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Chordality of graphs associated to commutative rings

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Abstract: We investigate when different graphs associated to commutative rings are chordal. In particular, we characterize commutative rings R with each of the following conditions: the total graph of R is chordal; the total dot product or the zero-divisor dot product graph of R is chordal; the comaximal graph of R is chordal; R is semilocal; and the unit graph or the Jacobson graph of R is chordal. Moreover, we state an equivalent condition for the chordality of the zero-divisor graph of an indecomposable ring and classify decomposable rings that have a chordal zero-divisor graph.

Key words: Chordal graph, zero-divisor graph, total graph, Jacobson graph, unit graph, comaximal graph, dot product graphs

1. Introduction

In this paper all rings are commutative with identity and R denotes a ring. Recently many researchers have tried to study the algebraic structure of R by associating some graphs to R, such as zero-divisor graphs, total graphs, or unit graphs; see [1–4, 6, 9–11, 16–19]. The interrelation of graph theoretic properties of these graphs and the algebraic structure of R has been the focus of research on this topic. In particular, many have tried to find graph theoretic invariants of these graphs, such as diameter, girth, and chromatic number, from the algebraic structure of R. Some have also investigated when these graphs have some specific graph theoretic properties, such as being connected, bipartite, or Eulerian.

On the other hand, some algebraic properties and invariants of R can be found from these graphs. For example, it is proved that if R and S are two finite *reduced rings* (that is, without any nilpotent elements), then R and S are isomorphic if and only if their zero-divisor graphs are isomorphic [2, Theorem 4.1]. Also, [3, Theorem 1.1] shows that the number of minimal prime ideals of a Noetherian reduced ring can be deduced from its compressed zero-divisor graph. Another interesting result is [11, Theorem 5.1], which states that an atomic integral domain R is a unique factorization domain if and only if for each nonzero nonunit $x \in R$ the irreducible divisor graph of x is connected.

In this paper, we try to characterize when some of these graphs associated to R are chordal. A simple graph G is called *chordal* when it has no induced cycle with length greater than 3. In other words, if C is a cycle in G with length greater than 3, then there is an edge of G not in C that connects two vertices of C (this edge is called a *chord* of C). Chordal graphs are long and well studied in graph theory and have nice properties and many applications in optimization and computation; see, for example, [13, 14, 20].

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Chordal graphs also play an important role in combinatorial commutative algebra. Assume that G is a finite simple graph with vertex set $\{1,\dots,n\}$ and edge set E. Let K be a field and $S = K[x_1,\dots,x_n]$. The ideal $I(G)$ of S generated by $\{x_ix_j | \{i,j\} \in E\}$ is called the *edge ideal* of G. By a theorem of Fröberg [12], the minimal free resolution of this ideal is linear if and only if $\overline{G} =$ the complement of G is a chordal graph. Also, if G is chordal, there are known combinatorial conditions equivalent to Cohen–Macaulayness of $S/I(G)$. To see these conditions and more on edge ideals of chordal graphs, see [15, Chapter 9].

Here, in Section 2, we investigate the chordality of some of the graphs that are constructed based on the structure of maximal ideals of a ring. In particular, we characterize all rings that have a chordal comaximal graph and also rings that are semilocal (for example, finite rings) and have chordal Jacobson or unit graphs.

Then, in Section 3, we pay attention to graphs that are constructed from the zero-divisor structure of a ring. Particularly, we classify all rings with chordal total graphs, chordal total dot product graphs, or chordal zero-divisor dot product graphs. We also characterize reduced or decomposable rings that have a chordal zero-divisor graph and state a condition equivalent to chordality of the zero-divisor graph of an indecomposable ring, which reduces the classification of chordal zero-divisor graphs to that of chordal compressed zero-divisor graphs.

All graphs considered in this paper are undirected and do not have multiple edges, but some of the graphs have loops. If G is a graph with some loops, then by saying that G is chordal, we mean that the simple graph obtained by deleting all loops from G is chordal. Also, for convenience, we assume that a graph with no vertices is chordal. We denote the set of vertices of G with $\mathrm{V}(G)$, and if $x, y \in \mathrm{V}(G)$ are adjacent, we write $x \sim y$. Moreover, if $V \subseteq \mathrm{V}(G)$, we denote the subgraph of G with vertex set V, which has all edges of G with endpoints in V by $G[V]$, and call it the *induced subgraph of G on V*. Furthermore, by $\mathrm{U}(R)$, $\mathrm{J}(R)$, $\mathrm{N}(R)$, and $\mathrm{Z}(R)$ we mean the set of units of R, the Jacobson radical of R, the nilradical of R, and the set of zero-divisors of R, respectively. In addition, for any set $A \subseteq R$, we denote $A \setminus \{0\}$ by A^*. Any undefined notation is as in [8] or [21].

2. Chordality of graphs based on maximal ideals

In this section, we investigate chordality of unit graphs, comaximal graphs, and Jacobson graphs of R, which are based on the structure of maximal ideals of R. First we consider the comaximal graph of R. In [18], a graph $G(R)$ is assigned to the ring R, where $\mathrm{V}(G(R)) = R$ and the set of edges of $G(R)$ is $\{\{a,b\} | a,b \in R, Ra + Rb = R\}$. Let $\mathrm{CG}(R) = G(R)[R \setminus (\mathrm{U}(R) \cup \mathrm{J}(R))]$ be the graph obtained by deleting all unit elements and all elements of $\mathrm{J}(R)$ from the vertex set of $G(R)$. Since unit elements are adjacent to all vertices in $G(R)$ and elements in $\mathrm{J}(R)$ are adjacent exactly to the vertices representing the unit elements, and as mentioned in [16], we see that the structure of $G(R)$ is determined by the structure of $\mathrm{CG}(R)$. For example, it is easy to see that $G(R)$ is chordal if and only if $\mathrm{CG}(R)$ is chordal (recall that a graph with no vertices is considered chordal). Consequently, we study $\mathrm{CG}(R)$, and following [16], we call it the *comaximal graph* of R.

Theorem 2.1 *The comaximal graph of R is chordal if and only if R satisfies one of the following:*

(i) *R is a local ring.*

(ii) *$R \cong \mathbb{Z}_2 \times F$, for a field F.*

(iii) *$R \cong \mathbb{Z}_2^3$.*

Proof ($\Rightarrow$): First assume that $R \cong \prod_{i=1}^{4} R_i$. Then we have the chordless cycle $(1,1,0,0) \sim (0,0,1,1) \sim (1,1,1,0) \sim (0,1,1,1) \sim (1,1,0,0)$ with length 4 in $\mathrm{CG}(R)$. From this contradiction, we deduce that the longest decomposition of R as a product of rings has length at most 3. Therefore, R has a decomposition $R = \prod_{i=1}^{n} R_i$ with each R_i an indecomposable ring and $n \leq 3$. Assume that $\mathrm{CG}(R_1)$ is not chordal and $a_1 \sim \cdots \sim a_m \sim a_1$ is a chordless cycle in $\mathrm{CG}(R_1)$ with $m > 3$. Then, setting $R' = \prod_{i=2}^{n} R_i$, $(a_1, 1_{R'}) \sim \cdots \sim (a_m, 1_{R'}) \sim (a_1, 1_{R'})$ is an induced cycle in $\mathrm{CG}(R)$, against its chordality. Thus, every $\mathrm{CG}(R_i)$ is chordal.

Suppose that R_1 is not local and $\mathfrak{M}_1 \neq \mathfrak{M}_2$ are two maximal ideals of R_1. Then as $\mathfrak{M}_1 + \mathfrak{M}_2 = R_1$, there are $m_1 \in \mathfrak{M}_1$ and $m_2 \in \mathfrak{M}_2$ such that $R_1 m_1 + R_1 m_2 = R_1$. Note that $m_1, m_2 \notin \mathrm{J}(R_1) \cup \mathrm{U}(R_1)$ and in particular they are nonzero and not equal to 1_{R_1}. By [8, Proposition 1.16] we see that for each pair of positive integers i, j we have $R_1 m_1^i + R_1 m_2^j = R_1$. If $m_1 \neq m_1^2$ and $m_2 \neq m_2^2$, then we get the following chordless cycle in $\mathrm{CG}(R_1)$: $m_1 \sim m_2 \sim m_1^2 \sim m_2^2 \sim m_1$. Thus, $m_i = m_i^2$ for $i = 1$ or 2, but then m_i is a nontrivial idempotent in R_1 and $R_1 \cong R_1 m_1 \times R_1(1 - m_1)$, contradicting the indecomposability of R_1. Hence, R_1 and similarly every R_i are local rings.

If $n = 1$, then R is local. Thus, assume $1 < n$ and $\mathfrak{M} \neq \mathfrak{M}'$ are two maximal ideals of R. Choose $m \in \mathfrak{M}$ and $m' \in \mathfrak{M}'$ such that $Rm + Rm' = R$. If $0 \neq j \in \mathrm{J}(R)$, then $m \sim m' \sim m + j \sim m' + j \sim m$ is an induced cycle with length 4 in $\mathrm{CG}(R)$, a contradiction. Therefore, $\mathrm{J}(R) = 0$ and hence $\mathrm{J}(R_i) = 0$ for each i. Since each R_i is a local ring, this means that each R_i is in fact a field.

Suppose that we can find $1 \leq i \neq j \leq n$ such that $2 < |R_i|, |R_j|$, say $i = 1, j = 2$. Then there are $0, 1 \neq a \in R_1$ and $0, 1 \neq b \in R_2$. Now the following induced cycle in $\mathrm{CG}(R)$ shows that $\mathrm{CG}(R)$ is not chordal (if $n = 2$, drop the last component in each vertex): $(a, 0, 1) \sim (0, 1, 1) \sim (1, 0, 1) \sim (0, b, 1) \sim (a, 0, 1)$. We deduce that each R_i is isomorphic to $\mathbb{Z}_2$ except for possibly one i, that is, either $R \cong \mathbb{Z}_2 \times F$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times F$ where F is a field. To complete the proof, we just need to show that in the latter case, $F \cong \mathbb{Z}_2$. Suppose not and $0, 1 \neq a \in F$. Then the chordless cycle $(1, 0, a) \sim (0, 1, a) \sim (1, 0, 1) \sim (0, 1, 1) \sim (1, 0, a)$ provides the required contradiction, and the result is established.

($\Leftarrow$): Easy. $\square$

Next we consider the unit graph of R. The *unit graph* of R is the graph with vertex set R in which distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in \mathrm{U}(R)$ ([6]). We denote this graph by $\mathrm{UG}(R)$. It should be mentioned that $\mathrm{UG}(R)$ is a subgraph of $G(R)$ defined above, though not an induced subgraph. Thus, it is possible that $G(R)$ is chordal but $\mathrm{UG}(R)$ is not. In fact, this is the case for any local ring R that is not a field, according to 2.1 and 2.4. To classify chordal unit graphs we need the following lemmas.

Lemma 2.2 *Assume that F is a field with $|F| > 5$ or $|F| = 4$; then there are distinct nonzero elements $a, b, c \in F$ such that $a \neq -b, -c$ and $b \neq -c$.*

Proof Easy and left to the reader. $\square$

Lemma 2.3 *If $\mathrm{UG}(R)$ is a chordal graph, then $\mathrm{J}(R) = 0$.*

Proof Else if $0 \neq j \in \mathrm{J}(R)$, then the following is a cycle of length > 3 in $\mathrm{UG}(R)$: $0 \sim 1 \sim j \sim 1 + j \sim 0$. By chordality, this cycle should have a chord and hence we have $1 \sim 1 + j$, that is, $2 + j \in \mathrm{U}(R)$ or equivalently $2 \in \mathrm{U}(R)$. However, then $1 \neq -1$ and the cycle $0 \sim 1 \sim j \sim -1 \sim 0$ has no chords, a contradiction. $\square$

Recall that a *semilocal ring* means a ring with finitely many maximal ideals.

Theorem 2.4 *Suppose that $\overline{R} = R/\mathrm{J}(R)$ is a product of fields (for example, if R is finite or, more generally, semilocal). Then $\mathrm{UG}(R)$ is chordal if and only if R is isomorphic to one of the following:*

(i) $(\mathbb{Z}_2)^I$ *for some index set I;*

(ii) *a field with characteristic 2;*

(iii) $\mathbb{Z}_3$.

Proof ($\Rightarrow$): According to 2.3 we can assume that $R = \overline{R} = \prod_{i\in I} F_i$, where each F_i is a field and I is an index set. Suppose that $|I| > 1$. Assume $F_i \not\cong \mathbb{Z}_2$ for some i, say $i = 1$, and set $R' = \prod_{1\neq i\in I} F_i$. If $|F_1| = 4$ or $|F_1| > 5$ and a, b, c are elements of F_1 provided by 2.2, then the cycle $(a, 0_{R'}) \sim (b, 1_{R'}) \sim 0_R \sim (c, -1_{R'}) \sim (a, 0_{R'})$ is a chordless cycle of length > 3 in $\mathrm{UG}(R)$. If $F_1 \cong \mathbb{Z}_5$, then $(1, 1_{R'}) \sim (2, 0_{R'}) \sim (4, -1_{R'}) \sim (3, 0_{R'}) \sim (1, 1_{R'})$ is such a cycle. If $F_1 \cong \mathbb{Z}_3$, then we have the following chordless cycle: $0_R \sim 1_R \sim (1, 0_{R'}) \sim (0, -1_{R'}) \sim (2, 0_{R'}) \sim (2, 1_{R'}) \sim 0_R$. Thus, in all cases we get a contradiction and it follows that if $|I| > 1$, then $R \cong (\mathbb{Z}_2)^I$.

Now assume that $|I| = 1$ and R is a field with characteristic $\neq 2$. If $|R| \geq 5$, then there is an $a \in R$ with $a \neq 0, -1, 1$, but then $1 \sim a \sim -1 \sim -a \sim 1$ is an induced cycle of length > 3, against chordality. Thus, we should have $R \cong \mathbb{Z}_3$, as claimed.

($\Leftarrow$): Just note that $\mathrm{UG}(\mathbb{Z}_2^I)$ is a *matching* (that is, a set of edges, no two of which have a common endpoint) and the unit graph of a field with characteristic 2 is complete. $\square$

It is easy to see that rings such as $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}_2$ also have chordal unit graphs. Thus, the assumption on $\overline{R}$ in 2.4 is necessary. The last graph that we study in this section is the Jacobson graph of R defined in [9]. The *Jacobson graph* $\mathrm{JG}(R)$ of R has vertex set $R \setminus \mathrm{J}(R)$ and two (not necessarily distinct) vertices x, y are adjacent when $1 - xy \notin \mathrm{U}(R)$. Note that in the definition of Azimi et al. [9], adjacency was defined only for distinct vertices and loops were not allowed. We allow loops so that we can have the following lemma. It should be noted that in the following lemma, if $x = y$, then the result follows from parts (2) and (3) of [9, Lemma 2.1], and if $x \neq y$, then ($\Leftarrow$) is proved in part (1) of that lemma. We give a simple proof for completeness.

Lemma 2.5 *Suppose that $\overline{R} = R/\mathrm{J}(R)$ and $\overline{x}$ denotes the image in $\overline{R}$ of an $x \in R$. Then for all $x, y \in \mathrm{V}(\mathrm{JG}(R))$, we have $x \sim y$ if and only if $\overline{x} \sim \overline{y}$ in $\mathrm{JG}(\overline{R})$.*

Proof We have $x \sim y$ if and only if there is a maximal ideal $\mathfrak{M}$ of R such that $1 - xy \in \mathfrak{M}$ if and only if there exists a maximal ideal $\mathfrak{M}$ of R such that $1 - \bar{x}\bar{y} \in \overline{\mathfrak{M}}$ if and only if $\overline{x} \sim \overline{y}$. $\square$

This lemma shows that we can construct $\mathrm{JG}(R)$ form $\mathrm{JG}(\overline{R})$ and $|\mathrm{J}(R)|$, where $\overline{R} = R/\mathrm{J}(R)$. Indeed, we put $|\mathrm{J}(R)|$ vertices instead of each vertex of $\mathrm{JG}(\overline{R})$ and then draw the edges according to the above lemma. Using this, we get the following theorem on the chordality of $\mathrm{JG}(R)$.

Theorem 2.6 *Suppose that $\overline{R} = R/\mathrm{J}(R)$ is a product of fields (for example, if R is finite or, more generally, semilocal). Then $\mathrm{JG}(R)$ is chordal if and only if either R is a field or $\overline{R}$ is isomorphic to one of the following:* $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof ($\Rightarrow$): If V_1 is an irredundant set of representatives of cosets of $\mathrm{J}(R)$ in R (that is, V_1 contains exactly one element from each coset $x + \mathrm{J}(R)$), then by 2.5, we see that the induced graph $\mathrm{JG}(R)[V_1]$ is isomorphic to $\mathrm{JG}(\overline{R})$. Thus, $\mathrm{JG}(\overline{R})$ is chordal. Assume that $\overline{R} = \prod_{i\in I} F_i$. If $|I| > 3$, then we can assume that $\{1,2,3,4\} \subseteq I$. Set $R' = \prod_{i\in I\setminus\{1,2,3,4\}} F_i$. Then, viewing $\overline{R}$ as the product $\prod_{i=1}^4 F_i \times R'$, the following is a cycle in $\mathrm{JG}(\overline{R})$ that has no chords: $(1,1,0,0,0) \sim (0,1,1,0,0) \sim (0,0,1,1,0) \sim (1,0,0,1,0) \sim (1,1,0,0,0)$. This contradicts the chordality of $\mathrm{JG}(\overline{R})$; hence, we must have $n = |I| \le 3$.

Suppose that $n = 1$ (that is, R is local) and R is not a field. Then there exists a $0 \neq j \in \mathrm{J}(R)$. If $\overline{R} \not\cong \mathbb{Z}_2$ and $\overline{R} \not\cong \mathbb{Z}_3$, then there is an $f \in R\setminus \mathrm{J}(R)$ such that $\overline{1} \neq \overline{f}^2$. Now we get the following cycle in $\mathrm{JG}(R)$ without any chords, a contradiction: $f \sim f^{-1} \sim f + j \sim f^{-1} + j \sim f$. Consequently, if $n = 1$, then either R is a field or $\overline{R}$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Next assume that $n > 1$ and $\overline{R} \cong \prod_{i=1}^n F_i$. If, for example, $|F_1| > 3$, then there is a $0 \neq f \in F_1$ such that $f^2 \neq 1$. Setting $R'' = \prod_{i=2}^n F_i$ and viewing $\overline{R}$ as $F_1 \times R''$, we get the following induced cycle in $\mathrm{JG}(\overline{R})$, a contradiction: $(f,0) \sim (f^{-1},0) \sim (f,1) \sim (f^{-1},1) \sim (f,0)$. From this contradiction we deduce that all F_is are either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. If $\overline{R} \cong \mathbb{Z}_3^2$, then the following cycle shows that $\mathrm{JG}(\overline{R})$ is not chordal: $(1,1) \sim (1,2) \sim (2,2) \sim (2,1) \sim (1,1)$. A similar cycle rules out the cases $\overline{R} \cong \mathbb{Z}_3^3$ or $\mathbb{Z}_3^2 \times \mathbb{Z}_2$ and the cycle $(1,0,1) \sim (1,1,0) \sim (2,1,0) \sim (2,0,1) \sim (1,0,1)$ rules out the case $\overline{R} \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, the only possibilities for $\overline{R}$ are those claimed in the statement of the theorem.

($\Leftarrow$): If R is a field, then $\mathrm{JG}(R)$ is union of some loops and a matching, hence chordal. Therefore, assume that $\overline{R}$ is one of the rings stated in the theorem. By drawing $\mathrm{JG}(\overline{R})$ we see that in all cases $\mathrm{JG}(\overline{R})$ is chordal and all of its vertices have a loop. Suppose that $C:\ x_0 \sim \cdots \sim x_{n-1} \sim x_0$ is a cycle with length > 3 in $\mathrm{JG}(R)$ without any chords. If $\overline{x}_i = \overline{x}_j$ for some $i \neq j \pm 1 \pmod{n}$, then as every vertex of $\mathrm{JG}(\overline{R})$ has a loop and by 2.5, we have that $x_i \sim x_j$ is a chord for C, a contradiction. If $\overline{x}_i = \overline{x}_j$ with $i = j + 1 \pmod{n}$ and if $k = j - 1 \pmod{n}$, then $\overline{x}_i = \overline{x}_j \sim \overline{x}_k$ because $x_j \sim x_k$ and by 2.5 $x_i \sim x_k$, against C being chordless. Consequently, $\overline{x}_i \neq \overline{x}_j$ for each $i \neq j$ and hence, by 2.5, the image of C in $\mathrm{JG}(\overline{R})$ is a chordless cycle, against chordality of $\mathrm{JG}(\overline{R})$. Thus, we conclude that such a C cannot exist and $\mathrm{JG}(R)$ is chordal. □

Again simple examples such as $R = \mathbb{Z}$ show that the assumption of the previous theorem on $\overline{R}$ is necessary.

3. Chordality of graphs based on zero-divisors

In this section, we study chordality of some of the graphs associated to R that are constructed based on the structure of $\mathrm{Z}(R)$, such as the zero-divisor graph, the total graph, or the dot product graph of R. We start with the *total graph* $\mathrm{TG}(R)$ of R, which has R as its vertex sets and distinct vertices x, y are adjacent in $\mathrm{TG}(R)$ if and only if $x + y \in \mathrm{Z}(R)$. This graph was introduced in [1]. Also, in [7], commutative rings whose total graph (or its complement) is in some known classes of graphs are characterized. In particular, they answered the question when $\mathrm{TG}(R)$ is a cycle. We need a lemma in order to classify rings with chordal total graph.

Lemma 3.1 **(i)** *If $x \neq \pm y$ are two elements of R such that $x \sim y$ in $\mathrm{TG}(R)$ and $\mathrm{TG}(R)$ is chordal, then $x \sim -y$ in $\mathrm{TG}(R)$.*

(ii) *If $a \in \mathrm{Z}^*(R)$ and $\mathrm{TG}(R)$ is chordal, then $a - 2b \in \mathrm{Z}(R)$ for all $b \in R$.*

Proof (i): If $x=-x$, then as $x\sim y$, we have $x+(-y)=-(y+x)\in \mathrm{Z}(R)$ and the claim follows. Thus, we can assume $x\neq -x$ and similarly $y\neq -y$. By chordality of $\mathrm{TG}(R)$, the cycle $y\sim x\sim -x\sim -y\sim y$ has a chord. Thus, either $y\sim -x$ or $-y\sim x$ and in both cases (which are indeed equivalent) the result follows.

(ii): Since $a\neq 0$, we have $a-b\neq -b$. If $a-b=b$, then $a-2b=0\in \mathrm{Z}(R)$. Thus, we may assume $a-b\neq \pm b$ and apply (i) with $x=a-b$ and $y=b$ to see that $x\sim -y$ in $\mathrm{TG}(R)$. From this, $a-2b=x-y\in \mathrm{Z}(R)$. □

In what follows $K^{\alpha,\beta}$ denotes the *complete bipartite graph* with partition sizes α and β for cardinal numbers α,β (that is, the graph in which the vertex set can be partitioned into two parts A and B, such that $|A|=\alpha$, $|B|=\beta$, every vertex of A is adjacent to every vertex of B, and no pair of vertices in one part are adjacent).

Theorem 3.2 *The graph* $\mathrm{TG}(R)$ *is chordal if and only if* $\mathrm{Z}(R)$ *is an ideal of* R *and either* $\mathrm{char}\frac{R}{\mathrm{Z}(R)}=2$ *or* $\mathrm{Z}(R)=0$*.*

Proof First we show that if $\mathrm{TG}(R)$ is chordal, then $\mathrm{Z}(R)$ is an ideal. As $Rz\subseteq \mathrm{Z}(R)$ for all $z\in \mathrm{Z}(R)$, we have to show that if $x,y\in \mathrm{Z}(R)$, then $x+y\in \mathrm{Z}(R)$. This holds clearly if $x=0$ or $y=0$ or $x=\pm y$. Thus, assume $x\neq 0\neq y$ and $x\neq \pm y$. By 3.1(ii), it follows that both $x+2y$ and $2x+y$ are zero-divisors. Hence, we have the following cycle of length 4 in $\mathrm{TG}(R)$: $0\sim x\sim x+y\sim y\sim 0$. By chordality of $\mathrm{TG}(R)$, this cycle has a chord and either $x\sim y$ or $0\sim x+y$. In both cases $x+y\in \mathrm{Z}(R)$, and $\mathrm{Z}(R)$ is an ideal of R.

Now, according to [1, Theorem 2.2], if $\overline{R}=R/\mathrm{Z}(R)$ has characteristic 2, then $\mathrm{TG}(R)$ is a disjoint union of complete graphs and hence is chordal. Also by the same theorem, if $\mathrm{char}\overline{R}\neq 2$, then $\mathrm{TG}(R)$ is a disjoint union of some copies of $K^{\alpha,\alpha}$, where $\alpha=|\mathrm{Z}(R)|$. If $\alpha>1$ then this graph has an induced cycle of length 4, so in the case that $\mathrm{char}\overline{R}\neq 2$, $\mathrm{TG}(R)$ is chordal if and only if $\alpha=1$ if and only if $\mathrm{Z}(R)=0$. □

In [10], the dot product graphs were introduced and studied. Let n be a positive integer and consider the dot product $\cdot : R^n\to R$ defined by $(r_i)\cdot(r'_i)=\sum_{i=1}^{n} r_i r'_i$. Construct a graph by letting every nonzero element of R^n be a vertex and joining two vertices $\mathbf{x}$ and $\mathbf{y}$ by an edge when $\mathbf{x}\cdot\mathbf{y}=0$. This graph is called the n*th total dot product graph* of R and we denote it by $\mathrm{TD}_n(R)$. Also, $\mathrm{ZD}_n(R)$ is the induced subgraph of $\mathrm{TD}_n(R)$ on the set $\mathrm{Z}^*(R^n)$ and is called the n*th zero-divisor dot product graph* of R.

In the case that $n=1$, the graph $\mathrm{ZD}_n(R)$ equals the usual *zero-divisor graph* of R with vertex set $\mathrm{Z}^*(R)$, in which, two distinct vertices x and y are adjacent if and only if $xy=0$. Moreover, $\mathrm{TD}_1(R)$ is the disjoint union of $\mathrm{ZD}_1(R)$ and a set of isolated vertices. Thus, the dot product graphs can be viewed as generalizations of the zero-divisor graph. The zero-divisor graph of R is denoted by $\Gamma(R)$. We will investigate chordality of $\Gamma(R)$ later in this section. Here we consider the case $n\geq 2$.

Theorem 3.3 *Let* $n\geq 2$ *be a positive integer.*

- **(i)** $\mathrm{ZD}_n(R)$ *is chordal if and only if* $R=\mathbb{Z}_2$ *and* $n=2$ *or 3.*
- **(ii)** $\mathrm{TD}_n(R)$ *is chordal if and only if* $R=\mathbb{Z}_2$ *and* $n=2$*.*

Proof (i): For ($\Leftarrow$) just draw the graphs and observe that they are chordal. ($\Rightarrow$): If $R\ncong \mathbb{Z}_2$, then we have the following induced cycle of length 4 in $\mathrm{ZD}_n(R)$: $(0,1,0,0,\ldots)\sim(1,0,0,0,\ldots)\sim(0,a,0,0,\ldots)\sim$

$(a,0,0,0,\ldots) \sim (0,1,0,0,\ldots)$, where $0,1 \neq a \in R$. Thus, $R \cong \mathbb{Z}_2$. If $n \geq 4$, then the following cycle shows that $\mathrm{ZD}_n(R)$ is not chordal: $(1,0,0,1) \sim (0,1,1,0) \sim (1,0,0,0) \sim (0,0,1,0) \sim (1,0,0,1)$ (if $n > 4$, add zeros at the end of the vertices of this cycle).

(ii): Note that if $\mathrm{TD}_n(R)$ is chordal, then its induced subgraph $\mathrm{ZD}_n(R)$ is also chordal, so $R \cong \mathbb{Z}_2$ and $n = 2$ or 3, but $\mathrm{TD}_3(\mathbb{Z}_2)$ is not chordal: $(1,1,1) \sim (0,1,1) \sim (1,0,0) \sim (0,0,1) \sim (1,1,0) \sim (1,1,1)$ is an induced cycle of length 5. □

As mentioned above, one can view $\mathrm{ZD}_n(R)$ as a generalization of $\Gamma(R)$ and use it to study the zero-divisor structure of R. However, when $n \geq 2$ the entries of elements in $\mathrm{Z}^*(R^n)$ are not necessarily in $\mathrm{Z}(R)$. Thus, it seems rational to restrict the vertices to those elements of R^n with all entries in $\mathrm{Z}(R)$. We study the subgraphs of $\mathrm{ZD}_n(R)$ induced by $\mathrm{Z}(R)^n \setminus \{(0,\ldots,0)\}$ and $(\mathrm{Z}^*(R))^n$, which are denoted by $\Gamma_n(R)$ and $\Gamma'_n(R)$, respectively. Note that $\Gamma_1(R) = \Gamma'_1(R) = \mathrm{ZD}_1(R) = \Gamma(R)$. To characterize all R and $n \geq 2$ with the property that $\Gamma_n(R)$ or $\Gamma'_n(R)$ is chordal, we need a couple of lemmas.

Lemma 3.4 *Let n and k be two positive integers and assume that $\mathrm{Z}(R) \neq 0$. Then $\Gamma'_n(R)$ is isomorphic to an induced subgraph of $\Gamma_n(R)$ and also an induced subgraph of $\Gamma'_{n+2k}(R)$. Moreover, if R is not reduced, then $\Gamma'_n(R)$ is isomorphic to an induced subgraph of $\Gamma'_m(R)$ for all $m \geq n$.*

Proof By definition, $\Gamma'_n(R)$ is an induced subgraph of $\Gamma_n(R)$. Now let $a \in \mathrm{Z}^*(R)$ and V be the set of vertices $\mathbf{x} = (x_1,\ldots,x_{n+2})$ of $\Gamma'_{n+2}(R)$ with the property that $x_1 = a$ and $x_2 = -a$. If we set $\overline{\mathbf{x}} = (x_3,\ldots,x_{n+2})$ for such an $\mathbf{x}$, then $\mathbf{x} \sim \mathbf{y}$ in $\Gamma'_{n+2}(R)$ if and only if $\overline{\mathbf{x}} \sim \overline{\mathbf{y}}$ in $\Gamma_n(R)$, for all pairs $\mathbf{x},\mathbf{y} \in V$. Therefore, $\Gamma_n(R)$ is isomorphic to $\Gamma'_{n+2}(R)[V]$. By induction, we see that $\Gamma'_n(R)$ is isomorphic to an induced subgraph of $\Gamma'_{n+2k}(R)$. Now assume that R has a nonzero nilpotent element and $m \geq n$. Then there is $0 \neq a \in R$ with $a^2 = 0$. If we set V' to be the set of vertices of $\Gamma'_m(R)$ that have a on their first $m-n$ coordinates, then $\Gamma_n(R)$ is isomorphic to $\Gamma_n(R)[V']$, as required. □

Lemma 3.5 *Let R be a commutative ring.*

- **(i)** *For each $a,b \in \mathrm{Z}^*(R)$ with $a \neq \pm b$ we have $ab = 0$ if and only if either $\mathrm{Z}(R)^2 = 0$ or R is isomorphic to one of the following: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$.*
- **(ii)** *Suppose that R is reduced. For each $a,b \in \mathrm{Z}^*(R)$ with $a \neq \pm b$ we have $ab = 0$ or $\mathrm{Ann}(a)\mathrm{Ann}(b) = 0$ if and only if R is an integral domain or is isomorphic to one of the following: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof (i)($\Leftarrow$) and (ii)($\Leftarrow$): Clear. (i)($\Rightarrow$): Assume that $\mathrm{Z}(R)^2 \neq 0$ and $aa' \neq 0$ for some $a,a' \in \mathrm{Z}(R)$. Then by assumption $0 \neq a = \pm a'$ and hence $a^2 \neq 0$. If $a \neq \pm a^2$, then the hypothesis gives us $a^3 = 0$. On the other hand, $a + a^2 \neq a$ and $a + a^2 \neq 0$. Applying the hypothesis on $a, a+a^2$ we get $a^2 = -a^3 = 0$, a contradiction. Thus, we must have $a = \pm a^2$. If $a = -a^2$, then $(-a)^2 = -a$ and thus by replacing a with $-a$ if necessary, we can assume that $0 \neq a = a^2$. Let $x \in Ra$, say $x = ra$. If $x \neq \pm a$, then $0 = xa = ra^2 = ra = x$. Therefore, $Ra = \{0, a, -a\}$ is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ as a ring. Since a is an idempotent, $R \cong Ra \times R(1-a)$. Note that $1-a$ is also a nonzero idempotent in $\mathrm{Z}(R)$ and by a similar argument $R(1-a)$ is isomorphic to one of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ and the result follows.

(ii)($\Rightarrow$): Assume that there are $a,b \in \mathrm{Z}^*(R)$ such that $a \neq \pm b$ and $ab \neq 0$. If both $a = \pm ab$ and $b = \pm ab$, then $a = \pm b$, against our assumption. We can assume that, for example, $a \neq \pm ab$. Therefore, by hypothesis we should have $a^2b = 0$ or $\mathrm{Ann}(a)\mathrm{Ann}(ab) = 0$. In the former case, $(ab)^2 = 0$, and as R is reduced, $ab = 0$, a contradiction. In the latter case, since $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(ab)$, we get $\mathrm{Ann}(a)^2 = 0$ and hence $\mathrm{Ann}(a) = 0$, again a contradiction. Consequently, for each $a, b \in \mathrm{Z}^*(R)$ with $a \neq \pm b$ we have $ab = 0$, and the claim follows from (i). □

Suppose that $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. We write elements of R and vertices of $\mathrm{TD}_n(R)$ as strings of elements of $\mathbb{Z}_2$ and vectors with entries in R, respectively. For example, $(10, 01) \in \mathrm{V}(\Gamma'_2(R))$ and $(00, 01) \in \mathrm{V}(\Gamma_2(R)) \setminus \mathrm{V}(\Gamma'_2(R))$ and $(10, 01) \cdot (11, 01) = 11 \in R$. We use similar notations for other decomposable rings.

Proposition 3.6 **(i)** *If $R = \mathbb{Z}_2 \times \mathbb{Z}_3$ or $R = \mathbb{Z}_3 \times \mathbb{Z}_3$, then for all $n \geq 2$, neither $\Gamma_n(R)$ is chordal nor $\Gamma'_n(R)$.*

(ii) *If $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then all $\Gamma_n(R)$s and $\Gamma'_n(R)$s are not chordal except for $\Gamma'_2(R)$.*

Proof (i): In $\Gamma'_2(R)$ we have the cycle $(01, 01) \sim (01, 02) \sim (02, 02) \sim (02, 01) \sim (01, 01)$ and $\Gamma'_3(R)$ has the induced cycle $C:$ $(10, 10, 10) \sim (01, 01, 01) \sim (01, 10, 02) \sim (10, 01, (-1)0) \sim (10, 10, 10)$, which show that these graphs are not chordal. Now the claim follows from 3.4.

(ii): It is straightforward to check that $\Gamma_2(R)$ is not chordal but $\Gamma'_2(R)$ is chordal. Also, if we change the only 2 in cycle C of (i) to 1, then we get an induced cycle of length 4 in $\Gamma'_3(R)$. Moreover, $\Gamma'_4(R)$ is not chordal because we have $(01, 10, 10, 10) \sim (10, 01, 01, 01) \sim (01, 01, 10, 01) \sim (10, 10, 01, 10) \sim (01, 10, 10, 10)$. Now the proof is concluded by 3.4. □

Theorem 3.7 *Suppose that R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then the following are equivalent.*

(i) *There exist an $n \geq 2$ such that either $\Gamma_n(R)$ or $\Gamma'_n(R)$ is chordal.*

(ii) $\mathrm{Z}(R)^2 = 0$.

(iii) *For all $n \geq 2$, both $\Gamma_n(R)$ and $\Gamma'_n(R)$ are chordal.*

Proof If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then according to 3.6, all of (i)–(iii) are incorrect. Assume that $R \ncong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_3$. Since (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) is trivial, we give the proof of (i) $\Rightarrow$(ii).

Suppose that (i) holds. According to 3.4, we can assume that $\Gamma'_n(R)$ is chordal for some $n \geq 2$. If R is not reduced, then by 3.4, $\Gamma'_2(R)$ is chordal. If R is reduced, then by the same lemma, either $\Gamma'_2(R)$ is chordal or $\Gamma'_3(R)$ is. First assume that R is reduced and $\Gamma'_3(R)$ is chordal. Let $a, b \in \mathrm{Z}^*(R)$ be such that $a \neq \pm b$ and $ab \neq 0$. We show that $\mathrm{Ann}(a)\mathrm{Ann}(b) = 0$ and then the result, in this case, follows by 3.5(ii).

If $a'b' \neq 0$ for some $a' \in \mathrm{Ann}(a)$ and $b' \in \mathrm{Ann}(b)$, then in $\Gamma'_3(R)$ we have $C : (b', a, a) \sim (a, -b', a'b') \sim (a', -b, b) \sim (b, a', a'b') \sim (b', a, a)$. Because R is reduced and $ab \neq 0$, we must have $a \neq a', b'$ and $b \neq a', b'$. Thus, it follows that all of the vertices in C are distinct and C is a cycle of length 4 in the chordal graph $\Gamma'_3(R)$. Hence, either we have $(b', a, a) \sim (a', -b, b)$ or $(a, -b', a'b') \sim (b, a', a'b')$. The former case means that $a'b' = (b', a, a) \cdot (a', -b, b) = 0$, against the choice of a' and b'. It follows that $(a, -b', a'b') \sim (b, a', a'b')$, that is, $ab - a'b' + (a'b')^2 = 0$. Consequently, $(ab)^2 = ab(a'b' - (a'b')^2) = 0$ and hence $ab = 0$, for R is reduced. However, this is a contradiction, from which the claim follows.

Now suppose that $\Gamma'_2(R)$ is chordal (and R is not necessarily reduced). By 3.5(i), we just need to show that $ab = 0$ for each $a, b \in \mathrm{Z}^*(R)$ with $a \neq \pm b$. Assume that this does not hold for some a, b. If

$\mathrm{Ann}(a) \subseteq \{0, a\}$ and $\mathrm{Ann}(b) \subseteq \{0, b\}$, then $0 \neq ab \in \mathrm{Ann}(a) \cap \mathrm{Ann}(b)$, which forces $a = b$, against the choice of a, b. We can assume that $\mathrm{Ann}(a) \not\subseteq \{0, a\}$, say $a \neq a' \in \mathrm{Ann}^*(a)$. Also let $b' \in \mathrm{Ann}^*(b)$. Note that $a \notin \{\pm b, a', \pm b'\}$ and $b \neq \pm a'$. These relations guarantee distinctness of vertices of the following cycle in $\Gamma_2'(R)$: $(b', a) \sim (a, -b') \sim (a', -b) \sim (b, a') \sim (b', a)$. Since this cycle has length > 3 in a chordal graph, it must have a chord that is either $(a', -b) \sim (b', a)$ or $(a, -b') \sim (b, a')$. In both cases, we deduce that $ab = a'b'$ and hence $a^2b = ab^2 = 0$. Thus, $ab \in \mathrm{Ann}(a) \cap \mathrm{Ann}(b)$ and either $a \neq ab$ or $b \neq ab$.

Notice that in the above paragraph, we indeed proved that if $a, b \in \mathrm{Z}^*(R)$, $a \neq \pm b$, $a'a = 0 = b'b$ for nonzero a' and b' such that either $a \neq a'$ or $b \neq b'$, then $ab = a'b'$. Applying this with $a' = b' = ab$, we get $ab = (ab)^2 = 0$, a contradiction. From this contradiction, it follows that $ab = 0$ for each $a, b \in \mathrm{Z}^*(R)$ with $a \neq \pm b$ and the result is established. □

Summing up the results on $\Gamma_n(R)$ and $\Gamma_n'(R)$, we get the following:

Corollary 3.8 *Suppose that $n \geq 2$ is an integer.*

- **(i)** *$\Gamma_n(R)$ is chordal if and only if $\mathrm{Z}(R)^2 = 0$.*
- **(ii)** *$\Gamma_n'(R)$ is chordal if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $n = 2$ or $\mathrm{Z}(R)^2 = 0$.*

The last graph we study in this article is the zero-divisor graph $\Gamma(R)$. First we consider the case where R is decomposable; that is, R decomposes as a product of nontrivial rings.

Proposition 3.9 *Assume that R is decomposable. The zero-divisor graph of R is chordal if and only if $R \cong \mathbb{Z}_2 \times R'$ where $R' \cong \mathbb{Z}_2^2$ or $\mathrm{Z}(R')^2 = 0$.*

Proof ($\Rightarrow$): Suppose that $R \cong R_1 \times R_2$ for nontrivial rings R_1 and R_2. If both $|R_1| \geq 3$ and $|R_2| \geq 3$ and $1 \neq a_i \in R_i^*$, then in $\Gamma(R)$ we have $(1, 0) \sim (0, 1) \sim (a_1, 0) \sim (0, a_2) \sim (1, 0)$, which forms an induced cycle of length 4. Thus, at least one of the R_is, say R_1, is isomorphic to $\mathbb{Z}_2$. Suppose that $R_2 \cong R_2' \times R_3'$ is itself decomposable. Then $R \cong (R_1 \times R_2') \times R_3'$ and since $|R_1 \times R_2'| \geq 4$ and by the above argument, we see that $R_3' \cong \mathbb{Z}_2$. Similarly, $R_2' \cong \mathbb{Z}_2$ and $R' \cong \mathbb{Z}_2^2$.

Now assume that R' is indecomposable but not an integral domain (note that a domain satisfies $\mathrm{Z}(R')^2 = 0$). If $\mathrm{Z}(R')^2 \neq 0$, then by [5, Remark 2.9(a)] $\Gamma(R')$ is not a complete graph, but it is known that $\Gamma(R')$ is a connected graph (see [4, Theorem 2.3]). Thus, there are vertices r_1, r_2 and r_3 of $\Gamma(R')$, such that $r_1 \sim r_2 \sim r_3$ but $r_1 \nsim r_3$ in $\Gamma(R')$. It follows that $(0, r_1) \sim (1, 0) \sim (0, r_3) \sim (1, r_2) \sim (0, r_1)$ is a chordless cycle in $\Gamma(R)$, a contradiction. Therefore, $\mathrm{Z}(R')^2 = 0$.

($\Leftarrow$): Set $A = \{1\} \times \mathrm{Z}(R')$, $B = \{0\} \times \mathrm{Z}^*(R')$ and $C = \{0\} \times (R' \setminus \mathrm{Z}(R'))$. Then $\mathrm{V}(\Gamma(R)) = A \cup B \cup C$. It is easy to see that in all cases, the vertices in C are adjacent only to $(1, 0)$ and are not contained in any cycle. Also, $\Gamma(R)[A]$ has no edges and $\Gamma(R)[B]$ is complete. Consequently, any induced cycle with length > 3 should have at least one vertex from A, say a. Then the neighbors of a in this cycle should be from B, say b_1 and b_2, but $b_1 \sim b_2$, a contradiction. Thus, $\Gamma(R)$ is chordal. □

Next we present a condition on an indecomposable ring R, equivalent to chordality of $\Gamma(R)$. For this we need a lemma.

Lemma 3.10 *Suppose that R is indecomposable, $\Gamma(R)$ is chordal, and $y, y' \in \mathrm{Z}^*(R)$ with $yy' \neq 0$. Then $(\mathrm{Ann}(y) \cap \mathrm{Ann}(y'))^2 = 0$.*

Proof Let $x, z \in (\mathrm{Ann}(y) \cap \mathrm{Ann}(y'))^*$. If $x \neq z$, then the cycle $x \sim y \sim z \sim y' \sim x$ in $\Gamma(R)$ has a chord by assumption. As $yy' \neq 0$, we must have $zx = 0$. Therefore, we just need to show that $x^2 = 0$. Suppose this is not the case. Note that $x \neq x^2$ because R is indecomposable. By applying the above argument with x^2 instead of z, we deduce that $x^3 = 0$. If $x + x^2 = 0$, then $(-x)^2 = -x$ is a nonzero idempotent in $\mathrm{Z}(R)$, which means that R is decomposable, against our hypothesis. Hence, $x \neq x + x^2 \in (\mathrm{Ann}(y) \cap \mathrm{Ann}(y'))^*$, and by the above argument, $x(x + x^2) = 0$. Then $x^2 = -x^3 = 0$, a contradiction, from which the result follows. □

The following theorem reduces chordality of $\Gamma(R)$ to chordality of $\Gamma_E(R)$, the *compressed zero-divisor graph* of R defined in [17] and further studied in [3]. The vertices of $\Gamma_E(R)$ are equivalence classes of elements of $\mathrm{Z}^*(R)$ under the relation $x \simeq y$ if and only if $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and two vertices $[x]$ and $[y]$ are adjacent if and only if $xy = 0$.

Theorem 3.11 *Assume that R is indecomposable. Then $\Gamma(R)$ is chordal if and only if $\Gamma_E(R)$ is chordal and for each $y \in \mathrm{Z}^*(R)$ either $y^2 = 0$ or $(\mathrm{Ann}(y))^2 = 0$.*

Proof ($\Rightarrow$): Let V be an irredundant set of representatives of vertices of $\Gamma_E(R)$. By definition, $x \sim y$ in $\Gamma(R)$ if and only if $[x] \sim [y]$ in $\Gamma_E(R)$. Hence, $\Gamma_E(R)$ is isomorphic to the induced subgraph $\Gamma(R)[V]$ of $\Gamma(R)$ and is chordal. Now let $y \in \mathrm{Z}^*(R)$ with $y^2 \neq 0$. Consider $y' = y + y^2 \neq y$. If $y' = 0$, then $(-y)^2 = -y$ is a nontrivial idempotent in R, against indecomposablity of R, so $y' \neq 0$. If $yy' = 0$, then $y^2 = -y^3$, from which $y^4 = y^2$ and y^2 is a nontrivial idempotent in R. Thus, $yy' \neq 0$, and as $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(y')$, the result follows from 3.10.

($\Leftarrow$): Let $C : x_0 \sim \cdots \sim x_{n-1} \sim x_0$ be an induced cycle of length $n > 3$ in $\Gamma(R)$. If $[x_i] \neq [x_j]$ for all $i \neq j$, then $[x_0] \sim \cdots \sim [x_{n-1}] \sim [x_0]$ is an induced cycle of the same length in $\Gamma_E(R)$, against the hypothesis. Thus, there are $i \neq j$ with $[x_i] = [x_j]$. If $x_i^2 = 0$, then $x_i \in \mathrm{Ann}(x_i) = \mathrm{Ann}(x_j)$ and $x_i \sim x_j$, from which x_i and x_j must be two consecutive vertices in C, say $j = i + 1 \pmod{\mathrm{n}}$, but then $x_k \in \mathrm{Ann}(x_i) = \mathrm{Ann}(x_j)$ for $k = i - 1 \pmod{\mathrm{n}}$ and $x_k \sim x_j$ is a chord for C, a contradiction. Consequently, we can assume $x_i^2 \neq 0$, so $x_i \nsim x_j$, or else $x_i \in \mathrm{Ann}(x_j) = \mathrm{Ann}(x_i)$. If $n \geq 5$, then there is a vertex of C adjacent to x_i but not x_j, which is impossible because $[x_i] = [x_j]$. Thus, $n = 4$, and if we assume $i = 0$, then $j = 2$. Now $x_1, x_3 \in \mathrm{Ann}(x_i)$ and by the assumption of the theorem $\mathrm{Ann}(x_i)^2 = 0$, so $x_1 \sim x_3$ is a chord for C, a contradiction. Therefore, no such cycle C exists and $\Gamma(R)$ is chordal. □

Thus, to characterize rings with chordal zero-divisor graphs, it suffices to characterize rings with chordal compressed zero-divisor graphs. Although we could not achieve this goal, in some cases we can utilize 3.11 to characterize chordal zero-divisor graphs for some classes of rings. For example, if R is reduced and $y \in \mathrm{Z}^*(R)$, then neither $y^2 = 0$ nor $(\mathrm{Ann}(y))^2 = 0$. Hence, we get the following corollary of 3.9 and 3.11.

Corollary 3.12 *Suppose that R is reduced. Then $\Gamma(R)$ is chordal if and only if either R is an integral domain or $R \cong \mathbb{Z}_2^3$ or $R \cong \mathbb{Z}_2 \times D$ for an integral domain D.*

At the end of the paper, we present some examples of rings R with chordal $\Gamma(R)$. Some simple examples are $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/\langle x^2 \rangle$ or more generally any local ring $(R, \mathfrak{M})$ with $\mathfrak{M}^2 = 0$. Another class is presented in the following example. Note that in this example, we may have $(\mathrm{Ann}(y))^2 = 0$ for some $y \in \mathrm{Z}^*(R)$ and $y^2 = 0$ for some other $y \in \mathrm{Z}^*(R)$.

Example 3.13 *Let R be a* special principal ideal ring *(SPIR), that is, a principal ideal ring with exactly one prime ideal $\langle p \rangle$ such that $p^n = 0$ for some positive integer n. For example, $R = \mathbb{Z}_{p^n}$ for a prime number p. Then $\Gamma_E(R)$ has $n-1$ vertices $[p], \ldots, [p^{n-1}]$ and $[p^i] \sim [p^j]$ if and only if $i + j \geq n$. It is routine to check that this graph is chordal. Also, if $y \in \langle p^i \rangle \setminus \langle p^{i+1} \rangle$ for some $0 < i < n$, then $\mathrm{Ann}(y) = \langle p^{n-i} \rangle$. Thus, either $y^2 = 0$ or $(\mathrm{Ann}(y))^2 = 0$. Therefore, by 3.11, $\Gamma(R)$ is chordal.*

We can extend this example to get non-Noetherian rings with chordal zero-divisor graphs.

Example 3.14 *Let $X = \{x_i | i \in I\}$ be a family of indeterminates, where I is an arbitrary indexing set. Then $R = \frac{D[X]}{\mathfrak{M}^k}$ has a chordal zero-divisor graph, where D is an integral domain, $\mathfrak{M}$ is the ideal of $D[X]$ generated by all x_is, and k is a positive integer.*

Proof Let $0 \neq f, g \in D[X]$ and set $\mathrm{Ord}(f)$ to be the smallest degree of a term present in f. Then $fg \in \mathfrak{M}^k$ if and only if $\mathrm{Ord}(f) + \mathrm{Ord}(g) = \mathrm{Ord}(fg) \geq k$. It follows that $\mathrm{Ann}(\overline{f}) = \overline{\mathfrak{M}}^{k-\mathrm{Ord}(f)}$ for each $f \notin \mathfrak{M}^k$. Thus, either $\mathrm{Ann}(\overline{f})^2 = 0$ or $\overline{f}^2 = 0$. Also, the vertices of $\Gamma_E(R)$ are $[\overline{x}_1], [\overline{x}_1{}^2], \ldots, [\overline{x}_1{}^{k-1}]$ and $\Gamma_E(R)$ is isomorphic to the compressed zero-divisor graph of the SPIR $\frac{D[x_1]}{\langle x_1^k \rangle}$ and is chordal. Thus, by 3.11, $\Gamma(R)$ is chordal. □

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