinterval of the real axis;
- ii) the set of spectral singularities of (1.1)–(1.2) is bounded and its linear Lebesgue measure is zero.

4. Naimark conditions

We will denote the set of all limit points of T_1 by T_3 and the set of all zeros of M_{22} with infinity multiplicity in $\overline{\mathbb{C}}_+$ by T_4 . We know from the uniqueness theorem of analytic functions that

$$T_3 \subset T_2, \quad T_4 \subset T_2, \quad \mu(T_3) = 0, \quad \mu(T_4) = 0, \tag{4.1}$$

and if we consider the continuity of all derivatives of M_{22} on the real axis, we find that

$$T_3 \subset T_4. \tag{4.2}$$

Now we suppose that the analog form of the Naimark condition

$$q \in AC(-\infty, \infty), \quad \lim_{x \rightarrow \pm\infty} q(x) = 0, \quad \int_{-\infty}^{\infty} e^{\varepsilon|x|} |q'(x)| dx < \infty, \quad \varepsilon > 0 \tag{4.3}$$

holds [21].

Theorem 4.1 Under the condition (4.3) the sets of bound states and spectral singularities of (1.1)–(1.2) have a finite number of elements; moreover, each of them is of finite multiplicity.

Proof Combining (1.7), (2.7), and (4.3), we obtain that

$$|f(t)| \leq ce^{-\frac{\varepsilon}{2}t}, \tag{4.4}$$

where $c > 0$ is a constant. From (2.5) and (4.4), we see that M_{22} has an analytical continuation to the half-plane $\text{Im}\lambda > -\frac{\varepsilon}{2}$. Using the uniqueness theorem of analytic functions [14], we get that $T_4 = \emptyset$ and it is obvious that, from (4.2), $T_3 = \emptyset$. Therefore, the sets of bound states and spectral singularities of (1.1)–(1.2) have no limit points, which yields that the sets of bound states and spectral singularities of (1.1)–(1.2) have a finite number of elements with a finite multiplicity. □

In the following lemma, we will show the same result of Theorem 4.1 by using a weaker condition than (4.3). In this case, we will give a different method for the proof.

Lemma 4.2 Under the following conditions,

$$q \in AC(-\infty, \infty), \quad \lim_{x \rightarrow \pm\infty} q(x) = 0, \quad \sup_{x \in \mathbb{R}} \left[e^{\varepsilon\sqrt{|x|}} |q'(x)| \right] < \infty, \quad \varepsilon > 0, \tag{4.5}$$

we have $T_4 = \emptyset$.

Proof Using (2.5) and (4.5), we see that M_{22} is analytic in \mathbb{C}_+ and all of its derivatives are continuous in $\overline{\mathbb{C}}_+$. We obtain that

$$\left| \frac{d^n}{d\lambda^n} M_{22}(\lambda) \right| \leq \int_0^\infty t^n |f(t)| e^{-t\text{Im}\lambda} dt.$$

From (1.7), we get the following inequality:

$$\int_0^\infty t^n |K_t^-(0, -t)| e^{-\text{Im}\lambda t} dt \leq \frac{1}{4} \int_0^\infty t^n \left| q\left(-\frac{t}{2}\right) \right| e^{-\text{Im}\lambda t} dt + c \int_0^\infty t^n \left| \sigma^-\left(\frac{t}{2}\right) \right| e^{-\text{Im}\lambda t} dt,$$

and by using (4.5) we see that

$$\begin{aligned} \int_0^\infty t^n \left| q\left(-\frac{t}{2}\right) \right| e^{-\text{Im}\lambda t} dt &= (-2)^n \int_{-\infty}^0 u^n |q(u)| e^{2u\text{Im}\lambda} du \\ &= (-2)^n \int_{-\infty}^0 u^n |q(u)| e^{\varepsilon\sqrt{|u|}} e^{-\varepsilon\sqrt{|u|}} e^{2u\text{Im}\lambda} du \\ &\leq c_\varepsilon (-2)^n \int_{-\infty}^0 u^n e^{-\varepsilon\sqrt{|u|}} du \\ &= c_\varepsilon \frac{2^n}{\varepsilon^{2n+2}} \int_0^\infty s^{2n+1} e^{-s} ds \\ &= c_\varepsilon \alpha^n \Gamma(2n + 2). \end{aligned}$$

It is clear from the last inequality that

$$\int_0^\infty t^n \left| \sigma^-\left(\frac{t}{2}\right) \right| e^{-\text{Im}\lambda t} dt \leq c_\varepsilon \alpha^n \int_0^\infty s^{2n+1} e^{-s} ds = c_\varepsilon \alpha^n \Gamma(2n + 2).$$

By using the gamma function, we write

$$c_\varepsilon \alpha^n \Gamma(2n + 2) \leq c_\varepsilon \alpha^n n! n^n, \quad |\lambda| \leq K, \quad \lambda \in \overline{\mathbb{C}}_+, \quad n = 0, 1, 2, \dots$$

where K is a sufficiently large number and c_ε is a constant. We can apply a similar argument to the integrals in M_{22} . Then we find

$$\left| \frac{d^n}{d\lambda^n} M_{22}(\lambda) \right| \leq c_\varepsilon \alpha^n n! n^n, \quad |\lambda| \leq K, \quad \lambda \in \overline{\mathbb{C}}_+, \quad n = 0, 1, 2, \dots \tag{4.6}$$

Since M_{22} is not equal to zero, it is obvious from Pavlov's theorem that

$$\int_0^h \ln H(s) d\mu(T_4, s) > -\infty, \tag{4.7}$$

where $H(s) = c_\varepsilon \inf_n (\alpha ns)^n$ and $\mu(T_4, s)$ is the linear Lebesgue measure of the s -neighborhood of T_4 [23]. It is clear that

$$H(s) \leq c_\varepsilon \exp(-\alpha^{-1} e^{-1} s^{-1}). \quad (4.8)$$

Combining (4.7) and (4.8), we find

$$\int_0^h \frac{1}{s} d\mu(T_4, s) < \infty. \quad (4.9)$$

It follows from (4.9) that $\mu(T_4, s) = 0$ and it gives s , so $T_4 = \emptyset$. \square

Theorem 4.3 *Assume (4.5). Then (1.1)–(1.2) has a finite number of bound states and spectral singularities, and each of them is of finite multiplicity.*

Proof Using (4.2) and Lemma 3.1, we find that $T_3 = \emptyset$. Since T_1 has no limit points, the function M_{22} has only a finite number of zeros in $\overline{\mathbb{C}}_+$. Also, since $T_4 = \emptyset$, these zeros are of finite multiplicity. \square

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