

Existence of maximal ideals in Leavitt path algebras

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Abstract: Let E be an arbitrary directed graph and let L be the Leavitt path algebra of the graph E over a field K . The necessary and sufficient conditions are given to assure the existence of a maximal ideal in L and also the necessary and sufficient conditions on the graph that assure that every ideal is contained in a maximal ideal are given. It is shown that if a maximal ideal M of L is nongraded, then the largest graded ideal in M , namely $gr(M)$, is also maximal among the graded ideals of L . Moreover, if L has a unique maximal ideal M , then M must be a graded ideal. The necessary and sufficient conditions on the graph for which every maximal ideal is graded are discussed.

Key words: Leavitt path algebras, arbitrary graphs, maximal ideals

1. Introduction and preliminaries

It is well known that in a ring with identity, any ideal is contained in a maximal ideal; however, for a nonunital ring, the existence of a maximal ideal is not always guaranteed. Also, any maximal ideal is not necessarily a prime ideal in a nonunital case. In this study, for the particular example of a Leavitt path algebra (which is nonunital if the number of vertices of the graph on which it is constructed is infinite), we discuss the existence of maximal ideals and its characterization via the graph properties.

The outline of the paper is as follows: we give preliminary definitions in the introduction, and in Section 2, we discuss maximal and prime ideals in a nonunital ring. Although a Leavitt path algebra does not necessarily have a unit, any maximal ideal is always a prime ideal. It is shown that if a maximal ideal M of L is nongraded, then the largest graded ideal in M , namely $gr(M)$, is also maximal among the graded ideals of L . In Section 3, first we prove that in a Leavitt path algebra, a maximal ideal exists if and only if there is a maximal hereditary and saturated subset of E^0 . We also prove the necessary and sufficient conditions on the graph that assure that every ideal is contained in a maximal ideal. Theorem 3.6 states that if L has a unique maximal ideal M , then M must be a graded ideal. Finally, we give the necessary and sufficient conditions on the graph for which every maximal ideal is graded.

Leavitt path algebras were introduced about a decade ago by [2] and [5] as an algebra defined over a directed graph and, since then, it has been a fruitful structure of study because of its close connection to the graph C^* -algebras.

For the general notation, terminology, and results in Leavitt path algebras, we refer to the recently published book [1], and for basic results in associative rings and modules, we refer to [4].

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A directed graph $E = (E^0, E^1, r, s)$ consists of E^0 , the set of *vertices*, and E^1 , the set of *edges*, together with the *range* and the *source* maps $r, s : E^1 \rightarrow E^0$.

A vertex v is called a *sink* if it emits no edges and a vertex v is called a *regular vertex* if it emits a nonempty finite set of edges. An *infinite emitter* is a vertex that emits infinitely many edges. A graph E is called *finite* if both E^0 and E^1 are finite, and it is called *row-finite* if the number of edges that each vertex emits is finite.

For each $e \in E^1$, we call e^* a *ghost edge*. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A *path* μ of length $|\mu| = n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \dots, n - 1$. In this case, $\mu^* = e_n^* \cdots e_2^* e_1^*$ is the corresponding ghost path. A vertex is considered a path of length 0. The set of all vertices on the path μ is denoted by μ^0 .

A path $\mu = e_1 \dots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. A closed path μ is called *simple* provided that it does not pass through its base more than once, i.e. $s(e_i) \neq s(e_1)$ for all $i = 2, \dots, n$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

An *exit* for a path $\mu = e_1 \dots e_n$ is an edge e such that $s(e) = s(e_i)$ for some i and $e \neq e_i$. We say that the graph E satisfies *Condition (L)* if every cycle in E has an exit. The graph E is said to satisfy *Condition (K)* if every vertex that is the base of a closed path c is also a base of another closed path c' different from c . A cycle c in a graph E is called a *cycle without K* if no vertex on c is the base of another distinct cycle in E (where distinct cycles possess different sets of edges and different sets of vertices).

If there is a path from vertex u to a vertex v , we write $u \geq v$.

A subset M of E^0 is said to be a *maximal tail* if it satisfies the following conditions:

(MT-1) If $v \in M$ and $u \in E^0$ with $u \geq v$, then $u \in M$.

(MT-2) If $v \in M$ is a regular vertex, then there exists an edge e , with $s(e) = v$ such that $r(e) \in M$.

(MT-3) for any two $u, v \in M$ there exists $w \in M$ such that $u \geq w$ and $v \geq w$.

A subset D of E^0 is called *downward directed* if it satisfies the MT-3 property.

For any vertex v , define $M(v) = \{w \in E^0 : w \geq v\}$.

A subset H of E^0 is called *hereditary* if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is *saturated* if, for any regular vertex v , $r(s^{-1}(v)) \subseteq H$ implies $v \in H$. The set of all hereditary saturated subsets of E^0 is denoted by \mathcal{H}_E , which is also a partially ordered set by set inclusion.

Given an arbitrary graph E and a field K , the *Leavitt path algebra* $L_K(E)$ is defined to be the K -algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ that satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.

(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.

(3) (CK-1 relations) For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.

(4) (CK-2 relations) For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1, s(e)=v} ee^*.$$

Recall that a ring R is said to have a set of local units F , where F is a set of idempotents in R having the property that, for each finite subset r_1, \dots, r_n of R , there exists $f \in F$ with $fr_i f = r_i$ for all $1 \leq i \leq n$.

In the case of Leavitt path algebras, for each $x \in L_K(E)$ there exists a finite set of distinct vertices $V(x)$ for which $x = fxf$, where $f = \sum_{v \in V(x)} v$. When E^0 is finite, $L_K(E)$ is a ring with unit element $1 = \sum_{v \in E^0} v$.

Otherwise, $L_K(E)$ is not a unital ring, but is a ring with local units consisting of sums of distinct elements of E^0 .

Every Leavitt path algebra $L_K(E)$ is a \mathbb{Z} -graded algebra $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n$ induced by defining, for all $v \in E^0$ and $e \in E^1$, $\deg(v) = 0$, $\deg(e) = 1$, $\deg(e^*) = -1$. Furthermore, for each $n \in \mathbb{Z}$, the homogeneous component L_n is given by

$$L_n = \{ \sum k_i \alpha_i \beta_i^* \in L : |\alpha_i| - |\beta_i| = n \}.$$

An ideal I of $L_K(E)$ is said to be a *graded ideal* if $I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n)$.

Throughout this paper, E will denote an arbitrary directed graph with no restriction on the number of vertices and the number of edges emitted by each vertex and K will denote an arbitrary field. For convenience in notation, we will denote, most of the time, the Leavitt path algebra $L_K(E)$ by L .

All ideals of concern in the sequel will be two-sided ideals. In this paper, we use both the lattice of all proper ideals, denoted by \mathcal{L} , and the lattice of all graded proper ideals, denoted by \mathcal{L}_{gr} . Hence, ‘maximal ideal’ means a maximal element in \mathcal{L} , which can be either a graded or a nongraded ideal, whereas a graded maximal ideal is a maximal element in \mathcal{L}_{gr} .

We shall be using the following concepts and results from [9]. A *breaking vertex* of a hereditary saturated subset H is an infinite emitter $w \in E^0 \setminus H$ with the property that $0 < |s^{-1}(w) \cap r^{-1}(E^0 \setminus H)| < \infty$. The set of all breaking vertices of H is denoted by B_H . For any $v \in B_H$, v^H denotes the element $v - \sum_{s(e)=v, r(e) \notin H} ee^*$. Given a hereditary saturated subset H and a subset $S \subseteq B_H$, the pair (H, S) is called an *admissible pair*. The set \mathbf{H} of all admissible pairs becomes a lattice under a partial order \leq' under which $(H_1, S_1) \leq' (H_2, S_2)$ if $H_1 \subseteq H_2$ and $S_1 \subseteq H_2 \cup S_2$ (see also [1, Proposition 2.5.6]). Given an admissible pair (H, S) , the ideal generated by $H \cup \{v^H : v \in S\}$ is denoted by $I(H, S)$.

It was shown in [9] that the graded ideals of $L_K(E)$ (i.e. elements of \mathcal{L}_{gr}) are precisely the ideals of the form $I(H, S)$ for some admissible pair (H, S) . By [1, Proposition 2.5.4], for $(H_1, S_1), (H_2, S_2) \in \mathbf{H}$, we have

$$(H_1, S_1) \leq' (H_2, S_2) \iff I(H_1, S_1) \subseteq I(H_2, S_2).$$

In particular, \subseteq is a partial order on \mathcal{L}_{gr} .

Moreover, $L_K(E)/I(H, S) \cong L_K(E \setminus (H, S))$. Here $E \setminus (H, S)$ is the *quotient graph of E* in which $(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{v' : v \in B_H \setminus S\}$ and $(E \setminus (H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \setminus S\}$ and r, s are extended to $(E \setminus (H, S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$. The necessary and sufficient condition for all ideals of $L_K(E)$ to be graded is that the graph E satisfies condition (K) [9, Theorem 6.16]. On the other hand, a description of nongraded ideals of $L_K(E)$ is given in [8].

A useful observation is that every element a of $L_K(E)$ can be written as $a = \sum_{i=1}^n k_i \alpha_i \beta_i^*$, where $k_i \in K$, α_i, β_i are paths in E and n is a suitable integer. Moreover, $L_K(E) = \bigoplus_{v \in E^0} L_K(E)v = \bigoplus_{v \in E^0} vL_K(E)$. Another useful fact is that if $p^*q \neq 0$, where p, q are paths, then either $p = qr$ or $q = ps$ where r, s are suitable paths in E .

2. Maximal ideals versus prime ideals

The ring of our concern in this sequel, namely the Leavitt path algebra, is unital if E^0 is finite, and nonunital otherwise.

In a unital ring, any maximal ideal is also a prime ideal. This is not necessarily true for a nonunital ring, as the following example shows. For completeness of the argument, we will cite the discussion from [7, pp. 86–87] on the maximal and the prime ideals in nonunital Leavitt path algebras.

Example 2.1 Take the nonunital ring $2\mathbb{Z}$ and its ideal $4\mathbb{Z}$; clearly, $4\mathbb{Z}$ is a maximal ideal but not a prime ideal in $2\mathbb{Z}$.

However, there are nonunital rings where any maximal ideal is also a prime ideal, and a Leavitt path algebra is an example of such.

Proposition 2.2 *If R is a ring satisfying $R^2 = R$, then any maximal ideal is a prime ideal in R .*

Proof Suppose $R^2 = R$, and let M be a maximal ideal of R such that $A \not\subseteq M$ and $B \not\subseteq M$ for some ideals A, B of R . Then $R = R^2 = (M + A)(M + B) = M^2 + AM + MB + AB \subseteq M + AB$. Then $M + AB = R$, and $AB \not\subseteq M$. Thus, M is a prime ideal. \square

As any Leavitt path algebra L is a ring with local units, $L^2 = L$ is satisfied, and the immediate corollary follows.

Corollary 2.3 *In a Leavitt path algebra, any maximal ideal is a prime ideal.*

Hence, in order to study the maximal ideals of a Leavitt path algebra L , we will turn our attention to the prime spectrum of L . Prime ideals of Leavitt path algebras were studied in [7] and the characterization of both graded and nongraded prime ideals was given. We state the result that we will frequently use throughout this paper.

Theorem 2.4 ([7, Theorem 3.12]) *Let E be a graph and P be an ideal of L with $P \cap E^0 = H$. Then P is a prime ideal if and only if P satisfies one of the following conditions:*

- (i) $P = I(H, B_H)$ where $E^0 \setminus H$ satisfies the MT-3 condition;
- (ii) $P = I(H, S)$ where $S = B_H - \{u\}$ for some $u \in B_H$ and $E^0 \setminus H = M(u)$;
- (iii) $P = I(H, B_H) + \langle f(c) \rangle$ where c is a cycle without K in E based at a vertex u , $E^0 \setminus H = M(u)$, and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.

Recall that for any ideal N of L , $gr(N)$ denotes the largest graded ideal contained in N . It was proved in [7, Lemma 3.6] that $gr(N)$ is the ideal generated by the admissible pair (H, S) where $H = N \cap E^0$, and $S = \{v \in B_H \mid v^H \in N\}$. Note that if a maximal element N in \mathcal{L} is a graded ideal, then clearly $N = gr(N)$

is also a maximal element in \mathcal{L}_{gr} . We start our discussion by showing that if a maximal element N in \mathcal{L} is nongraded, then $gr(N)$ is a maximal element in \mathcal{L}_{gr} .

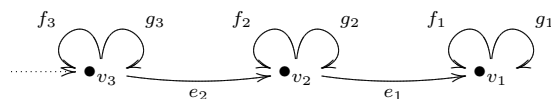
Theorem 2.5 *If M is a maximal ideal of L that is nongraded, then $gr(M)$ is maximal among the graded ideals of L .*

Proof Now, M is a prime ideal, and so we can write $M = I(H, B_H) + \langle p(c) \rangle$ where $H = M \cap E^0$, c is a cycle without \mathbf{K} in E bases at a vertex u , $p(x) \in K[x, x^{-1}]$ is an irreducible polynomial, and $E^0 \setminus H = M(u)$ is therefore downward directed. If $N = I(H', S')$ is a graded ideal such that $N \supsetneq I(H, B_H) = gr(M)$, then necessarily $H' \supsetneq H$ and so there is a vertex $v \in H' \setminus H$. Since $v \geq u$, c and hence $p(c) \in N$. This means $M \subsetneq N$ (note $M \neq N$ as M is nongraded) and, by maximality of M , $N = L$. Thus, $gr(M)$ is maximal among the graded ideals of L . \square

3. Existence of maximal ideals

Recall that in a unital ring, it is always true that any ideal is contained in a maximal ideal. Hence, maximal ideals always exist in unital rings. However, this is not necessarily true in a nonunital ring. In this section we are going to study when maximal ideals exist and also the conditions on the graph E for which every ideal of L is contained in a maximal ideal. We first state an example of a Leavitt path algebra having no maximal ideals. This example is given in [7, Example p.87], [6, Example 2.8].

Example 3.1 Let E be a graph with $E^0 = \{v_i : i = 1, 2, \dots\}$. For each i , there is an edge e_i with $r(e_i) = v_i$ and $s(e_i) = v_{i+1}$ and at each v_i there are two loops f_i, g_i so that $v_i = s(f_i) = r(f_i) = s(g_i) = r(g_i)$. Thus, E is the following graph:



Clearly, E is a row-finite graph and the nonempty proper hereditary saturated subsets of vertices in E are the sets $H_n = \{v_1, \dots, v_n\}$ for some $n \geq 1$ and form an infinite chain under set inclusion. Graph E satisfies condition (K), so all ideals are graded, generated by H_n for some n , and they form a chain under set inclusion. As the chain of ideals does not terminate, $L_K(E)$ does not contain any maximal ideals. Note also that $E^0 \setminus (H_n, \emptyset)$ is downward directed for each n ; thus, all ideals of L are prime ideals.

The necessary and sufficient conditions for the existence of a maximal ideal in a Leavitt path algebra are given in the following lemmas. Moreover, descriptions of the maximal ideals in both graded and nongraded cases are stated.

Lemma 3.2 *Let E be any graph and $H \in \mathcal{H}_E$. Then $I(H, B_H)$ is a maximal ideal in $L_K(E)$ if and only if H is a maximal element in \mathcal{H}_E and the quotient graph $E \setminus (H, B_H)$ has Condition (L).*

Proof Assume $I(H, B_H)$ is a maximal ideal in $L_K(E)$. Notice that $L/I(H, B_H) \cong L(F)$ is simple where $F = E \setminus (H, B_H)$. By [3, Theorem 3.1], [9, Theorem 6.18] the graph F has Condition (L) and the only hereditary saturated subsets of F are the trivial ones. Suppose H is not a maximal element in \mathcal{H}_E . There exists H' in \mathcal{H}_E containing H . Then $I(H', B_{H'})/I(H, B_H)$ is a proper ideal of $L/I(H, B_H)$, which contradicts the simplicity.

Conversely, take a maximal element $H \in \mathcal{H}_E$ such that $E \setminus (H, B_H)$ has Condition (L). Suppose that there exists a proper ideal N containing $I(H, B_H)$. Consider the graded part of N , $gr(N) = I(H_1, S_1)$ with $H_1 \in \mathcal{H}_E$ and $S_1 \subseteq B_{H_1}$. Hence,

$$I(H, B_H) \subseteq gr(N) = I(H_1, S_1) \subseteq N.$$

By [1, Proposition 2.5.4], $(H, B_H) \leq' (H_1, S_1)$. Therefore, $H \subset H_1$ and $B_H \subseteq S_1 \cup H_1 \subseteq B_{H_1} \cup H_1$. Since H is maximal in \mathcal{H}_E , it follows that $H = H_1$ and so $B_H \subseteq S_1 \cup H \subseteq B_H \cup H$ implies $S_1 = B_H$. Hence, $I(H, B_H) = gr(N)$. On the other hand, by [8, Theorem 4], N is of the form

$$H \cup \{v^H \mid v \in S \subseteq B_H\} \cup Y$$

where Y is a set of mutually orthogonal elements of the form $(u + \sum_{i=1}^n k_i g^{r_i})$ in which g is a (unique) cycle with no exits in $E^0 \setminus H$ based at a vertex u in $E^0 \setminus H$ and $k_i \in K$ with at least one $k_i \neq 0$. As $E \setminus (H, B_H)$ has Condition (L), $Y = \emptyset$ and $I(H, B_H) = N$. Hence, $I(H, B_H)$ is a maximal ideal in $L_K(E)$. \square

For the existence of a maximal nongraded ideal, \mathcal{H}_E needs to have a maximal element whose quotient graph does not satisfy condition (L).

Lemma 3.3 *For any graph E , H is a maximal element in \mathcal{H}_E with $E \setminus (H, B_H)$ not satisfying Condition (L), if and only if there is a maximal nongraded ideal M containing $I(H, B_H)$ with $H = M \cap E^0$.*

Proof Take a maximal element H in \mathcal{H}_E with $E \setminus (H, B_H)$ containing a cycle g with no exit and based at a vertex u in $E^0 \setminus H$. Then construct the ideal N generated by

$$H \cup \{v^H \mid v \in B_H\} \cup Y$$

where Y is a set of mutually orthogonal elements of the form $(u + \sum_{i=1}^n k_i g^{r_i})$ in which $k_i \in K$ with at least one $k_i \neq 0$.

Now we want to show that N is not contained in a maximal graded ideal M , since, otherwise, M is of the form (H', S') where H' is a hereditary saturated subset of E^0 and S' is a subset of B_H . Since every maximal ideal in a Leavitt path algebra is prime, M is a graded prime ideal. By [7, Theorem 3.12], $M = I(H', S')$ with either $S' = B_{H'}$ and $E^0 \setminus H'$ is MT-3 or $S' = B_{H'} - \{u\}$ with $E^0 \setminus H' = M(u)$ for some $u \in B_{H'}$. However, notice that if $S' = B_{H'} - \{u\}$ for some $u \in B_{H'}$ then M cannot be a maximal ideal as it is contained in the proper ideal generated by $(H', B_{H'})$. Thus, $M = I(H', B_{H'})$. Now, by Lemma 3.2, H' is a maximal element in \mathcal{H}_E and $E \setminus (H', B_{H'})$ has Condition (L). This contradicts the fact that H is a maximal element and $I(H, B_H) = gr(N) \subset N \subsetneq I(H', B_{H'}) = M$.

If N is maximal, then we are done. If not, then there exists a proper ideal N_1 containing $N_0 := N$ properly. If N_1 is a maximal ideal, then it is a nongraded ideal and we are done. Otherwise, continuing in this manner, either there is a maximal nongraded ideal N_i containing N_{i-1} or else there is an infinite chain of proper ideals $\{N_i\}_{i \in \mathbb{N}}$, with $N_0 = N$, $N_i \subset N_{i+1}$. The graded part of N_i satisfies $gr(N_i) \subseteq N_i \subsetneq N_{i+1}$ and hence $gr(N_i) \subseteq gr(N_{i+1})$ for all i . By the maximality of H , we conclude that $I(H, B_H) = gr(N) = gr(N_i)$ for all i , and N_i is generated by $I(H, B_H) \cup \langle f_i(g) \rangle$ for some polynomial $f_i(x) \in K[x]$. This yields to a sequence of polynomials $f_i(x)$, with $f_0 = f$ and $f_{i+1} | f_i$. As there are only finitely many factors of $(1 + \sum_{i=1}^n k_i x^{r_i})$, the

sequence stabilizes at an irreducible polynomial f that divides $(1 + \sum_{i=1}^n k_i x^{r_i})$. Hence, $I(H, B_H) \cup \langle f(g) \rangle$ is a maximal nongraded ideal.

Conversely, assume M is a maximal nongraded ideal. By [7, Theorem 3.12], $M = I(H, B_H) + \langle p(c) \rangle$ where c is a cycle without \mathbf{K} in E based at a vertex u , $E^0 \setminus H = M(u)$, and $p(x)$ is an irreducible polynomial in $K[x, x^{-1}]$. By Theorem 2.5, $I(H, B_H)$ is a maximal element in \mathcal{L}_{gr} . If $H \subseteq H'$, then $I(H', B_{H'})$ contains $I(H, B_H)$, by maximality in \mathcal{L}_{gr} , $I(H', B_{H'}) = I(H, B_H)$. Hence, H is a maximal element in \mathcal{H}_E . Now, clearly $E \setminus (H, B_H)$ cannot satisfy Condition (L); otherwise, by Lemma 3.2, $I(H, B_H)$ will be a maximal ideal of $L_K(E)$. \square

From Lemma 3.3 and Lemma 3.2, we deduce that there is a maximal element in \mathcal{H}_E if and only if there exists a maximal ideal in L . We record this result as a theorem below. Note that the graph in Example 3.1 did not have any maximal elements in \mathcal{H}_E and hence L did not have any maximal ideals.

Theorem 3.4 (*Existence Theorem*) $L_K(E)$ has a maximal ideal if and only if \mathcal{H}_E has a maximal element.

Proof By Lemma 3.3 and Lemma 3.2, the result follows. \square

Now we prove the main result that states the condition when every ideal of a Leavitt path algebra is contained in a maximal ideal.

Theorem 3.5 Assume that for every element $X \in \mathcal{H}_E$ there exists a maximal element $Z \in \mathcal{H}_E$ such that $X \subseteq Z$, if and only if every ideal in $L_K(E)$ is contained in a maximal ideal.

Proof Let N be an ideal; then N is generated by $I(H, B_H) \cup Y$ where Y is a set of mutually orthogonal elements of the form $(u + \sum_{i=1}^n k_i g^{r_i})$ in which there is a cycle g with no exits in $E^0 \setminus H$ based at a vertex u in $E^0 \setminus H$ and $k_i \in K$ with at least one $k_i \neq 0$. By hypothesis, there is a maximal element H' containing H .

If $E \setminus (H', B_{H'})$ satisfies Condition (L), by Lemma 3.2, $I(H', B_{H'})$ is a graded maximal ideal containing N . If $E \setminus (H', B_{H'})$ does not satisfies Condition (L), by Lemma 3.3, there is a nongraded maximal ideal containing $I(H', B_{H'})$ containing N .

Conversely, take any $X \in \mathcal{H}_E$, and form the ideal $I(X, B_X)$. By assumption, $I(X, B_X)$ is contained in a maximal ideal M . Moreover, $I(X, B_X) \subseteq gr(M)$. Then by Lemma 3.2 and Lemma 3.3, $gr(M) = I(Z, B_Z)$ for some maximal element Z of \mathcal{H}_E . Hence, $X \subset Z$ and the desired result is achieved. \square

Now we prove that a unique maximal element in \mathcal{L} has to be a graded ideal. Hence, the unique maximal element in \mathcal{L}_{gr} .

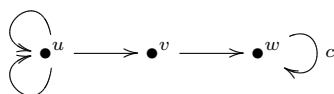
Theorem 3.6 If L has a unique maximal ideal M , then M must be a graded ideal.

Proof Clearly, M is a prime ideal. Assume, by the way of contradiction, that M is a nongraded ideal with $M \cap E^0 = H$. Then, by [7, Theorem 3.12], $M = I(H, B_H) + \langle p(c) \rangle$ where c is a cycle without \mathbf{K} in E based at a vertex u , $E^0 \setminus H = M(u)$, and $p(x)$ is an irreducible polynomial in $K[x, x^{-1}]$. By Theorem 2.5, $I(H, B_H)$ is a maximal graded ideal of L , so $L/I(H, B_H) \cong L_K(E \setminus (H, B_H))$ has no proper graded maximal ideal of L . Thus, $(E \setminus (H, B_H))^0 = E^0 \setminus H$ has no proper nonempty hereditary saturated subsets and so no proper ideal of $L_K(E \setminus (H, B_H))$ contains any vertices. Moreover, c is a cycle without exits based at a vertex u in $E^0 \setminus H$

and $v \geq u$ for all $v \in E^0 \setminus H$. The last condition ($E^0 \setminus H = M(u)$) implies that c is the only cycle without exits in $E^0 \setminus H$. Using Lemma 3.5 of [7], we then conclude that every proper ideal of $L/I(H, B_H)$ is an ideal of the form $\langle f(c) \rangle$ where $f(x) \in K[x]$. Therefore, if $q(x)$ is an irreducible polynomial in $K[x]$ different from $p(x)$, then $\langle q(c) \rangle$ will be a maximal ideal of $L/I(H, B_H)$ different from $\langle p(c) \rangle$. Then $Q = I(H, B_H) + \langle q(c) \rangle$ will be a maximal ideal of L not equal to M , contradicting the uniqueness of M . Hence, M must be a graded ideal. \square

Let us illustrate the above theorem with different graphs. We start with a finite graph whose corresponding Leavitt path algebra has infinitely many ideals but only a unique maximal ideal.

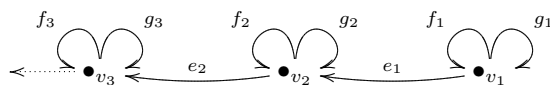
Example 3.7 Let E be the following graph:



Then E does not satisfy Condition (K), so the Leavitt path algebra on E has both graded and nongraded ideals. Let Q be the graded ideal generated by the hereditary saturated set $H = \{v, w\}$. Q is a maximal ideal as L/Q is isomorphic to the simple Leavitt algebra $L(1, 2)$. By using [7, Theorem 3.12], we classify the prime ideals in L . There are infinitely many nongraded prime ideals, each generated by $f(c)$ where $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$, which are all contained in Q . Also, the trivial ideal $\{0\}$ is prime as E satisfies condition MT-3. The prime spectrum of L has a unique maximal element Q .

We now give an example of a graph with infinitely many hereditary saturated sets and the corresponding Leavitt path algebra has a unique maximal ideal that is graded.

Example 3.8 Let E be a graph with $E^0 = \{v_i : i = 1, 2, \dots\}$. For each i , there is an edge e_i with $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$, and at each v_i there are two loops f_i, g_i so that $v_i = s(f_i) = r(f_i) = s(g_i) = r(g_i)$. Thus E is the following graph:



Now E is a row-finite graph and the nonempty proper hereditary saturated subsets of vertices in E are the sets $H_n = \{v_n, v_{n+1}, \dots\}$ for some $n \geq 2$ and $H_{n+1} \subsetneq H_n$ form an infinite chain under set inclusion and $H_2 = \{v_2, v_3, \dots\}$ is the maximal element in \mathcal{H}_E . The graph E satisfies Condition (K), so all ideals are graded, generated by H_n for some n . Thus, $L_K(E)$ contains a unique maximal ideal $I(H_2)$. Note also that $E^0 \setminus H_n$ is downward directed for each n and thus all ideals of L are prime ideals.

We pause here to note an obvious corollary, which is easily deducible.

Corollary 3.9 If $I(H, B_H)$ is a maximal ideal in $L_K(E)$, then $E^0 \setminus H$ satisfies condition MT-3.

Proof By Corollary 2.3 and Theorem 2.4, the result follows. \square

The converse of this corollary is not true. Considering Example 3.8, and $H_n = \{v_n, v_{n+1}, \dots\}$ for $n \geq 3$, $E^0 \setminus H_n = \{v_1, v_2, \dots, v_{n-1}\}$ satisfies the MT-3 condition. (In fact, $E^0 \setminus H_n = M(v_{n-1})$.) However, $I(H_n)$ for $n \geq 3$ is not a maximal ideal.

We finish this paper with a discussion on the necessary and sufficient conditions for all maximal ideals to be graded. In [7, Corollary 3.13] it was proved that every prime ideal is graded if and only if the graph E satisfies Condition (K). A natural follow-up question is to determine the conditions on the graph so that every maximal ideal is graded. Example 3.1 shows that Condition (K) is not sufficient to assure the existence of any maximal ideals. Moreover, Condition (K) on the graph is not even necessary, as Example 3.7 shows. We are ready to state the necessary and sufficient conditions on the graph to assure that all maximal ideals are graded.

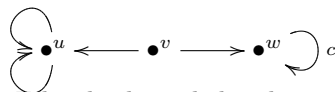
Theorem 3.10 *Let E be any graph. Then every maximal ideal is graded in $L_K(E)$ if and only if for every maximal element H in \mathcal{H}_E , $E \setminus (H, B_H)$ satisfies Condition (L).*

Proof (\Leftarrow) Suppose that, for every maximal element H in \mathcal{H}_E , $E \setminus (H, B_H)$ satisfies Condition (L). Then, by Lemma 3.2, $I(H, B_H)$ is a maximal ideal. However, if there exists a maximal nongraded ideal N , then by Lemma 3.3, $H := M \cap E^0$ is a maximal element in \mathcal{H}_E with $E \setminus (H, B_H)$ not satisfying Condition (L), which contradicts the hypothesis.

(\Rightarrow) Take a maximal element H in \mathcal{H}_E with $E \setminus (H, B_H)$ containing a cycle g with no exit. Then by Lemma 3.3 there exists a nongraded maximal ideal. This gives the required contradiction. \square

We give an example that has both graded and nongraded maximal ideals to show that the converse of Theorem 3.6 is false.

Example 3.11 Let E be the following graph:



Then the Leavitt path algebra on E has both graded and nongraded maximal ideals. The set \mathcal{H}_E is finite and hence any ideal is contained in a maximal ideal. The trivial ideal $\{0\}$, which is a graded ideal generated by the empty set, is not prime as E does not satisfy condition MT-3. There are infinitely many nongraded prime ideals, each generated by $f(c)$ where $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$, which all contain $\{0\}$. Let N be the graded ideal generated by the hereditary saturated set $H = \{u\}$ and, in this case, the quotient graph $E \setminus H$ does not satisfy condition (L). By Lemma 3.3, there are infinitely many maximal nongraded ideals, each generated by $f(c)$ where $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$, which all contain N . Also, let Q be the graded ideal generated by the hereditary saturated set $H = \{w\}$. In this case, the quotient graph $E \setminus H$ satisfies condition (L). By Lemma 3.2, Q is a maximal ideal. The prime spectrum of L has infinitely many maximal ideals; one of them is graded, namely Q , and infinitely many are nongraded ideals whose graded part is N .

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