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Two asymptotic results of solutions for nabla fractional (q,h)-difference equations

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Abstract: In this paper we study the Caputo and Riemann–Liouville nabla (q,h)-fractional difference equation and obtain the following two main results:

Theorem A *Assume* $0<\alpha<1$ *and there is a constant* b *such that* $c(t)\le b<0$, *for* $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$. *Then any solution,* $x(t)$, *of the nabla Caputo* (q,h)*-fractional difference equation*

$$ {}_a^C\nabla_{(q,h)}^{\alpha}x(t)=c(t)x(t), \qquad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \tag{0.1}$$

with $x(a)>0$ *satisfies*

$$\lim_{t\to\infty}x(t)=0.$$

Theorem B *Assume* $0<\alpha<1$, $c(t)\le 0$, $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)}$, *and* $x(t)$ *is a solution of the equation*

$$ {}_a\nabla_{(q,h)}^{\alpha}x(t)=c(t)x(t), \qquad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)}, \tag{0.2}$$

satisfying $x(\sigma(a))>0$. *Then* $x(t)>0$, $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ *and*

$$\lim_{t\to\infty}x(t)=0.$$

Theorem A and Theorem B extend the results in other recent works of the authors.

Key words: Nabla fractional difference, (q,h)-calculus, monotonicity, asymptotic behavior

1. Introduction

In recent years, fractional calculus has attracted increasing interest. Although several results of fractional differential/difference equations are already published [1–6, 13–16, 22, 23, 25–30, 32, 33, 35, 40–44, 46, 47], the development of a qualitative theory for fractional difference equations is still in its beginning due to the memory effects of fractional operators.

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The extension of the basic notions of discrete fractional calculus to the (q,h)-calculus setting appear in [10, 12]. The (q,h)-calculus reduces to discrete nabla-calculus (see [18, Chapter 3]) when $q=1=h$ and q-difference calculus (quantum-calculus) (see[18, Chapter 4]) when $h=0$. Some interesting results concerning fractional (q,h)-calculus can be found in [36–38].

In 1996, Matignon [34] gave a classical stability condition for homogeneous linear fractional differential equations. This criterion was developed by several authors. In [31], Chen et al. proposed a fractional Lyapunov direct method. Later, Chen et al. [13] used the Schauder fixed point theorem in the asymptotic stability of nonlinear fractional difference equations. In 2013, Cermak et al. [11] investigated the stability regions for linear fractional difference systems using Laplace transform. In 2015, Jia et al. [25] established comparison theorems to extend the corresponding asymptotic result in [4]. In 2016, Mozyrska et al. [35] presented linearization to decide the stability of fractional difference systems. In 2017, Jia et al. [20] constructed a Liapunov functional to decide the stability of fractional (q,h)-difference equations.

Some efforts have been made in stability of nabla fractional q-difference equations [19, 25] and nabla fractional h-difference equations (see [11, 21], [39, Chapter 6]). Other efforts have been made in stability of delta fractional difference equations ($h=1$) [9]. However, [20] is the only result on the asymptotic behavior of nabla fractional (q,h)-difference equations. The formal solutions (when $c(t)=\lambda$ is a constant) of (0.1) and (0.2) are given through (q,h)-Mittag-Leffler function (see details in [10, 38]),

$$E^{s,\lambda}_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\lambda^k\hat{h}_{\alpha k+\beta-1}(t,s),$$

which converges (absolutely) if $|\lambda|<(\nu(t))^{-\alpha}$, which means λ only can be zero when $q>1$ and $h>0$ (since $\lim\limits_{t\to\infty}\nu(t)=\infty,\ \ t\in\mathbb{T}^{t_0}_{(q,h)}$). This illustrates that there are substantial differences between the h-time scale and q-time scale (or (q,h)-time scale). Thus, we cannot directly use (q,h)-Mittag-Leffler function to study their asymptotic behavior.

The aim of this paper is to further investigate asymptotic stability of the nabla fractional (q,h)-difference equations. Theorem A is motivated by Jia et al. [23], who obtained asymptotic results for the nabla Caputo fractional difference equation

$$\begin{aligned}&\nabla^{\nu}_{a*}x(t)=c(t)x(t), \qquad t\in\mathbb{N}_{a+1},\\ &x(a)>0,\end{aligned}\tag{1.1}$$

where $0<\nu<1$, $c(t)\le b_1<0$, and motivated by Baleanu et al. [6], who obtained the solution's monotonicity for $\lambda<0$ and asymptotic stability of fractional Caputo-like h-difference equation

$$\begin{aligned}&{}^{C}_{h}\Delta^{\nu}_{a}u(t)=\lambda u(t+\nu h), \quad 0<\nu\le 1,\ t\in(h\mathbb{N})_{a+(1-\nu)h},\\ &u(a)>0.\end{aligned}\tag{1.2}$$

Theorem B is motivated by Jia et al. [25] and Goodrich and Peterson [18], who obtained asymptotic results for the nabla fractional q-difference equation

$$\begin{aligned}&\nabla^{\alpha}_{q,p}x(t)=c(t)x(t), \qquad t\in q^{\mathbb{N}_1},\\ &x(1)>0,\end{aligned}\tag{1.3}$$

where $p=q^{-1}$, $c(t)\le 0$.

This paper is organized as follows. In Section 2, some preliminaries about (q,h)-discrete time scales are introduced. In Section 3, Theorem A is proved and we show that the solution of the nabla Caputo (q,h)-fractional difference equation

$$ {}_a^C\nabla^{\alpha}_{(q,h)}x(t) = c(t)x(t), \quad c(t) \le b < 0, \quad 0 < \alpha < 1, $$

has similar asymptotic behavior with the solution of the ordinary nabla Caputo difference equation

$$ {}_a^C\nabla^{\alpha}_{(1,1)}x(t) = c(t)x(t), \quad c(t) \le b < 0, \quad 0 < \alpha < 1. $$

In Section 4, Theorem B is given and we show that the solution of nabla (q,h)-fractional difference equation ${}_a\nabla^{\alpha}_{(q,h)}x(t) = cx(t), \quad c < 0, \quad 0 < \alpha < 1,$ has similar asymptotic behavior with the solution of the first-order nabla q-difference equation $\nabla_q x(t) = cx(t), c < 0, t \in q^{\mathbb{N}_1}$. Four numerical examples are given in Section 5 and asymptotic stability of numerical solution is shown to support our the qualitative analysis.

2. Preliminaries

First we introduce some notation used in (q,h)-calculus (see [10] and [12]). For any real number α and $q > 0, q \neq 1$, we set $[\alpha]_q := \frac{q^{\alpha}-1}{q-1}$. Then we have the q-analogy of $n!$ in the form $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$ for $n = 1, 2, \cdots$, whereas for $n = 0$ we put $[0]_q! := 1$. If $q = 1$, then $[\alpha]_1 := \alpha$ and $[n]_1!$ becomes the standard factorial. Further, the q-binomial coefficient is introduced by use of relations

$$ \begin{bmatrix} \alpha \\ 0 \end{bmatrix}_{\tilde{q}} := 1, $$

$$ \begin{bmatrix} \alpha \\ n \end{bmatrix}_{\tilde{q}} := \frac{[\alpha]_{\tilde{q}}[\alpha-1]_{\tilde{q}} \cdots [\alpha-n+1]_{\tilde{q}}}{[n]_{\tilde{q}}!}, \tag{2.1} $$

where $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_0$. The extension of the q-binomial coefficient to noninteger value n is allowed via the $\Gamma_{\tilde{q}}(t)$ function defined for $0 < \tilde{q} < 1$ as

$$ \Gamma_{\tilde{q}}(t) := \frac{(\tilde{q},\tilde{q})_{\infty}(1-\tilde{q})^{1-t}}{(\tilde{q}^t,\tilde{q})_{\infty}}, \quad 0 < \tilde{q} < 1, \tag{2.2} $$

where $(a,\tilde{q})_{\infty} = \prod_{j=0}^{\infty}(1-a\tilde{q}^j)$ and $t \in \mathbb{R} \setminus \{0,-1,-2,\cdots\}$. It is easy to check that $\Gamma_{\tilde{q}}$ satisfies the functional relation $\Gamma_{\tilde{q}}(t+1) = [t]_{\tilde{q}}\Gamma_{\tilde{q}}(t)$ and $\Gamma_{\tilde{q}}(1) = 1$. The q-binomial coefficient was introduced as

$$ \begin{bmatrix} x \\ k \end{bmatrix}_{\tilde{q}} = \frac{\Gamma_{\tilde{q}}(x+1)}{\Gamma_{\tilde{q}}(k+1)\Gamma_{\tilde{q}}(x-k+1)}, \quad x \in \mathbb{R}, \quad k \in \mathbb{Z}. \tag{2.3} $$

From (2.1), it is easy to derive the recursion formula

$$ \begin{bmatrix} x \\ k \end{bmatrix}_{\tilde{q}} = \frac{1-\tilde{q}^{x+1-k}}{1-\tilde{q}^k}\begin{bmatrix} x \\ k-1 \end{bmatrix}_{\tilde{q}}, \qquad \begin{bmatrix} x \\ 0 \end{bmatrix}_{\tilde{q}} = 1. \tag{2.4} $$

The q-analogue of the power function is introduced as

$$(t-s)_{\tilde{q}}^{(\alpha)} = t^{\alpha}\frac{(s/t,\tilde{q})_{\infty}}{(\tilde{q}^{\alpha}s/t,\tilde{q})_{\infty}}, \qquad t \neq 0,\ 0<\tilde{q}<1,\ \alpha\in\mathbb{R}.$$

For $\alpha = n$, a positive integer, this expression reduces to

$$(t-s)_{\tilde{q}}^{(n)} = t^{n}\prod_{j=0}^{n-1}(1-\tilde{q}^{j}\frac{s}{t}).$$

We consider here the (q,h)-time scale (for details, see [10] and [12]):

$$\mathbb{T}_{(q,h)}^{t_0} = \{t_0q^k+[k]_qh, k\in\mathbb{Z}\}\cup\left\{\frac{h}{1-q}\right\}, t_0>0, q\geq 1, h\geq 0, q+h>1.$$

Note that if $q=1$ then the cluster point $h/(1-q)=-\infty$ is not involved in $\mathbb{T}_{(q,h)}^{t_0}$. The forward and backward jump operator is the linear function $\sigma(t)=qt+h$ and $\rho(t)=q^{-1}(t-h)$, respectively. Similarly, the forward and backward graininess is given by $\mu(t)=(q-1)t+h$ and $\nu(t)=q^{-1}\mu(t)$, respectively. We use the standard notation

$$\sigma^{k+1}(t):=\sigma^{k}(\sigma(t)), \qquad \rho^{k+1}(t):=\rho^{k}(\rho(t)), \quad k=0,1,\cdots.$$

It is easy to show that

$$\sigma^{k}(t)=q^{k}t+[k]_qh \ \text{ and } \ \rho^{k}(t)=q^{-k}(t-[k]_qh).$$

The relation

$$\nu\big(\rho^{k}(t)\big)=q^{-k}\nu(t)$$

holds for $t\in\mathbb{T}_{(q,h)}^{t_0}$.

Let $a\in\mathbb{T}_{(q,h)}^{t_0}, a>h/(1-q)$ be fixed. Then we introduce restrictions of the time scale $\mathbb{T}_{(q,h)}^{t_0}$ by relation

$$\tilde{\mathbb{T}}_{(q,h)}^{\sigma^{i}(a)} = \{t\in\mathbb{T}_{(q,h)}^{t_0}, t\geq\sigma^{i}(a)\}, \quad i=0,1,\cdots,$$

where the symbol σ^{i} stands for the ith iterate of σ (analogously, we use the symbol ρ^{i}). To simplify the notation, we put $\tilde{q}=1/q$ whenever considering the time scale $\mathbb{T}_{(q,h)}^{t_0}$ or $\tilde{\mathbb{T}}_{(q,h)}^{\sigma^{i}(a)}$.

The nabla (q,h)-derivative of the function $f:\mathbb{T}_{(q,h)}^{t_0}\to\mathbb{R}$ is defined by

$$\nabla_{(q,h)}f(t):=\frac{f(t)-f(\rho(t))}{\nu(t)}=\frac{f(t)-f(\tilde{q}(t-h))}{(1-\tilde{q})t+\tilde{q}h},$$

where $\tilde{q}=q^{-1}$.

The power functions and the nabla fractional (q,h)-Taylor monomial of degree α on $\mathbb{T}_{(q,h)}^{t_0}$ (see [10]) are defined by

$$(t-s)_{(\tilde{q},h)}^{(\alpha)} := ([t]-[s])_{\tilde{q}}^{(\alpha)}, \tag{2.5}$$

$$\hat{h}_\alpha(t,s) := \frac{(t-s)^{(\alpha)}_{(\tilde{q},h)}}{\Gamma_{\tilde{q}}(\alpha+1)}, \quad \alpha \in \mathbb{R}, \tag{2.6}$$

respectively, where the square bracket of t, $[t]$ is defined by

$$[t] := t + h\tilde{q}/(1-\tilde{q}). \tag{2.7}$$

It follows that

$$\nu(t) := [t](1-\tilde{q}). \tag{2.8}$$

The following definitions appear in [10].

Definition 2.1 *Assume $x : \mathbb{T}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$ and $t = \sigma^n(a)$, $n \geq 1$. Then the nabla (q,h)-integral of x from a to t is defined by*

$$\int_a^t x(\tau)\nabla\tau := \sum_{i=1}^{n} x(\sigma^i(a))\nu(\sigma^i(a))$$

with the standard convention that $\int_a^a x(\tau)\nabla\tau = 0$. We also define the nabla (q,h)-fractional integral of order $\alpha > 0$ by

$$_a\nabla^{-\alpha}_{(q,h)}x(t) := \int_a^t \hat{h}_{\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau. \tag{2.9}$$

Definition 2.2 *The nabla (q,h)-fractional difference (in the Riemann–Liouville sense) of order $\alpha > 0$ is defined by*

$$_a\nabla^{\alpha}_{(q,h)}x(t) := \nabla^m_{(q,h)}\ {}_a\nabla^{-(m-\alpha)}_{(q,h)}x(t),$$

where $m = \lceil\alpha\rceil$; that is, m is the ceiling of α.

Definition 2.3 *The nabla (q,h)-fractional difference (in the sense of Caputo) of order $\alpha > 0$ is defined by*

$$^C_a\nabla^{\alpha}_{(q,h)}x(t) := {}_a\nabla^{-(m-\alpha)}_{(q,h)}\nabla^m_{(q,h)}x(t).$$

3. Asymptotic behavior for Caputo (q,h)-fractional difference equation

From the definition of the q-gamma function (2.2), it is easy to prove the following lemma.

Lemma 3.1 (Sign of q-gamma function)

1. *If $x > 0$ or $x \in (-2n-2, -2n-1), n \in \mathbb{N}_0$, then $\Gamma_{\tilde{q}}(x) > 0$.*
2. *If $x \in (-2n-1, -2n), n \in \mathbb{N}_0$, then $\Gamma_{\tilde{q}}(x) < 0$.*

The following Figure 1 illustrates the above lemma.

The following lemma gives a very nice form of the Taylor monomials on $\mathbb{T}^{t_0}_{(q,h)}$ (see [10]).

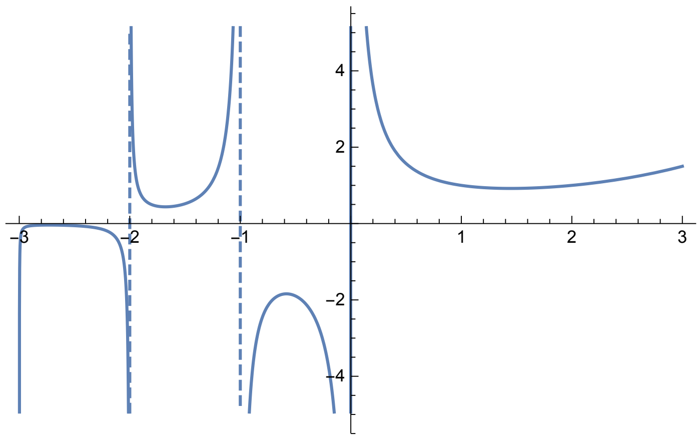

Figure 1. The q-gamma function on interval $(-3,3)$, $\tilde{q}=0.5$.

Lemma 3.2 *Let* $\alpha\in\mathbb{R}, s,t\in\mathbb{T}_{(q,h)}^{t_0}$ *and* $n\in\mathbb{N}_0$ *such that* $t=\sigma^n(s)$ *Then:*

$$\hat{h}_\alpha(t,s)=(\nu(t))^\alpha\begin{bmatrix}\alpha+n-1\\ n-1\end{bmatrix}_{\tilde{q}}. \tag{3.1}$$

Lemma 3.3 *Let* $\alpha\in\mathbb{R}, s,t\in\mathbb{T}_{(q,h)}^{t_0}$ *and* $n\in\mathbb{N}_0$ *such that* $t=\sigma^n(s)$. *Then:*

(i) $\hat{h}_\alpha(t,t)=0,\ \alpha\neq 0.$

(ii) $\hat{h}_0(t,s)=1$, *for* $n\in\mathbb{N}_0$.

(iii) *for* $k\in\mathbb{N}_1$,

$$\hat{h}_{-k}(t,s)=\begin{cases}(\nu(t))^{-k}, & n=1,\\ (\nu(t))^{-k}\dfrac{[n-1-k]_{\tilde{q}}[n-2-k]_{\tilde{q}}\cdots[1-k]_{\tilde{q}}}{\Gamma_{\tilde{q}}(n)}, & n\in\mathbb{N}_2,\end{cases} \tag{3.2}$$

when $1\leq k\leq n-1$ *and* $n\in\mathbb{N}_2$, $\hat{h}_{-k}(t,s)=0.$

Proof (i) If $\alpha\neq 0,-1,-2,\cdots$, then

$$\begin{aligned}\hat{h}_\alpha(t,t)&=(\nu(t))^\alpha\begin{bmatrix}\alpha+0-1\\ 0-1\end{bmatrix}_{\tilde{q}}\\ &=(\nu(t))^\alpha\frac{\Gamma_{\tilde{q}}(\alpha+0)}{\Gamma_{\tilde{q}}(0)\Gamma_{\tilde{q}}(\alpha+1)}\\ &=(\nu(t))^\alpha\frac{1}{\Gamma_{\tilde{q}}(0)[\alpha]_{\tilde{q}}}\\ &=0.\end{aligned}$$

If $\alpha = -1, -2, \cdots$, then

$$\begin{aligned}
\hat{h}_{\alpha}(t,t) &= \lim_{\delta\to\alpha} (\nu(t))^{\delta} \begin{bmatrix} \delta+0-1 \\ 0-1 \end{bmatrix}_{\tilde{q}} \\
&= \lim_{\delta\to\alpha} (\nu(t))^{\delta} \frac{\Gamma_{\tilde{q}}(\delta+0)}{\Gamma_{\tilde{q}}(0)\Gamma_{\tilde{q}}(\delta+1)} \\
&= \lim_{\delta\to\alpha} (\nu(t))^{\delta} \frac{1}{\Gamma_{\tilde{q}}(0)[\delta]_{\tilde{q}}} \\
&= \lim_{\delta\to\alpha} 0 = 0.
\end{aligned}$$

(ii) If $n = 0$, then

$$\begin{aligned}
\hat{h}(t,t) &= (\nu(t))^{0} \begin{bmatrix} -1 \\ -1 \end{bmatrix}_{\tilde{q}} \\
&= \lim_{\gamma\to -1} (\nu(t))^{0} \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}_{\tilde{q}} \\
&= 1.
\end{aligned}$$

If $n \in \mathbb{N}_1$, then

$$\begin{aligned}
\hat{h}_{0}(t,s) &= (\nu(t))^{0} \begin{bmatrix} 0+n-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \\
&= (\nu(t))^{0} \frac{\Gamma_{\tilde{q}}(0+n)}{\Gamma_{\tilde{q}}(n)\Gamma_{\tilde{q}}(1)} \\
&= 1.
\end{aligned}$$

(iii) If $n = 1$, then

$$\begin{aligned}
\hat{h}_{-k}(t,s) &= \lim_{\alpha\to k} \hat{h}_{-\alpha}(t,s) \\
&= \lim_{\alpha\to k} (\nu(t))^{-\alpha} \begin{bmatrix} -\alpha+n-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \\
&= \lim_{\alpha\to k} (\nu(t))^{-\alpha} \frac{\Gamma_{\tilde{q}}(-\alpha+n)}{\Gamma_{\tilde{q}}(n)\Gamma_{\tilde{q}}(1-\alpha)} \\
&= \lim_{\alpha\to k} (\nu(t))^{-\alpha} \\
&= (\nu(t))^{-k}.
\end{aligned}$$

If $n \in \mathbb{N}_2$, then

$$\begin{aligned}
\hat{h}_{-k}(t,s) &= \lim_{\alpha \to k} \hat{h}_{-\alpha}(t,s) \\
&= \lim_{\alpha \to k} (\nu(t))^{-\alpha} \begin{bmatrix} -\alpha + n - 1 \\ n-1 \end{bmatrix}_{\tilde{q}} \\
&= \lim_{\alpha \to k} (\nu(t))^{-\alpha} \frac{\Gamma_{\tilde{q}}(-\alpha + n)}{\Gamma_{\tilde{q}}(n)\Gamma_{\tilde{q}}(1-\alpha)} \\
&= \lim_{\alpha \to k} (\nu(t))^{-\alpha} \frac{[n-1-\alpha]_{\tilde{q}}[n-2-\alpha]_{\tilde{q}} \cdots [1-\alpha]_{\tilde{q}}}{\Gamma_{\tilde{q}}(n)} \\
&= (\nu(t))^{-k} \frac{[n-1-k]_{\tilde{q}}[n-2-k]_{\tilde{q}} \cdots [1-k]_{\tilde{q}}}{\Gamma_{\tilde{q}}(n)}.
\end{aligned}$$

Note that if $1 \leq k \leq n-1$ and $n \in \mathbb{N}_2$, then $\hat{h}_{-k}(t,s) = 0$. □

Next we give nabla power rule formulas as an extension of results that appear in [24] and [25].

Lemma 3.4 *Assume $q > 1$, $\tilde{q} = q^{-1}$, $\alpha \in \mathbb{R}$, and let $t = \sigma^n(s)$, $n \in \mathbb{N}_0$, where $s \in \mathbb{T}_{(q,h)}^{t_0}$. Then:*

1. *The nabla (q,h)-difference of the function $\hat{h}_{-\alpha}(t,s)$ with respect to t is given by*

$$_t\nabla_{(q,h)}\hat{h}_{-\alpha}(t,s) = \hat{h}_{-\alpha-1}(t,s).$$

2. *The nabla (q,h)-difference of the function $\hat{h}_{-\alpha}(t,s)$ with respect to s is given by*

$$_s\nabla_{(q,h)}\hat{h}_{-\alpha}(t,s) = -\hat{h}_{-\alpha-1}(t,\rho(s)).$$

Proof These results follow easily from Lemma 2.2 in [17]. □

Lemma 3.5 *Assume $c(t) < (\nu(t))^{-\alpha}$, $0 < \alpha < 1$. Then any solution of*

$$_a^C\nabla_{(q,h)}^{\alpha} x(t) = c(t)x(t), \qquad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \tag{3.3}$$

satisfying $x(a) > 0$ is positive on $\tilde{\mathbb{T}}_{(q,h)}^{a}$.

Proof Using integration by parts and

$$_s\nabla_{(q,h)}\hat{h}_{-\alpha}(t,s) = -\hat{h}_{-\alpha-1}(t,\rho(s)),$$

we have

$$
\begin{aligned}
{}_a^C\nabla_{(q,h)}^{\alpha}x(t) &\overset{\text{Def.2.3}}{=} {}_a\nabla_{(q,h)}^{-(1-\alpha)}\nabla_{(q,h)}x(t)\\
&\overset{\text{Def.2.1}}{=} \int_a^t \hat{h}_{-\alpha}(t,\rho(s))\nabla_{(q,h)}x(s)\nabla s\\
&= \hat{h}_{-\alpha}(t,s)x(s)|_{s=a}^{t} + \int_a^t \hat{h}_{-\alpha-1}(t,\rho(s))x(s)\nabla s\\
&= -\hat{h}_{-\alpha}(t,a)x(a) + \sum_{i=1}^{k}\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a))x(\sigma^i(a))\nu(\sigma^i(a)).
\end{aligned}
$$

Taking $t=\sigma^k(a), k\geq 1$, we have

$$
\begin{aligned}
{}_a^C\nabla_{(q,h)}^{\alpha}x(t) &= {}_a^C\nabla_{(q,h)}^{\alpha}x(\sigma^k(a))\\
&= -\hat{h}_{-\alpha}(t,a)x(a) + \hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{k-1}(a))x(\sigma^k(a))\nu(\sigma^k(a))\\
&\quad + \sum_{i=1}^{k-1}\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a))x(\sigma^i(a))\nu(\sigma^i(a))\\
&= -\hat{h}_{-\alpha}(t,a)x(a) + x(\sigma^k(a))(\nu(\sigma^k(a)))^{-\alpha}\\
&\quad + \sum_{i=1}^{k-1}\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a))x(\sigma^i(a))\nu(\sigma^i(a)).
\end{aligned}
$$

Using $x(t)$ as a solution of (3.3), we get

$$
\begin{aligned}
x(\sigma^k(a)) =& \frac{1}{(\nu(\sigma^k(a)))^{-\alpha}-c(\sigma^k(a))}\Big\{\hat{h}_{-\alpha}(t,a)x(a)\\
&-\sum_{i=1}^{k-1}\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a))x(\sigma^i(a))\nu(\sigma^i(a))\Big\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\hat{h}_{-\alpha}(t,a) &= \Big(\nu(t)\Big)^{-\alpha}\begin{bmatrix}-\alpha+k-1\\ k-1\end{bmatrix}_{\tilde{q}}\\
&= \Big(\nu(t)\Big)^{-\alpha}\frac{\Gamma_{\tilde{q}}(-\alpha+k)}{\Gamma_{\tilde{q}}(k)\Gamma_{\tilde{q}}(1-\alpha)}\\
&> 0,
\end{aligned}
\tag{3.4}
$$

for $k\geq 1$.

$$
\begin{aligned}
\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a)) &= \Big(\nu(t)\Big)^{-\alpha-1}\begin{bmatrix}-\alpha-1+k-i\\ k-i\end{bmatrix}_{\tilde{q}}\\
&= \Big(\nu(t)\Big)^{-\alpha-1}\frac{\Gamma_{\tilde{q}}(-\alpha+k-i)}{\Gamma_{\tilde{q}}(k-i+1)\Gamma_{\tilde{q}}(-\alpha)}\\
&< 0,
\end{aligned}
\tag{3.5}
$$

for $1\leq i\leq k-1$.

Using the strong induction principle, (3.4), (3.5), and $x(a)>0$, it is easy to see that $x(\sigma^i(a))>0$ for $i\in\mathbb{N}_0$, where we use the convention $\sum_{i=1}^{k-1}\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a))x(\sigma^i(a))\nu(\sigma^i(a))=0$ for $k<2$. □

The following comparison theorem plays an important role in proving our main results.

Theorem 3.6 *Assume* $c_2(t)\le c_1(t)<(\nu(t))^{-\alpha}$, $0<\alpha<1$, *and* $x(t),y(t)$ *are solutions of the equations*

$$ {}_a^C\nabla^{\alpha}_{(q,h)}x(t)=c_1(t)x(t), \tag{3.6}$$

and

$$ {}_a^C\nabla^{\alpha}_{(q,h)}y(t)=c_2(t)y(t), \tag{3.7}$$

respectively, for $t\in\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ *satisfying* $x(a)\ge y(a)>0$. *Then*

$$x(t)\ge y(t),$$

for $t\in\tilde{\mathbb{T}}^{a}_{(q,h)}$.

Proof Similar to the first part of the proof of Lemma 3.5, taking $t=\sigma^k(a), k\ge 1$, we have

$$\begin{aligned} x(\sigma^k(a))=&\frac{1}{(\nu(\sigma^k(a)))^{-\alpha}-c_1(\sigma^k(a))}\Big\{\hat{h}_{-\alpha}(t,a)x(a)\\ &-\sum_{i=1}^{k-1}\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a))x(\sigma^i(a))\nu(\sigma^i(a))\Big\}. \end{aligned}\tag{3.8}$$

$$\begin{aligned} y(\sigma^k(a))=&\frac{1}{(\nu(\sigma^k(a)))^{-\alpha}-c_2(\sigma^k(a))}\Big\{\hat{h}_{-\alpha}(t,a)y(a)\\ &-\sum_{i=1}^{k-1}\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a))y(\sigma^i(a))\nu(\sigma^i(a))\Big\}. \end{aligned}\tag{3.9}$$

We will prove $x(\sigma^k(a))\ge y(\sigma^k(a))$ for $k\in\mathbb{N}_0$ by using the principle of strong induction. By assumption $x(a)\ge y(a)>0$, so the base case holds. Now assume that $x(\sigma^i(a))\ge y(\sigma^i(a))$, for $i=0,1\cdots,k-1$.

Since $c_2(t)\le c_1(t)<(\nu(t))^{-\alpha}$,

$$\hat{h}_{-\alpha}(t,a)>0$$

and

$$\hat{h}_{-\alpha-1}(t,\sigma^{i-1}(a))<0,\qquad \text{for } 1\le i\le k-1.$$

From (3.8) and (3.9) we have

$$x(\sigma^k(a))\ge y(\sigma^k(a))>0.$$

This completes the proof. □

The following result appears in [38].

Lemma 3.7 *([38]) Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ so that $n-1 < \alpha \leq n$. Then*

$$ {}_a\nabla_{(q,h)}^{-\alpha}\, {}_a^C\nabla_{(q,h)}^{\alpha} x(t) = x(t) - \sum_{k=0}^{n-1} \hat{h}_k(t,a) \nabla_{(q,h)}^k x(a). \tag{3.10} $$

Since $\hat{h}_0(t,a) = 1$ we get the following Lemma.

Lemma 3.8 *Assume that $0 < \alpha < 1$ and $x(t)$ is a solution of the fractional equation*

$$ {}_a^C\nabla_{(q,h)}^{\alpha} x(t) = c(t)x(t), \qquad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \tag{3.11} $$

satisfying $x(a) > 0$. Then $x(t)$ satisfies the integral equation

$$ x(t) = x(a) + \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))c(s)x(s)\nabla s. \tag{3.12} $$

Remark 3.9 *Letting $t = \sigma^k(a), k \in \mathbb{N}_0$, (3.12) can be rewritten explicitly as*

$$ \begin{aligned} x(\sigma^k(a)) &= \frac{x(a) + \sum_{i=1}^{k-1} \hat{h}_{\alpha-1}\big(\sigma^k(a), \sigma^{i-1}(a)\big) c(\sigma^i(a)) \nu\big(\sigma^i(a)\big) x(\sigma^i(a))}{1 - (\nu(\sigma^k(a)))^\alpha c(\sigma^k(a))} \\ &= \frac{x(a) + \sum_{i=1}^{k-1} (\nu(\sigma^k(a)))^{\alpha-1} \begin{bmatrix} \alpha - 1 + k - i \\ k - i \end{bmatrix}_{\tilde{q}} c(\sigma^i(a)) \nu\big(\sigma^i(a)\big) x(\sigma^i(a))}{1 - (\nu(\sigma^k(a)))^\alpha c(\sigma^k(a))}, \end{aligned} \tag{3.13} $$

for $k \geq 1$, where we used formula (3.1).

Lemma 3.10

$$ \lim_{n\to\infty} \frac{\Gamma_{\tilde{q}}(n+\alpha)}{\Gamma_{\tilde{q}}(n)} = (1-\tilde{q})^{-\alpha}, \ 0 < \tilde{q} < 1, \alpha \in \mathbb{R}. \tag{3.14} $$

Proof From Lemma 3.1, we know that if $\alpha \in \mathbb{R}, a, t \in \mathbb{T}_{(q,h)}^{t_0}$ and $n \in \mathbb{N}_0$ such that $t = \sigma^n(a)$, then

$$ \begin{aligned} \hat{h}_\alpha(t,a) &= (\nu(t))^\alpha \begin{bmatrix} \alpha + n - 1 \\ n - 1 \end{bmatrix}_{\tilde{q}} \\ &= (\nu(t))^\alpha \frac{\Gamma_{\tilde{q}}(\alpha+n)}{\Gamma_{\tilde{q}}(\alpha+1)\Gamma_{\tilde{q}}(n)} \\ &= [t]^\alpha (1-\tilde{q})^\alpha \frac{\Gamma_{\tilde{q}}(\alpha+n)}{\Gamma_{\tilde{q}}(\alpha+1)\Gamma_{\tilde{q}}(n)}. \end{aligned} $$

On the other hand, from (2.6),

$$\begin{aligned}
\hat{h}_\alpha(t,a) &= \frac{(t-a)^{(\alpha)}_{(\tilde{q},h)}}{\Gamma_{\tilde{q}}(\alpha+1)} \\
&= \frac{([t]-[a])^{(\alpha)}_{\tilde{q}}}{\Gamma_{\tilde{q}}(\alpha+1)} \\
&= [t]^\alpha \frac{([a]/[t],\tilde{q})_\infty}{(\tilde{q}^\alpha[a]/[t],\tilde{q})_\infty \Gamma_{\tilde{q}}(\alpha+1)} \\
&= [t]^\alpha \frac{(\tilde{q}^n,\tilde{q})_\infty}{(\tilde{q}^{\alpha+n},\tilde{q})_\infty \Gamma_{\tilde{q}}(\alpha+1)} \\
&= [t]^\alpha \frac{(1-\tilde{q}^{\alpha+1})\cdots(1-\tilde{q}^{\alpha+n-1})}{(1-\tilde{q})\cdots(1-\tilde{q}^{n-1})\Gamma_{\tilde{q}}(\alpha+1)} \frac{\prod_{j=0}^\infty(1-\tilde{q}^{j+1})}{\prod_{j=0}^\infty(1-\tilde{q}^{j+\alpha+1})} \\
&= [t]^\alpha \frac{(1-\tilde{q}^{\alpha+1})\cdots(1-\tilde{q}^{\alpha+n-1})}{(1-\tilde{q})\cdots(1-\tilde{q}^{n-1})\Gamma_{\tilde{q}}(\alpha+1)} \frac{\Gamma_{\tilde{q}}(\alpha+1)}{(1-\tilde{q})^{-\alpha}},
\end{aligned}$$

so

$$\frac{\Gamma_{\tilde{q}}(\alpha+n)}{\Gamma_{\tilde{q}}(n)} = \frac{(1-\tilde{q}^{\alpha+1})\cdots(1-\tilde{q}^{\alpha+n-1})}{(1-\tilde{q})\cdots(1-\tilde{q}^{n-1})}\Gamma_{\tilde{q}}(\alpha+1).$$

Letting n tend to infinity, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\Gamma_{\tilde{q}}(\alpha+n)}{\Gamma_{\tilde{q}}(n)} &= \lim_{n\to\infty} \frac{(1-\tilde{q}^{\alpha+1})\cdots(1-\tilde{q}^{\alpha+n-1})}{(1-\tilde{q})\cdots(1-\tilde{q}^{n-1})}\Gamma_{\tilde{q}}(\alpha+1) \\
&= \frac{\prod_{j=0}^\infty(1-\tilde{q}^{j+\alpha+1})}{\prod_{j=0}^\infty(1-\tilde{q}^{j+1})}\Gamma_{\tilde{q}}(\alpha+1) \\
&= \frac{(\tilde{q}^{\alpha+1},\tilde{q})_\infty}{(\tilde{q},\tilde{q})_\infty}\Gamma_{\tilde{q}}(\alpha+1) \\
&= (1-\tilde{q})^{-\alpha}.
\end{aligned}$$

□

Remark 3.11 *Note that*

$$\begin{aligned}
\lim_{n\to\infty} \frac{\Gamma_{\tilde{q}}(\alpha+n)}{\Gamma_{\tilde{q}}(n)} &= (1-\tilde{q})^{-\alpha} \\
&= \lim_{n\to\infty} \left(\frac{1-\tilde{q}^n}{1-\tilde{q}}\right)^\alpha \\
&= \lim_{n\to\infty} ([n]_{\tilde{q}})^\alpha.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \frac{\Gamma_{\tilde{q}}(n+\alpha)}{\Gamma_{\tilde{q}}(n)[n]^\alpha_{\tilde{q}}} = 1,$$

so this limit can be regarded as the q-analogue of the Stirling formula (see [45])

$$\lim_{x\to\infty} \frac{\Gamma(x+\alpha)}{\Gamma(x)x^{\alpha}} = 1.$$

Considering the monotonicity of the solution for the following linear Caputo fractional difference equation,

$$ {}_a^C\nabla_{(q,h)}^{\alpha} x(t) = cx(t), \ \ c<0, \tag{3.15}$$

we have the following result.

Lemma 3.12 *If $x(t)$ is a solution of equation (3.15), $x(a)>0$ and $c<0$, then $\nabla_{(q,h)}x(t)<0$ for $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$.*

Proof Letting $t=\sigma(a)$, from (3.12), we get

$$\begin{aligned} x(\sigma(a)) &= x(a) + \hat{h}_{\alpha-1}(\sigma(a),a)cx(\sigma(a))\nu(\sigma(a)) \\ &= x(a) + cx(\sigma(a))\nu^{\alpha}(\sigma(a)) \\ &< x(a), \end{aligned}$$

where we use $\hat{h}_{\alpha-1}(\sigma(a),a) = \Big(\nu(\sigma(a))\Big)^{\alpha-1}$, $x(\sigma(a))>0$ (from Lemma 3.5), and $c<0$.

If $\nabla_{(q,h)}x(t)<0$, for $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)}$, this completes the proof.

If not, we assume that there exists two points $t_1=\sigma^n(a), t_2=\sigma^m(a), t_1<t_2, 1\le n<m$ such that $\nabla_{(q,h)}x(\sigma^i(a))<0$ for $1\le i\le n<m$ and $\nabla_{(q,h)}x(\sigma^i(a))>0$ for $n+1\le i\le m$.

$$\begin{aligned} &{}_a^C\nabla_{(q,h)}^{\alpha}x(t_1) - {}_a^C\nabla_{(q,h)}^{\alpha}x(t_2) \\ &= \int_a^{t_1} \hat{h}_{-\alpha}(t_1,\rho(s))\nabla_{(q,h)}x(s)\nabla s - \int_a^{t_2} \hat{h}_{-\alpha}(t_2,\rho(s))\nabla_{(q,h)}x(s)\nabla s \\ &= \int_a^{t_1} \Big(\hat{h}_{-\alpha}(t_1,\rho(s)) - \hat{h}_{-\alpha}(t_2,\rho(s))\Big)\nabla_{(q,h)}x(s)\nabla s \\ &\quad - \int_{t_1}^{t_2} \hat{h}_{-\alpha}(t_2,\rho(s))\nabla_{(q,h)}x(s)\nabla s \\ &= \sum_{i=1}^{n} \Big(\hat{h}_{-\alpha}(\sigma^n(a),\sigma^{i-1}(a)) - \hat{h}_{-\alpha}(\sigma^m(a),\sigma^{i-1}(a))\Big)\nabla_{(q,h)}x(\sigma^i(a))\nu(\sigma^i(a)) \\ &\quad - \sum_{i=n+1}^{m} \hat{h}_{-\alpha}(\sigma^m(a),\sigma^{i-1}(a))\nabla_{(q,h)}x(\sigma^i(a))\nu(\sigma^i(a)). \end{aligned}$$

Since

$$\begin{aligned} \hat{h}_{-\alpha}(\sigma^m(a),\sigma^{i-1}(a)) &= \Big(\nu(\sigma^m(a))\Big)^{-\alpha} \begin{bmatrix} -\alpha+m-i \\ m-i \end{bmatrix}_{\tilde{q}} \\ &= \Big(\nu(\sigma^m(a))\Big)^{-\alpha} \frac{\Gamma_{\tilde{q}}(-\alpha+m-i+1)}{\Gamma_{\tilde{q}}(m-i+1)\Gamma_{\tilde{q}}(1-\alpha)} \\ &> 0, \end{aligned}$$

for $i \leq m$, similarly, we have

$$\hat{h}_{-\alpha}(\sigma^n(a), \sigma^{i-1}(a)) > 0, \quad \text{for } i \leq n.$$

Note that

$$\begin{aligned}
&\frac{\hat{h}_{-\alpha}(\sigma^n(a), \sigma^{i-1}(a))}{\hat{h}_{-\alpha}(\sigma^m(a), \sigma^{i-1}(a))} \\
&= \frac{(\nu(t_1))^{-\alpha}}{(\nu(t_2))^{-\alpha}} \cdot \frac{\Gamma_{\tilde{q}}(-\alpha+n-i+1)}{\Gamma_{\tilde{q}}(n-i+1)\Gamma_{\tilde{q}}(1-\alpha)} \cdot \frac{\Gamma_{\tilde{q}}(m-i+1)\Gamma_{\tilde{q}}(1-\alpha)}{\Gamma_{\tilde{q}}(-\alpha+m-i+1)} \\
&= \frac{(\nu(t_1))^{-\alpha}}{(\nu(t_2))^{-\alpha}} \cdot \frac{\Gamma_{\tilde{q}}(-\alpha+n-i+1)}{\Gamma_{\tilde{q}}(-\alpha+m-i+1)} \cdot \frac{\Gamma_{\tilde{q}}(m-i+1)}{\Gamma_{\tilde{q}}(n-i+1)} \\
&= \left(\frac{\nu(t_1)}{\nu(t_2)}\right)^{-\alpha} \frac{[m-i]_{\tilde{q}}[m-i-1]_{\tilde{q}} \cdots [n-i+1]_{\tilde{q}}}{[-\alpha+m-i]_{\tilde{q}}[-\alpha+m-i-1]_{\tilde{q}} \cdots [-\alpha+n-i+1]_{\tilde{q}}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\hat{h}_{-\alpha}(\sigma^n(a), \sigma^{i-1}(a))}{\hat{h}_{-\alpha}(\sigma^m(a), \sigma^{i-1}(a))} \\
&= \left(\frac{\nu(t_1)}{\nu(t_2)}\right)^{-\alpha} \cdot \frac{1-\tilde{q}^{m-i}}{1-\tilde{q}^{-\alpha+m-i}} \frac{1-\tilde{q}^{m-i-1}}{1-\tilde{q}^{-\alpha+m-i-1}} \cdots \frac{1-\tilde{q}^{n-i+1}}{1-\tilde{q}^{-\alpha+n-i+1}} \\
&= \left(\frac{(q-1)t_1+h}{(q-1)t_2+h}\right)^{-\alpha} \cdot \frac{1-\tilde{q}^{m-i}}{1-\tilde{q}^{-\alpha+m-i}} \frac{1-\tilde{q}^{m-i-1}}{1-\tilde{q}^{-\alpha+m-i-1}} \cdots \frac{1-\tilde{q}^{n-i+1}}{1-\tilde{q}^{-\alpha+n-i+1}} \\
&> 1,
\end{aligned}$$

for $1 \leq i \leq n < m$.
Thus,

$$\hat{h}_{-\alpha}(\sigma^n(a), \sigma^{i-1}(a)) - \hat{h}_{-\alpha}(\sigma^m(a), \sigma^{i-1}(a)) > 0, \quad \text{for } 1 \leq i \leq n < m.$$

We can derive that

$${}_a^C\nabla_{(q,h)}^{\alpha} x(t_1) - {}_a^C\nabla_{(q,h)}^{\alpha} x(t_2) < 0.$$

On the other hand, due to $c < 0$ and $x(t_1) < x(t_2)$, we have

$${}_a^C\nabla_{(q,h)}^{\alpha} x(t_1) - {}_a^C\nabla_{(q,h)}^{\alpha} x(t_2) = c(x(t_1) - x(t_2)) > 0,$$

which is a contradiction. □

From Corollary 4.8 in [38], we know that

$$x(t) = x(a) E_{\alpha,1}^{a,c}(t)$$

is the solution of (3.15). As an application of Lemma 3.12, we have the following corollary.

Corollary 3.13 *The* (q,h)*-Mittag–Leffler function*

$$E^{a,c}_{\alpha,1}(t)=\sum_{k=0}^{\infty}c^k\hat{h}_{\alpha k}(t,a)$$

is monotonically decreasing if $c<0$*.*

When $c(t)=c$ is a negative constant, we can give a simpler proof for Lemma 3.5.

Lemma 3.14 *Assume* $c<0$ *and* $x(t)$ *is a solution of equation* (3.15) *satisfying* $x(a)>0$*. Then* $x(t)>0$ *for* $t\in\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$*.*

Proof Let $t=\sigma^k(a), k\geq 1$. Note that

$$\begin{aligned}\hat{h}_{-\alpha}(\sigma^k(a),\sigma^{i-1}(a))&=\big(\nu(\sigma^k(a))\big)^{-\alpha}\begin{bmatrix}-\alpha+k-i\\ k-i\end{bmatrix}_{\tilde{q}}\\&=\big(\nu(\sigma^k(a))\big)^{-\alpha}\frac{\Gamma_{\tilde{q}}(-\alpha+k-i+1)}{\Gamma_{\tilde{q}}(k-i+1)\Gamma_{\tilde{q}}(1-\alpha)}\\&>0,\end{aligned}$$

for $1\leq i\leq k$. From Lemma 3.12, we have $\nabla_{(q,h)}x(\sigma^i(a))<0,\ i=1,2,\cdots,k$. Hence,

$$\begin{aligned}{}^{C}_{a}\nabla^{\alpha}_{(q,h)}x(t)&=\int_a^t\hat{h}_{-\alpha}(t,\rho(s))\nabla_{(q,h)}x(s)\nabla s\\&=\sum_{i=1}^{k}\hat{h}_{-\alpha}(\sigma^k(a),\sigma^{i-1}(a))\nabla_{(q,h)}x(\sigma^i(a))\nu(\sigma^i(a))\\&<0.\end{aligned}$$

As a result, $x(t)$ of the RHS of (3.15) should be positive when $c<0$. □

Theorem 3.15 *If* $c<0$ *and* $x(t)$ *is a solution of equation* (3.15) *satisfying* $x(a)>0$*, then*

$$\lim_{t\to\infty}x(t)=0.$$

Proof From Lemma 3.12 and Lemma 3.14, we know the limit exits. Arguing by contradiction, we assume $\lim\limits_{t\to\infty}x(t)=A>0$. By Lemma 3.12 we get that $x(t)$ is decreasing. Then, by equation (3.12), we have

$$\begin{aligned}x(t)-x(a)&=\int_a^t\hat{h}_{\alpha-1}(t,\rho(s))cx(s)\nabla s\\&<cx(t)\int_a^t\hat{h}_{\alpha-1}(t,\rho(s))\nabla s\\&=-cx(t)\hat{h}_{\alpha}(t,s)\Big|_{s=a}^{t}\\&=cx(t)\hat{h}_{\alpha}(t,a).\end{aligned}\tag{3.16}$$

Letting t tend to infinity, the LHS of (3.16) becomes

$$\lim_{t\to\infty} x(t) - x(a) = A - x(a) < 0,$$

while the RHS gives for $t = \sigma^k(a)$, $k \ge 1$,

$$\begin{aligned}
\lim_{t\to\infty} cx(t)\hat{h}_\alpha(t,a) &= cA \lim_{t\to\infty} \hat{h}_\alpha(t,a) \\
&= cA \lim_{t\to\infty} \Big(\nu(t)\Big)^\alpha \lim_{t\to\infty} \begin{bmatrix} \alpha+k-1 \\ k-1 \end{bmatrix}_{\tilde{q}} \\
&= cA \lim_{t\to\infty} \Big(\nu(t)\Big)^\alpha \lim_{k\to\infty} \begin{bmatrix} \alpha+k-1 \\ k-1 \end{bmatrix}_{\tilde{q}} \\
&\overset{(2.8)}{=} cA \lim_{t\to\infty} [\sigma^k(a)]^\alpha (1-\tilde{q})^\alpha \lim_{k\to\infty} \frac{\Gamma_{\tilde{q}}(\alpha+k)}{\Gamma_{\tilde{q}}(k)\Gamma_{\tilde{q}}(\alpha+1)} \\
&= cA \cdot (+\infty) \frac{1}{\Gamma_{\tilde{q}}(\alpha+1)} \\
&= -\infty,
\end{aligned}$$

where we use

$$\begin{aligned}
\lim_{k\to\infty} [\sigma^k(a)]^\alpha &= \lim_{k\to\infty} \left[q^k a + [k]_q h\right]^\alpha \\
&= \lim_{k\to\infty} \left(q^k a + [k]_q h + h\frac{\tilde{q}}{1-\tilde{q}}\right)^\alpha \\
&= \lim_{k\to\infty} \left(q^k a + \frac{1-q^k}{1-q} h + \frac{h}{q-1}\right)^\alpha \\
&= \lim_{k\to\infty} \left(q^k (a + \frac{h}{q-1})\right)^\alpha \\
&= +\infty
\end{aligned}$$

and

$$\lim_{k\to\infty} \frac{\Gamma_{\tilde{q}}(\alpha+k)}{\Gamma_{\tilde{q}}(k)} \overset{(3.14)}{=} (1-\tilde{q})^{-\alpha}.$$

This yields a contradiction and we conclude that $\lim_{k\to\infty} x(t) = 0$. □

Theorem A. Assume $0 < \alpha < 1$ and there is a constant b such that $c(t) \le b < 0$, for $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. Then any solution, $x(t)$, of the nabla Caputo (q,h)-fractional difference equation

$${}^C_a\nabla^\alpha_{(q,h)} x(t) = c(t)x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \tag{3.17}$$

with $x(a) > 0$ satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Proof Assume $y(t)$ is the solution of Caputo nabla fractional equation

$${}^C_a\nabla^\alpha_{(q,h)} y(t) = by(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$$

satisfying $y(a) = 2x(a) > 0$. From Theorem 3.6, we have

$$0 < x(t) < y(t),$$

for $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. From Theorem 3.15, we get that

$$0 \leq \lim_{t\to\infty} x(t) \leq \lim_{t\to\infty} y(t) = 0.$$

This completes the proof. □

Remark 3.16 *(see Example 5.2) If $c(t)$ in Theorem A is not a negative constant, then the solution of the equation (3.17) may not be monotonically decreasing.*

4. Asymptotic behavior for Riemann–Liouville (q,h)-fractional difference equation

From the Leibniz rule on time scales in [7, 8], it is easy to get the following nabla (q,h)-difference Leibniz rule.

Lemma 4.1 (Leibniz rule) *Assume $f : \tilde{\mathbb{T}}^{a}_{(q,h)} \times \tilde{\mathbb{T}}^{a}_{(q,h)} \to \mathbb{R}$. Then*

$$\nabla_{(q,h)} \int_a^t f(t,s)\nabla s = \int_a^t {}_t\nabla_{(q,h)} f(t,s)\nabla s + f(\rho(t),t)$$

for $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, where it is understood that the expression ${}_t\nabla_{(q,h)} f(t,s)$ in the last integral means the (partial) (q,h)-difference of f with respect to t.

Theorem 4.2 *Assume $0 < \alpha < 1$, $c(t) \leq 0$, and $t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}$. Then any solution of the equation*

$$ {}_a\nabla^{\alpha}_{(q,h)} x(t) = c(t)x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)} \tag{4.1}$$

satisfying $x(\sigma(a)) > 0$ is positive on $\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$.

Proof Since $m = \lceil \alpha \rceil = 1$, we have that

$$\begin{aligned}
{}_a\nabla^{\alpha}_{(q,h)} x(t) &\overset{\text{Def. } 2.2}{=} \nabla_{(q,h)}\, {}_a\nabla^{-(1-\alpha)}_{(q,h)} x(t) \\
&\overset{(2.9)}{=} \nabla_{(q,h)} \int_a^t \hat{h}_{-\alpha}\big(t,\rho(\tau)\big)x(\tau)\nabla\tau \\
&\overset{\text{Lem. } 4.1}{=} \int_a^t \hat{h}_{-\alpha-1}\big(t,\rho(\tau)\big)x(\tau)\nabla\tau + \hat{h}_{-\alpha}(\rho(t),\rho(t))x(t) \\
&= \int_a^t \hat{h}_{-\alpha-1}\big(t,\rho(\tau)\big)x(\tau)\nabla\tau.
\end{aligned}$$

Letting $t = \sigma^k(a), k \geq 2$, we get that

$$ {}_{a}\nabla^{\alpha}_{(q,h)}x(t) = \sum_{i=1}^{k} \hat{h}_{-\alpha-1}\big(t, \sigma^{i-1}(a)\big)x\big(\sigma^{i}(a)\big)\nu\big(\sigma^{i}(a)\big). $$

Using that $x(t)$ is a solution of (4.1), we get that for $t = \sigma^k(t)$, $k \geq 2$,

$$ \begin{aligned} &\Big(\hat{h}_{-\alpha-1}\big(t, \sigma^{k-1}(a)\big)\nu(t) - c(t)\Big)x\big(\sigma^{k}(a)\big) \\ &\qquad = -\sum_{i=1}^{k-1} \hat{h}_{-\alpha-1}\big(t\sigma^{i-1}(a)\big)x\big(\sigma^{i}(a)\big)\nu\big(\sigma^{i}(a)\big). \end{aligned} \tag{4.2} $$

We now show that since $0 < \alpha < 1$, $\hat{h}_{-\alpha-1}\big(t, \sigma^{i-1}(a)\big) < 0$ for $1 \leq i \leq k-1$. First note that

$$ \begin{aligned} \hat{h}_{-\alpha-1}\big(t, \sigma^{i-1}(a)\big) &\overset{(3.1)}{=} (\nu(t))^{-\alpha-1} \begin{bmatrix} -\alpha - 1 + k - i \\ k - i \end{bmatrix}_{\tilde{q}} \\ &\overset{(2.3)}{=} (\nu(t))^{-\alpha-1} \frac{\Gamma_{\tilde{q}}(-\alpha + k - i)}{\Gamma_{\tilde{q}}(k - i + 1)\Gamma_{\tilde{q}}(-\alpha)}. \end{aligned} $$

Also, since $\Gamma_{\tilde{q}}(-\alpha + k - i) > 0, \Gamma_{\tilde{q}}(k - i + 1) > 0$, for $k - i \geq 1$ and $\Gamma_{\tilde{q}}(-\alpha) < 0$, we have

$$ \hat{h}_{-\alpha-1}\big(t, \sigma^{i-1}(a)\big) < 0, $$

for $1 \leq i \leq k-1$.

Note that from Lemma 3.2 we have that

$$ \hat{h}_{-\alpha-1}\big(t, \sigma^{k-1}(a)\big) > 0. $$

Since $c(t) \leq 0, \nu(t) > 0$ and $\hat{h}_{-\alpha-1}\big(t, \sigma^{k-1}(a)\big) > 0$, the coefficient of $x(t) = x(\sigma^k(a))$ on the left hand side of (4.2) is positive. Since $\hat{h}_{-\alpha-1}\big(t, \sigma^{i-1}(a)\big) < 0$ and $\nu(t) > 0$ for $1 \leq i \leq k-1$, and $x(\sigma(a)) > 0$, it follows from (4.2) and the strong induction principle that $x(t) = x(\sigma^k(a)) > 0$, for $t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}$.
This completes the proof. □

Remark 4.3 *In the above theorem we proved that if $x(t)$ is a solution of the IVP*

$$ {}_{a}\nabla^{\alpha}_{(q,h)}x(t) = c(t)x(t), \quad x(\sigma(a)) = A, \tag{4.3} $$

then $x(t)$ is a solution of the summation equation (4.2) satisfying the initial condition $x(\sigma(a)) = A$. Since the steps in this calculation are reversible we have that if $x(t)$ is a solution of the summation equation (4.2) satisfying the initial condition $x(\sigma(a)) = A$, then $x(t)$ is a solution of the IVP (4.3). Hence, we say the IVP (4.3) is equivalent to the summation equation (4.2) along with the initial condition $x(\sigma(a)) = A$. Now solving the summation equation (4.2) for $x(\sigma^k(a))$, we obtain the summation equation

$$ \begin{aligned} x(\sigma^k(a)) &= -\frac{\sum_{i=1}^{k-1} \hat{h}_{-\alpha-1}\big(\sigma^k(a), \sigma^{i-1}(a)\big)x\big(\sigma^{i}(a)\big)\nu\big(\sigma^{i}(a)\big)}{\hat{h}_{-\alpha-1}\big(\sigma^k(a), \sigma^{k-1}(a)\big)\nu(\sigma^k(a)) - c(\sigma^k(a))} \\ &= -\frac{\sum_{i=1}^{k-1} (\nu(\sigma^k(a)))^{-\alpha-1} \begin{bmatrix} -\alpha - 1 + k - i \\ k - i \end{bmatrix}_{\tilde{q}} x\big(\sigma^{i}(a)\big)\nu\big(\sigma^{i}(a)\big)}{(\nu(\sigma^k(a)))^{-\alpha} - c(\sigma^k(a))}, \end{aligned} \tag{4.4} $$

for $k \geq 2$, where we used formula (3.1). Hence, the IVP (4.3) and the summation equation (4.4) with the initial condition $x(\sigma(a)) = A$ are equivalent. Note that if $x(\sigma(a)) = A$, then by the summation equation (4.4) with $k = 2$ we get that $x(\sigma^2(a))$ is uniquely determined, and then when $k = 3$ the values of $x(t)$ at $\sigma(a)$ and $\sigma^2(a)$ uniquely determine the value of $x(\sigma^3(a))$. Hence, it follows by strong mathematical induction that the solution of the IVP (4.3) exists and is unique on $\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. Since the IVP (4.3) and the summation equation with the the initial condition $x(\sigma(a)) = A$ have a unique solution, we have that the IVP (4.3) has a unique solution and this solution exists on $\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. Furthermore, it is easy to see that the nabla (q,h)-fractional equation has just one linearly independent solution $x_1(t)$ on $\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ and $x(t) := \alpha x_1(t)$ is a general solution of the nabla fractional equation (4.1).

Lemma 4.4 *Assume that $\alpha \in \mathbb{R}$, $0 < \tilde{q} < 1$. Then for $[t] \neq 0$,*

$$\nu(\sigma(a))\hat{h}_{-\alpha-1}(t,a) + \hat{h}_{-\alpha}(t,\sigma(a)) = \hat{h}_{-\alpha}(t,a). \tag{4.5}$$

Proof

On the one hand,

$${}_s\nabla_{(q,h)}\hat{h}_{-\alpha}(t,\sigma(a)) = \frac{\hat{h}_{-\alpha}(t,\sigma(a)) - \hat{h}_{-\alpha}(t,a)}{\nu(\sigma(a))}.$$

On the other hand, using Lemma 3.4 (2),

$${}_s\nabla_{(q,h)}\hat{h}_{-\alpha}(t,\sigma(a)) = -\hat{h}_{-\alpha-1}(t,a).$$

Thus,

$$\frac{\hat{h}_{-\alpha}(t,\sigma(a)) - \hat{h}_{-\alpha}(t,a)}{\nu(\sigma(a))} = -\hat{h}_{-\alpha-1}(t,a).$$

Hence, (4.5) holds. □

Lemma 4.5 *Assume $f : \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$, $0 < \alpha < 1$, $q > 1$. Then*

$${}_{\sigma(a)}\nabla^{-(1-\alpha)}_{(q,h)}\nabla_{(q,h)}f(t) = \nabla_{(q,h)}\ {}_{\sigma(a)}\nabla^{-(1-\alpha)}_{(q,h)}f(t) - f(\sigma(a))\hat{h}_{-\alpha}(t,\sigma(a)) \tag{4.6}$$

$$= {}_{\sigma(a)}\nabla^{\alpha}_{(q,h)}f(t) - f(\sigma(a))\hat{h}_{-\alpha}(t,\sigma(a)) \tag{4.7}$$

and

$${}_{\sigma(a)}\nabla^{-(1-\alpha)}_{(q,h)}\nabla_{(q,h)}f(t) = \nabla_{(q,h)}\ {}_{a}\nabla^{-(1-\alpha)}_{(q,h)}f(t) - f(\sigma(a))\hat{h}_{-\alpha}(t,a) \tag{4.8}$$

$$= {}_{a}\nabla^{\alpha}_{(q,h)}f(t) - f(\sigma(a))\hat{h}_{-\alpha}(t,a). \tag{4.9}$$

Proof Integrating by parts and using the Leibniz rule (Lemma 4.1), we have

$$
\begin{aligned}
&{}_{\sigma(a)}\nabla_{(q,h)}^{-(1-\alpha)}\nabla_{(q,h)}f(t)\\
=&\int_{\sigma(a)}^{t}\hat{h}_{-\alpha}(t,\rho(\tau))\nabla_{(q,h)}f(\tau)\nabla\tau\\
=&\Big(\hat{h}_{-\alpha}(t,\tau)f(\tau)\Big)\Big|_{\tau=\sigma(a)}^{t}+\int_{\sigma(a)}^{t}\hat{h}_{-\alpha-1}(t,\rho(\tau))f(\tau)\nabla\tau\\
=&{}_{\sigma(a)}\nabla_{(q,h)}^{\alpha}f(t)-f(\sigma(a))\hat{h}_{-\alpha}(t,\sigma(a)),
\end{aligned}
$$

and hence (4.7) holds. By the Leibniz rule (Lemma 4.1) we get that

$$
\nabla_{(q,h)\sigma(a)}\nabla_{(q,h)}^{-(1-\alpha)}={}_{\sigma(a)}\nabla_{(q,h)}^{\alpha}f(t). \tag{4.10}
$$

Using (4.10) and (4.7), we see that (4.6) holds.

Next we will show that (4.8) holds.

$$
\begin{aligned}
&\nabla_{(q,h)a}\nabla_{(q,h)}^{-(1-\alpha)}f(t)\\
=&\nabla_{(q,h)}\int_{a}^{t}\hat{h}_{-\alpha}(t,\rho(s))f(s)\nabla s\\
=&\int_{a}^{t}\hat{h}_{-\alpha-1}(t,\rho(s))f(s)\nabla s+\hat{h}_{-\alpha}(\rho(t),\rho(t))f(t)\\
=&\int_{a}^{t}\hat{h}_{-\alpha-1}(t,\rho(s))f(s)\nabla s\\
=&\int_{a}^{\sigma(a)}\hat{h}_{-\alpha-1}(t,\rho(s))f(s)\nabla s+\int_{\sigma(a)}^{t}\hat{h}_{-\alpha-1}(t,\rho(s))f(s)\nabla s\\
=&\hat{h}_{-\alpha-1}(t,a)f(\sigma(a))\nu(\sigma(a))+\int_{\sigma(a)}^{t}\hat{h}_{-\alpha-1}(t,\rho(s))f(s)\nabla s.
\end{aligned}
$$

Subtracting $f(\sigma(a))\hat{h}_{-\alpha}(t,a)$ from both sides of this last equation we get

$$
\begin{aligned}
&\nabla_{(q,h)a}\nabla_{(q,h)}^{-(1-\alpha)}f(t)-f(\sigma(a))\hat{h}_{-\alpha}(t,a)\\
=&\hat{h}_{-\alpha-1}(t,a)f(\sigma(a))\nu(\sigma(a))+\int_{\sigma(a)}^{t}\hat{h}_{-\alpha-1}(t,\rho(s))f(s)\nabla s-f(\sigma(a))\hat{h}_{-\alpha}(t,a)\\
=&\int_{\sigma(a)}^{t}\hat{h}_{-\alpha-1}(t,\rho(s))f(s)\nabla s+f(\sigma(a))\Big(\nu(\sigma(a))\hat{h}_{-\alpha-1}(t,a)-\hat{h}_{-\alpha}(t,a)\Big)\\
\overset{(4.5)}{=}&\int_{\sigma(a)}^{t}\hat{h}_{-\alpha-1}(t,\rho(s))f(s)\nabla s-f(\sigma(a))\hat{h}_{-\alpha}(t,\sigma(a))\\
\overset{(4.7)}{=}&{}_{\sigma(a)}\nabla_{(q,h)}^{-(1-\alpha)}\nabla_{(q,h)}f(t).
\end{aligned}
$$

Hence, (4.8) holds. Since by the Leibniz rule $\nabla_{(q,h)}\ {}_a\nabla_{(q,h)}^{-(1-\alpha)} = {}_a\nabla_{(q,h)}^{\alpha} f(t)$, we have that (4.8) implies (4.9) holds. $\square$

Replacing α by $1-\alpha$ in Lemma 4.5, we get the following corollary, which will be used later.

Corollary 4.6 *Assume* $f : \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \to \mathbb{R}$, $0 < \alpha < 1$, $q > 1$. *Then:*

$$ {}_{\sigma(a)}\nabla_{(q,h)}^{-\alpha}\nabla_{(q,h)} f(t) = \nabla_{(q,h)}\ {}_{\sigma(a)}\nabla_{(q,h)}^{-\alpha} f(t) - f(\sigma(a))\hat{h}_{\alpha-1}(t,\sigma(a)), \tag{4.11}$$

$$ {}_{\sigma(a)}\nabla_{(q,h)}^{-\alpha}\nabla_{(q,h)} f(t) = \nabla_{(q,h)}\ {}_{a}\nabla_{(q,h)}^{-\alpha} f(t) - f(\sigma(a))\hat{h}_{\alpha-1}(t,a). \tag{4.12}$$

Lemma 4.7 *Assume* $f : \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \times \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \to \mathbb{R}$, $0 < \alpha < 1$. *Then we have that*

$$\int_a^t \int_{\rho(\xi)}^t f(\eta,\xi)\nabla\eta\nabla\xi = \int_a^t \int_a^\eta f(\eta,\xi)\nabla\xi\nabla\eta. \tag{4.13}$$

Proof Let

$$\phi(t) := \int_a^t \int_{\rho(\xi)}^t f(\eta,\xi)\nabla\eta\nabla\xi - \int_a^t \int_a^\eta f(\eta,\xi)\nabla\xi\nabla\eta,$$

$$F_1(t,\xi) := \int_{\rho(\xi)}^t f(\eta,\xi)\nabla\eta, \text{ and } F_2(t,\eta) := \int_a^\eta f(t,\eta)\nabla\xi.$$

From Lemma 4.1, we have

$$\begin{aligned}
\nabla_{(q,h)}\phi(t) &= \nabla_{(q,h)}\int_a^t F_1(t,\xi)\nabla\xi - \nabla_{(q,h)}\int_a^t F_2(t,\eta)\nabla\eta \\
&= \int_a^t \nabla_{(q,h)}F_1(t,\xi)\nabla\xi + F_1(\rho(t),t) - \int_a^t f(t,\xi)\nabla\xi \\
&= \int_a^t \nabla_{(q,h)}F_1(t,\xi)\nabla\xi + \int_{\rho(t)}^{\rho(t)} f(\eta,t)\nabla\eta - \int_a^t f(t,\xi)\nabla\xi \\
&= \int_a^t f(t,\xi)\nabla\xi - \int_a^t f(t,\xi)\nabla\xi = 0,
\end{aligned}$$

and since $\phi(a) = 0$, we see that $\phi(t) \equiv 0$. This completes the proof. $\square$

The following lemma is from [10, page 14].

Lemma 4.8 *Let* $\alpha > 0$, $\beta \in \mathbb{R}$ *and* $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$. *Then*

$$ {}_a\nabla_{(q,h)}^{-\alpha}\hat{h}_\beta(t,a) = \hat{h}_{\alpha+\beta}(t,a). \tag{4.14}$$

A special case of the composition rule in the next lemma is Theorem 3.107 [18]. Also, Cermak and Nechvatal [12] pointed out that this composition rule follows from a very general formula that they proved. With the aid of Lemma 4.6 and Lemma 4.7 we now give a straightforward proof of this result for the convenience of the reader.

Lemma 4.9 *Let* $\alpha, \beta \in \mathbb{R}^+$ *and* $f : \tilde{\mathbb{T}}^a_{(q,h)} \to \mathbb{R}$. *Then*

$$ {}_a\nabla^{-\alpha}_{(q,h)}\ {}_a\nabla^{-\beta}_{(q,h)} f(t) = {}_a\nabla^{-(\alpha+\beta)}_{(q,h)} f(t). $$

Proof For $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ we have that

$$
\begin{aligned}
{}_a\nabla^{-\alpha}_{(q,h)}\ {}_a\nabla^{-\beta}_{(q,h)} f(t) &\overset{\text{Def. 2.1}}{=} \int_a^t \hat{h}_{\alpha-1}\big(t, \rho(s)\big)\, {}_a\nabla^{-\beta}_{(q,h)} f(s) \nabla s \\
&= \int_a^t \hat{h}_{\alpha-1}\big(t, \rho(s)\big) \int_a^s \hat{h}_{\beta-1}(s, \rho(\tau)) f(\tau) \nabla\tau \nabla s \\
&= \int_a^t \int_a^s \Big(\hat{h}_{\alpha-1}\big(t, \rho(s)\big) \hat{h}_{\beta-1}(s, \rho(\tau)) f(\tau)\Big) \nabla\tau \nabla s \\
&\overset{(4.13)}{=} \int_a^t \int_{\rho(\tau)}^t \Big(\hat{h}_{\alpha-1}\big(t, \rho(s)\big) \hat{h}_{\beta-1}(s, \rho(\tau)) f(\tau)\Big) \nabla s \nabla\tau \\
&= \int_a^t f(\tau) \int_{\rho(\tau)}^t \Big(\hat{h}_{\alpha-1}\big(t, \rho(s)\big) \hat{h}_{\beta-1}(s, \rho(\tau))\Big) \nabla s \nabla\tau \\
&= \int_a^t f(\tau) \Big({}_{\rho(\tau)}\nabla^{-\alpha}_{(q,h)} \hat{h}_{\beta-1}(t, \rho(\tau))\Big) \nabla\tau \\
&\overset{(4.14)}{=} \int_a^t f(\tau) \hat{h}_{\alpha+\beta-1}(t, \rho(\tau)) \nabla\tau \\
&= {}_a\nabla^{-(\alpha+\beta)}_{(q,h)} f(t).
\end{aligned}
$$

Also,

$$ {}_a\nabla^{-\alpha}_{(q,h)}\ {}_a\nabla^{-\beta}_{(q,h)} f(a) = {}_a\nabla^{-(\alpha+\beta)}_{(q,h)} f(a). $$

□

Theorem B. Assume $0 < \alpha < 1$, $c(t) \leq 0$, $t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}$, and $x(t)$ is a solution of the equation

$$ {}_a\nabla^{\alpha}_{(q,h)} x(t) = c(t)x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)} \tag{4.15} $$

satisfying $x(\sigma(a)) > 0$. Then

$$ \lim_{t\to\infty} x(t) = 0. $$

Proof Applying the operator ${}_{\sigma(a)}\nabla^{-\alpha}_{(q,h)}$ to each side of equation (4.15), we obtain

$$ {}_{\sigma(a)}\nabla^{-\alpha}_{(q,h)}\ {}_a\nabla^{\alpha}_{(q,h)} x(t) = {}_{\sigma(a)}\nabla^{-\alpha}_{(q,h)} c(t)x(t), $$

which can be written in the form

$$ {}_{\sigma(a)}\nabla^{-\alpha}_{(q,h)} \nabla_{(q,h)}\ {}_a\nabla^{-(1-\alpha)}_{(q,h)} x(t) = {}_{\sigma(a)}\nabla^{-\alpha}_{(q,h)} c(t)x(t). $$

Using (4.12), we get that

$$\begin{aligned}&\nabla_{(q,h)}\ {}_a\nabla^{-\alpha}_{(q,h)}{}_a\nabla^{-(1-\alpha)}_{(q,h)}x(t)-\hat{h}_{\alpha-1}(t,a)\left({}_a\nabla^{-(1-\alpha)}_{(q,h)}x(t)\Big|_{t=\sigma(a)}\right)\\&={}_{\sigma(a)}\nabla^{-\alpha}_{(q,h)}c(t)x(t).\end{aligned}$$

Using

$$\begin{aligned}{}_a\nabla^{-(1-\alpha)}_{(q,h)}x(t)\Big|_{t=\sigma(a)}&=\int_a^{\sigma(a)}\hat{h}_{-\alpha}(\sigma(a),\rho(s))x(s)\nabla s\\&=\hat{h}_{-\alpha}(\sigma(a),a)x(\sigma(a))\nu(\sigma(a))\\&=(\nu(\sigma(a)))^{-\alpha}\begin{bmatrix}-\alpha\\0\end{bmatrix}_{\tilde{q}}x(\sigma(a))\nu(\sigma(a))\\&=[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha}x(\sigma(a)),\end{aligned}$$

we see that

$$\begin{aligned}\nabla_{(q,h)a}\nabla^{-\alpha}_{(q,h)a}\nabla^{-(1-\alpha)}_{(q,h)}x(t)=&\hat{h}_{\alpha-1}(t,a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha}x(\sigma(a))\\&+{}_{\sigma(a)}\nabla^{-\alpha}_{(q,h)}c(t)x(t).\end{aligned}$$

It is easy to see for $t\geq s$, letting $t=\sigma^n(s)=\sigma^{n+1}(\rho(s))$, for $n\geq 0$. Using the composition rule (Lemma 4.9), we have

$${}_a\nabla^{-\alpha}_{(q,h)}\ {}_a\nabla^{-(1-\alpha)}_{(q,h)}x(t)={}_a\nabla^{-1}_{(q,h)}x(t)$$

and

$$\begin{aligned}\nabla_{(q,h)}\ {}_a\nabla^{-1}_{(q,h)}x(t)&=\nabla_{(q,h)}\int_a^t\hat{h}_0(t,\rho(s))x(s)\nabla s\\&=\nabla_{(q,h)}\int_a^t x(s)\nabla s\\&=x(t),\end{aligned}$$

where we used

$$\begin{aligned}\hat{h}_0(t,\rho(s))&=(\nu(t))^0\begin{bmatrix}0+n+1-1\\n\end{bmatrix}_{\tilde{q}}\\&=1.\end{aligned}$$

Hence, we get that

$$\begin{aligned}x(t)=&\hat{h}_{\alpha-1}(t,a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha}x(\sigma(a))\\&+\int_{\sigma(a)}^t\hat{h}_{\alpha-1}(t,\rho(s))c(s)x(s)\nabla s.\end{aligned}\tag{4.16}$$

Since $x(\sigma(a)) > 0,\ 0 < \alpha < 1,\ c(t) \le 0$, and from Theorem 4.2, we have $x(t) > 0$, for $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$. Then

$$\begin{aligned}\hat{h}_{\alpha-1}(t,\rho(s)) &= (\nu(t))^{\alpha-1}\begin{bmatrix}\alpha-1+n\\ n\end{bmatrix}_{\tilde{q}}\\ &= (\nu(t))^{\alpha-1}\frac{\Gamma_{\tilde{q}}(\alpha+n)}{\Gamma_{\tilde{q}}(n)\Gamma_{\tilde{q}}(\alpha)}.\end{aligned}$$

Note that $\Gamma_{\tilde{q}}(\alpha+n) > 0,\ \Gamma_{\tilde{q}}(\alpha) > 0$, and $\Gamma_{\tilde{q}}(n+1) > 0$, so $\hat{h}_{\alpha-1}(t,\rho(s)) > 0$. It follows that

$$\int_{\sigma(a)}^{t}\hat{h}_{\alpha-1}(t,\rho(s))c(s)x(s)\nabla s < 0.$$

Since $c(s) \le 0$ and from (4.16), we get that (taking $t = \sigma^k(a)$)

$$0 < x(\sigma^k(a)) \le \hat{h}_{\alpha-1}(\sigma^k(a),a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha}x(\sigma(a)). \tag{4.17}$$

Note that

$$\begin{aligned}\hat{h}_{\alpha-1}(\sigma^k(a),a) &= (\nu(\sigma^k(a)))^{\alpha-1}\begin{bmatrix}\alpha-1+k-1\\ k-1\end{bmatrix}_{\tilde{q}}\\ &= (\nu(\sigma^k(a)))^{\alpha-1}\frac{\Gamma_{\tilde{q}}(\alpha-1+k)}{\Gamma_{\tilde{q}}(k)\Gamma_{\tilde{q}}(\alpha)}\\ &= [\sigma^k(a)]^{\alpha-1}(1-\tilde{q})^{\alpha-1}\frac{\Gamma_{\tilde{q}}(\alpha-1+k)}{\Gamma_{\tilde{q}}(k)\Gamma_{\tilde{q}}(\alpha)}\\ &\to 0,\end{aligned} \tag{4.18}$$

where we used

$$\begin{aligned}\lim_{k\to\infty}[\sigma^k(a)]^{\alpha-1} &= \lim_{k\to\infty}\left[q^k a + [k]_q h\right]^{\alpha-1}\\ &= \lim_{k\to\infty}\left(q^k a + [k]_q h + h\frac{\tilde{q}}{1-\tilde{q}}\right)^{\alpha-1}\\ &= \lim_{k\to\infty}\left(q^k a + \frac{1-q^k}{1-q}h + \frac{h}{q-1}\right)^{\alpha-1}\\ &= \lim_{k\to\infty}\left(q^k(a + \frac{h}{q-1})\right)^{\alpha-1}\\ &= 0,\end{aligned}$$

and

$$\lim_{k\to\infty}\frac{\Gamma_{\tilde{q}}(\alpha-1+k)}{\Gamma_{\tilde{q}}(k)} = \left(\frac{1}{1-\tilde{q}}\right)^{\alpha-1} \quad (\text{using}(3.14)).$$

From (4.17) and (4.18), we have the desired result:

$$\lim_{k\to\infty}x\big(\sigma^k(a)\big) = 0.$$

□

Remark 4.10 *(see Example 5.4) If $c(t)$ in Theorem A is not a constant, then there are solutions of the equation (4.15) satisfying $x(\sigma(a)) > 0$ that are not monotonically decreasing.*

We end this section by considering solutions $x(t)$ of the αth order nabla fractional (q,h)-difference equation

$$ {}_a\nabla^{\alpha}_{(q,h)}x(t) = c(t)x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)} $$

satisfying $x(\sigma(a)) < 0$. By making the transformation $x(t) = -y(t)$ and using Theorem A, we get the following theorem.

Theorem C. Assume $0 < \alpha < 1$, $c(t) \le 0$, $t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}$, and $x(t)$ is a solution of the equation

$$ {}_a\nabla^{\alpha}_{(q,h)}x(t) = c(t)x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)} $$

satisfying $x(\sigma(a)) < 0$. Then $x(t) < 0$ for $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ and

$$ \lim_{t\to\infty} x(t) = 0. $$

5. Examples

In this section we give four examples to illustrate Lemma 3.5, Theorem 3.15, Theorem A, Remark 3.16, Theorem 4.2, Theorem B, and Remark 4.10.

Using the recursion formula (2.4), we can get the corresponding numerical solutions through formulae (3.13) and (4.4) using MATLAB. For the sake of convenience in plotting figures, letting $x_k = x(\sigma^k(a))$, we plot the points (k, x_k) instead of the points $(t, x(t))$.

Example 5.1 *Consider the initial value problem in the form*

$$ \begin{aligned} &{}^C_1\nabla^{0.3}_{(1.2,0.5)}x(t) = -2x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma(1)}_{(1.2,0.5)}, \\ &x(1) = 0.1. \end{aligned} $$

The solution $x(t)$ is given in Figure 2. Note that $x(t)$ monotonically tends to zero as $t \to \infty$.

Example 5.2 *Consider the initial value problem in the form*

$$ \begin{aligned} &{}^C_1\nabla^{0.3}_{(1.2,0.5)}x(t) = -(0.1 + \sin^2 t)\cdot x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma(1)}_{(1.2,0.5)}, \\ &x(1) = 0.1. \end{aligned} $$

There is a constant $b = -0.1$ such that

$$ c(t) = -(0.1 + \sin^2 t) \le b. $$

It is evident that the solution $x(t)$ is asymptotically stable from Theorem A.

We plot the solution $x(t)$ in Figure 3. Note that $x(t)$ tends to zero as $t \to \infty$, but is not monotonically decreasing.

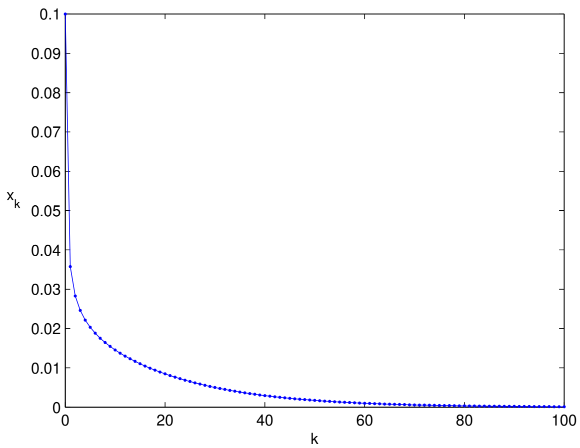

Figure 2. Asymptotic behavior of $x(t)$ for $a=1$, $\nu=0.3$, $q=1.2$, $h=0.5$, $x(a)=0.1$, $c(t)=-2$.

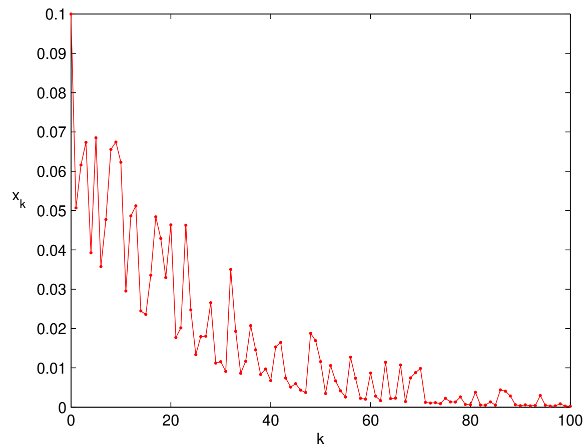

Figure 3. Asymptotic behavior of $x(t)$ for $a=1$, $\nu=0.3$, $q=1.2$, $h=0.5$, $x(a)=0.1$, $c(t)=-(0.1+\sin^2 t)$.

Example 5.3 *Consider the initial value problem in the form*

$$
\begin{aligned}
&{}_1\nabla^{0.7}_{(1.2,0.5)}x(t) = -0.00001x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma^2(1)}_{(1.2,0.5)},\\
&x(\sigma(1)) = 0.1.
\end{aligned}
$$

We plot the solution $x(t)$ in Figure 4. Note that $x(t)$ monotonically tends to zero as $t \to \infty$.

Example 5.4 *Consider the initial value problem in the form*

$$
\begin{aligned}
&{}_1\nabla^{0.9}_{(1.2,0.5)}x(t) = -\frac{1}{100}\sin^{10} t \cdot x(t), \qquad t \in \tilde{\mathbb{T}}^{\sigma^2(1)}_{(1.2,0.5)},\\
&x(\sigma(1)) = 0.1.
\end{aligned}
$$

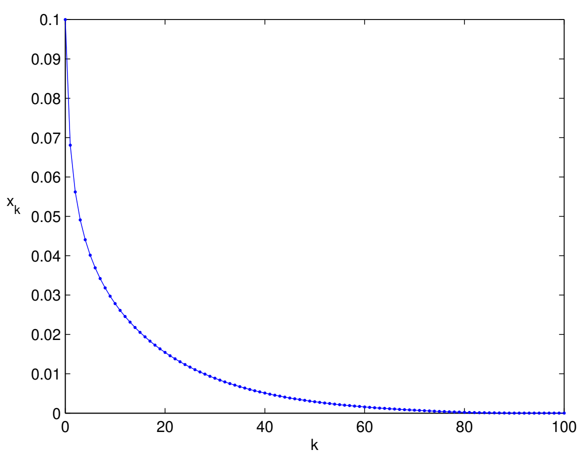

Figure 4. Asymptotic behavior of $x(t)$ for $a=1$, $\nu=0.7$, $q=1.2$, $h=0.5$, $x(\sigma(1))=0.1$, $c(t)=-0.00001$.

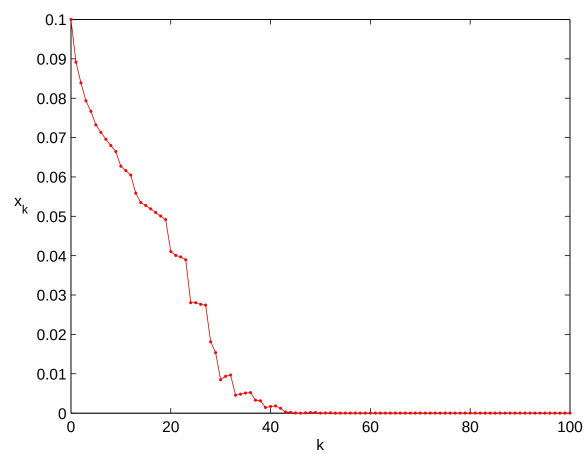

Figure 5. Asymptotic behavior of $x(t)$ for $a=1$, $\nu=0.9$, $q=1.2$, $h=0.5$, $x(\sigma(1))=0.1$, $c(t)=-\frac{1}{100}\sin^{10} t$.

It is easy to check that

$$c(t) = -\frac{1}{100}\sin^{10} t \leq 0.$$

Hence, the solution $x(t)$ is asymptotically stable according to Theorem B.

We plot the solution $x(t)$ in Figure 5. Note that $x(t)$ tends to zero as $t \to \infty$, but is not monotonically decreasing.

6. Conclusions

In this paper, we present two asymptotic results on linear nabla fractional difference equations originating from the recent papers as well as their new extensions on the (q,h)-time scale. Furthermore, the fractional difference equations' numerical solutions are given to support the established theories. It remains to discuss the asymptotic stability of (0.1) and (0.2) with $c(t) > 0$, which is still an open problem.

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