Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 2243 – 2259
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doi:10.3906/mat-1803-62

Approximation by Chlodowsky type of Szász operators based on Boas–Buck-type polynomials

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Received: 11.03.2018 • **Accepted/Published Online:** 11.06.2018 • **Final Version:** 27.09.2018

Abstract: A Chlodowsky variant of generalized Szász-type operators involving Boas–Buck-type polynomials is considered and some convergence properties of these operators by using a weighted Korovkin-type theorem are given. A Voronoskaja-type theorem is proved. The convergence properties of these operators in a weighted space of functions defined on $[0,\infty)$ are studied. The theoretical results are exemplified choosing the special cases of Boas–Buck polynomials, namely Appell-type polynomials, Laguerre polynomials, and Charlier polynomials.

Key words: Szász operators, modulus of continuity, rate of convergence, weighted space, Boas–Buck-type polynomials

1. Introduction and preliminaries

In recent years, there is an increasing interest to study linear positive operators based on certain polynomials, such as Appell polynomials, Laguerre polynomials, Charlier polynomials, Sheffer polynomials, and Hermite polynomials. In 1969, Jakimovski and Leviatan [16] introduced Szász-type operators by using Appell polynomials, as follows:

$$P_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_k(x)$, $k\geq 0$, are the Appell polynomials defined by $g(u)e^{ux}=\sum_{k=0}^{\infty}p_k(x)u^k$ and $g(u)=\sum_{k=0}^{\infty}a_ku^k$ is an analytic function in the disk $|u|<R$, $R>1$ and $g(1)\neq 0$. If $g(u)=1$, then $p_k(x)=\frac{x^k}{k!}$ (see [9]) and we obtain Szász–Mirakjan operators:

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right).$$

Very recently, the detailed approximation properties of the Szász-type operators were studied in [1, 2, 17]. The Stancu-type generalization of operators (1.1) was introduced by Atakut and Büyükyazici [5]. Ismail [14] obtained another generalization of the Szasz operators (1.1) by means of Sheffer polynomials. The bivariate Chlodowsky–Szász operators involving Appell polynomials were studied by Sidharth et al. [23]. Recently, Sucu

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2010 *AMS Mathematics Subject Classification:* 41A10, 41A25, 41A36

et al. [25] constructed positive linear operators with the help of Boas–Buck-type polynomials.The Boas–Buck-type polynomials [15] have generating functions of the form

$$A(t)B(xH(t)) = \sum_{k=0}^{\infty} p_k(x)t^k, \tag{1.2}$$

where A, B, and H are analytic functions:

$$A(t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0, \tag{1.3}$$

$$B(t) = \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0 \ (r \geq 0), \tag{1.4}$$

$$H(t) = \sum_{r=1}^{\infty} h_r t^r, \quad h_1 \neq 0. \tag{1.5}$$

We will restrict ourselves to the Boas–Buck-type polynomials satisfying:

i) $A(1) \neq 0, \quad H'(1) = 1, \quad p_k(x) \geq 0, \quad k = 0, 1, 2, \ldots,$

ii) $B : \mathbb{R} \to (0, \infty)$,

iii) The power series (1.2)–(1.5) converging for $|t| < R \ (R > 1)$.

Sucu et al. [25] introduced the following positive linear operators involving the Boas–Buck-type polynomials

$$B_n(f; x) := \frac{1}{A(1)B(nxH(1))} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.6}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

Let $H(t) = t$. Then operators (1.6) reduce to the operators given by Varma et al. [27]. If $B(t) = e^t$, operators (1.6) reduce to the operators given by Ismail [14]. For $H(t) = t$ and $B(t) = e^t$, one can get operators (1.1). In addition, if we choose $A(t) = 1$, we get the Szász–Mirakjan operators [26].

In this paper, we consider the Chlodowsky [8] variant of Szász-type operators given by (1.6), involving the Boas–Buck-type polynomials, as follows:

$$B_n^*(f; x) := \frac{1}{A(1)B\left(\frac{n}{b_n} xH(1)\right)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n} x\right) f\left(\frac{k}{n} b_n\right), \tag{1.7}$$

where (b_n) is a positive increasing sequence such that

$$\lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

The rest of the paper is organized as follows. In the next section some local approximation results by the generalized Szász operators are obtained. Furthermore, in Section 3, a Voronovskaja-type asymptotic theorem will be proved. Section 4 is devoted to studying some convergence properties of these operators in weighted spaces with weighted norm on the interval $[0,\infty)$ by using the weighted Korovkin-type theorems [11, 12]. In order to show the relevance of the results, in the last section some numerical examples are given.

Note that throughout the paper we will assume that the operators B_n^* are positive and we use the following test functions:

$$e_i(x) = x^i, \quad i \in \{0,1,2,3,4\}.$$

Also, we consider

$$\lim_{y\to\infty} \frac{B^{(k)}(y)}{B(y)} = 1, \text{ for } k \in \{1,2,3,\dots,r\}, r \in \mathbb{N}. \tag{1.8}$$

Some recent papers on the topic dealing with different classes of polynomials as well as Korovkin-type approximation theorems and Voronovskaja-type approximation theorems can be consulted by readers (cf. [6, 7, 10, 18–21, 24]).

2. Local approximation properties of B_n^*

We denote by $C_E(\mathbb{R}_0^+)$ the set of all continuous functions f on $\mathbb{R}_0^+ = [0,\infty)$ with the property that $|f(x)| \le \beta e^{\alpha x}$ for all $x \ge 0$ and some positive finite α and β. For a fixed $r \in \mathbb{N}$, we denote $C_E^*(\mathbb{R}_0^+) = \{f \in C_E(\mathbb{R}_0^+) : f', f^{(2)}, \cdots, f^{(r)} \in C_E(\mathbb{R}_0^+)\}$. Using equality (1.2) and the fundamental properties of the B_n^* operators, one can easily get the following lemmas:

Lemma 2.1 *For all $x \in [0,\infty)$, we have*

$$B_n^*(e_0;x) = 1,$$

$$B_n^*(e_1;x) = \frac{B'\big(\frac{n}{b_n}xH(1)\big)}{B\big(\frac{n}{b_n}xH(1)\big)}x + \frac{b_n}{n}\frac{A'(1)}{A(1)},$$

$$B_n^*(e_2;x) = \frac{B^{(2)}\big(\frac{n}{b_n}xH(1)\big)}{B\big(\frac{n}{b_n}xH(1)\big)}x^2 + \frac{b_n}{n}\frac{B'\big(\frac{n}{b_n}xH(1)\big)\big[A(1)+2A'(1)+H^{(2)}(1)A(1)\big]}{A(1)B\big(\frac{n}{b_n}xH(1)\big)}x + \frac{b_n^2}{n^2}\frac{A'(1)+A^{(2)}(1)}{A(1)},$$

$$\begin{aligned}B_n^*(e_3;x) &= \frac{B^{(3)}\big(\frac{n}{b_n}xH(1)\big)}{B\big(\frac{n}{b_n}xH(1)\big)}x^3 + \Big(3A'(1)+3H^{(2)}(1)A(1)+3A(1)\Big)\frac{B^{(2)}\big(\frac{n}{b_n}xH(1)\big)}{A(1)B\big(\frac{n}{b_n}xH(1)\big)}\frac{b_n}{n}x^2 \\ &+ \Big(3A^{(2)}(1)+3H^{(2)}(1)A'(1)+H^{(3)}(1)A(1)+6A'(1)+3H^{(2)}(1)A(1)+A(1)\Big)\frac{B'\big(\frac{n}{b_n}xH(1)\big)}{A(1)B\big(\frac{n}{b_n}xH(1)\big)}\frac{b_n^2}{n^2}x \\ &+ \Big(A^{(3)}(1)+3A^{(2)}(1)+A'(1)\Big)\frac{b_n^3}{A(1)n^3},\end{aligned}$$

$$B_n^*(e_4;x) = \frac{B^{(4)}\big(\frac{n}{b_n}xH(1)\big)}{B\big(\frac{n}{b_n}xH(1)\big)}x^4 + \Big(4A'(1)+6H^{(2)}(1)A(1)+6A(1)\Big)\frac{B^{(3)}\big(\frac{n}{b_n}xH(1)\big)}{A(1)B\big(\frac{n}{b_n}xH(1)\big)}\frac{b_n}{n}x^3$$
$$+\Big(6A^{(2)}(t)+12H^{(2)}(1)+A'(1)+4H^{(3)}(1)A(1)+3H^{(2)}(1)^2A(1)+18A'(1)+18H^{(2)}(1)A(1)+7A(1)\Big)$$
$$\cdot\frac{B^{(2)}\big(\frac{n}{b_n}xH(1)\big)}{A(1)B\big(\frac{n}{b_n}xH(1)\big)}\frac{b_n^2}{n^2}x^2 + \Big(4A^{(3)}(1)+6A^{(2)}(1)H^{(2)}(1)+4A'(1)H^{(3)}(1)+A(1)H^{(4)}(1)+18A^{(2)}(1)$$
$$+18H^{(2)}(1)A'(1)+6H^{(3)}(1)A(1)+14A'(1)+7H^{(2)}(1)A(1)+A(1)\Big)\frac{B'\big(\frac{n}{b_n}xH(1)\big)}{A(1)B\big(\frac{n}{b_n}xH(1)\big)}\frac{b_n^3}{n^3}x$$
$$+\Big(A^{(4)}(1)+6A^{(3)}(1)+7A^{(2)}(1)+A'(1)\Big)\frac{b_n^4}{A(1)n^4}.$$

Proof From the generating functions of the Boas–Buck-type polynomials given by (1.2), we obtain

$$\sum_{k=0}^{\infty}p_k\Big(\frac{n}{b_n}x\Big) = A(1)B\Big(\frac{n}{b_n}xH(1)\Big),$$
$$\sum_{k=0}^{\infty}kp_k\Big(\frac{n}{b_n}x\Big) = A'(1)B\Big(\frac{n}{b_n}xH(1)\Big) + \frac{n}{b_n}xA(1)B'\Big(\frac{n}{b_n}xH(1)\Big),$$
$$\sum_{k=0}^{\infty}k^2p_k\Big(\frac{n}{b_n}x\Big) = \frac{n^2}{b_n^2}x^2A(1)B^{(2)}\Big(\frac{n}{b_n}xH(1)\Big) + \frac{n}{b_n}x\big(A(1)+2A'(1)+H^{(2)}(1)A(1)\big)B'\Big(\frac{n}{b_n}xH(1)\Big)$$
$$+\big(A'(1)+A^{(2)}(1)\big)B\Big(\frac{n}{b_n}xH(1)\Big),$$

$$\sum_{k=0}^{\infty}k^3p_k\Big(\frac{n}{b_n}x\Big) = \frac{n^3}{b_n^3}x^3A(1)B^{(3)}\Big(\frac{n}{b_n}xH(1)\Big) + \frac{n^2}{b_n^2}x^2\big(3A'(1)+3H^{(2)}(1)A(1)+3A(1)\big)B^{(2)}\Big(\frac{n}{b_n}xH(1)\Big)$$
$$+\frac{n}{b_n}x\big(3A^{(2)}(1)+3H^{(2)}(1)A'(1)+H^{(3)}(1)A(1)+6A'(1)+3H^{(2)}(1)A(1)+A(1)\big)B'\Big(\frac{n}{b_n}xH(1)\Big)$$
$$+\big(A^{(3)}(1)+3A^{(2)}(1)+A'(1)\big)B\Big(\frac{n}{b_n}xH(1)\Big),$$
$$\sum_{k=0}^{\infty}k^4p_k\Big(\frac{n}{b_n}x\Big) = \frac{n^4}{b_n^4}x^4A(1)B^{(4)}\Big(\frac{n}{b_n}xH(1)\Big) + \frac{n^3}{b_n^3}x^3\big(4A'(1)+6H^{(2)}(1)A(1)+6A(1)\big)B^{(3)}\Big(\frac{n}{b_n}xH(1)\Big)$$
$$+\frac{n^2}{b_n^2}x^2\big(6A^{(2)}(t)+12H^{(2)}(1)+A'(1)+4H^{(3)}(1)A(1)+3H^{(2)}(1)^2A(1)+18A'(1)+18H^{(2)}(1)A(1)$$
$$+7A(1)\big)B^{(2)}\Big(\frac{n}{b_n}xH(1)\Big)+\Big(4A^{(3)}(1)+6A^{(2)}(1)H^{(2)}(1)]+4A'(1)H^{(3)}(1)+A(1)H^{(4)}(1)+18A^{(2)}(1)$$
$$+18H^{(2)}(1)A'(1)+6H^{(3)}(1)A(1)+14A'(1)+7H^{(2)}(1)A(1)+A(1)\Big)\frac{n}{b_n}xB'\Big(\frac{n}{b_n}xH(1)\Big)$$
$$+\big(A^{(4)}(1)+6A^{(3)}(1)+7A^{(2)}(1)+A'(1)\big)B\Big(\frac{n}{b_n}xH(1)\Big).$$

In view of these equalities, we get our desired results. □

Lemma 2.2 *The operators (1.7) verify:*

$$B_n^*\big((e_1-x);x\big)=\frac{B'\big(\frac{n}{b_n}xH(1)\big)-B\big(\frac{n}{b_n}xH(1)\big)}{B\big(\frac{n}{b_n}xH(1)\big)}x+\frac{b_n}{n}\frac{A'(1)}{A(1)},$$

$$B_n^*\big((e_1-x)^2;x\big)=\frac{B^{(2)}\big(\frac{n}{b_n}xH(1)\big)-2B'\big(\frac{n}{b_n}xH(1)\big)+B\big(\frac{n}{b_n}xH(1)\big)}{B\big(\frac{n}{b_n}xH(1)\big)}x^2$$

$$+\frac{b_n}{n}\frac{(A(1)+2A'(1)+A(1)H^{(2)}(1))B'\big(\frac{n}{b_n}xH(1)\big)-2A'(1)B\big(\frac{n}{b_n}xH(1)\big)}{A(1)B\big(\frac{n}{b_n}xH(1)\big)}x$$

$$+\frac{b_n^2}{n^2}\frac{A'(1)+A^{(2)}(1)}{A(1)},$$

$$B_n^*\big((e_1-x)^4;x\big)=\frac{x^4}{B\big(\frac{n}{b_n}xH(1)\big)}\left[B^{(4)}\left(\frac{n}{b_n}xH(1)\right)-4B^{(3)}\left(\frac{n}{b_n}xH(1)\right)+6B^{(2)}\left(\frac{n}{b_n}xH(1)\right)\right.$$

$$\left.-\ 4B'\left(\frac{n}{b_n}xH(1)\right)+B\left(\frac{n}{b_n}xH(1)\right)\right]+\frac{2x^3b_n}{nA(1)B\left(\frac{n}{b_n}xH(1)\right)}$$

$$\cdot\left[\left(2A'(1)+3A(1)H^{(2)}(1)+3A(1)\right)B^{(3)}\left(\frac{n}{b_n}xH(1)\right)-6\left(A'(1)+A(1)H^{(2)}(1)+A(1)\right)\right.$$

$$\cdot\ B^{(2)}\left(\frac{n}{b_n}xH(1)\right)+3\left(2A'(1)+A(1)H^{(2)}(1)+A(1)\right)B'\left(\frac{n}{b_n}xH(1)\right)$$

$$\left.-2A'(1)B\left(\frac{n}{b_n}xH(1)\right)\right]+\frac{x^2b_n^2}{n^2A(1)B\left(\frac{n}{b_n}xH(1)\right)}\left[\left(6A^{(2)}(1)+12A'(1)H^{(2)}(1)\right.\right.$$

$$\left.+\ 4A(1)H^{(3)}(1)+21A(1)H^{(2)}(1)+18A'(1)+7A(1)\right)B^{(2)}\left(\frac{n}{b_n}xH(1)\right)$$

$$+\left(-12A^{(2)}(1)-12A'(1)H^{(2)}(1)-4A(1)H^{(3)}(1)\right.$$

$$\left.\left.-24A'(1)-12A(1)H^{(2)}(1)-4A(1)\right)B'\left(\frac{n}{b_n}xH(1)\right)+6\left(A^{(2)}(1)+A'(1)\right)B\left(\frac{n}{b_n}xH(1)\right)\right]$$

$$+\frac{xb_n^3}{n^3A(1)B\left(\frac{n}{b_n}xH(1)\right)}\left[\left(4A^{(3)}(1)+6A^{(2)}(1)H^{(2)}(1)+A(1)H^{(4)}(1)+18A^{(2)}(1)\right.\right.$$

$$\left.+\ 18A'(1)H^{(2)}(1)+6A(1)H^{(3)}(1)+14A'(1)+7A(1)H^{(2)}(1)+A(1)+4A'(1)H^{(3)}(1)\right)B'\left(\frac{n}{b_n}xH(1)\right)$$

$$\left.+\left(-4A^{(3)}(1)-12A^{(2)}(1)-4A'(1)\right)B\left(\frac{n}{b_n}xH(1)\right)\right]+\frac{b_n^4}{n^4A(1)}\left(A^{(4)}(1)+6A^{(3)}(1)+7A^{(2)}(1)+A'(1)\right).$$

Theorem 2.1 *Let the condition (1.8) hold for* $k=1,2$*. Then for* $f\in C_E(\mathbb{R}_0^+)$*, the operators* B_n^* *converge uniformly to* f *on* $[0,a]$*,* $a>0$*, as* $n\to\infty$*.*

Proof According to Lemma 2.1, taking into account equality (1.8), we find

$$\lim_{n\to\infty} B_n^*(e_i;x) = e_i(x), \quad i \in \{0,1,2\}.$$

The above convergence is verified uniformly in each compact subset of $[0,\infty)$. Applying the Korovkin theorem [3], we obtain the desired result. □

3. Voronovskaja-type theorem

In order to study the Voronovskaja-type theorem for the Chlodowsky variant of Szász-type operators involving the Boas–Buck polynomials, we consider the following assumptions on the analytic functions A, B, and H:

$$\lim_{n\to\infty} \frac{n}{b_n} \frac{B'\left(\frac{n}{b_n}xH(1)\right) - B\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)} = l_1(x); \tag{3.1}$$

$$\lim_{n\to\infty} \frac{n}{b_n} \frac{B^{(2)}\left(\frac{n}{b_n}xH(1)\right) - 2B'\left(\frac{n}{b_n}xH(1)\right) + B\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)} = l_2(x); \tag{3.2}$$

$$\lim_{n\to\infty} \left(\frac{n}{b_n}\right)^2 \frac{1}{B\left(\frac{n}{b_n}xH(1)\right)} \left[B^{(4)}\left(\frac{n}{b_n}xH(1)\right) - 4B^{(3)}\left(\frac{n}{b_n}xH(1)\right) + 6B^{(2)}\left(\frac{n}{b_n}xH(1)\right)\right.$$
$$\left. - \; 4B'\left(\frac{n}{b_n}xH(1)\right) + B\left(\frac{n}{b_n}xH(1)\right)\right] = l_3(x); \tag{3.3}$$

$$\lim_{n\to\infty} \frac{n}{b_n} \frac{1}{B\left(\frac{n}{b_n}xH(1)\right)A(1)} \left[\left(2A'(1) + 3A(1)H^{(2)}(1) + 3A(1)\right) B^{(3)}\left(\frac{n}{b_n}xH(1)\right) - 6\left(A'(1)\right.\right.$$
$$\left. + \; A(1)H^{(2)}(1) + A(1)\right) B^{(2)}\left(\frac{n}{b_n}xH(1)\right) + 3\left(2A'(1) + A(1)H^{(2)}(1) + A(1)\right) B'\left(\frac{n}{b_n}xH(1)\right)$$
$$\left. -2A'(1)B\left(\frac{n}{b_n}xH(1)\right)\right] = l_4(x). \tag{3.4}$$

Using assumptions (3.1)–(3.4) and Lemma 2.2, we can obtain the following result:

Lemma 3.1 *The operators (1.7) verify:*

i) $\lim_{n\to\infty} \frac{n}{b_n} B_n^*\left(e_1 - x; x\right) = \eta_1(x)$,

ii) $\lim_{n\to\infty} \frac{n}{b_n} B_n^*\left((e_1 - x)^2; x\right) = \eta_2(x)$,

ii) $\lim_{n\to\infty} \left(\frac{n}{b_n}\right)^2 B_n^*\left((e_1 - x)^4; x\right) = \eta_3(x)$,

where

$$\eta_1(x) = xl_1(x) + \frac{A'(1)}{A(1)}, \ \eta_2(x) = x^2 l_2(x) + x\left(1 + H^{(2)}(1)\right),$$

$$\eta_3(x) = x^4 l_3(x) + 2x^3 l_4(x) + 3x^2\left(H^{(2)}(1)^2 + 2H^{(2)}(1) + 1\right).$$

Theorem 3.1 *(Voronovskaja-type theorem) For every* $f \in C_E(\mathbb{R}_0^+)$ *such that* $f', f^{(2)} \in C_E(\mathbb{R}_0^+)$, *we have*

$$\lim_{n\to\infty} \frac{n}{b_n}[B_n^*(f;x) - f(x)] = \left(xl_1(x) + \frac{A'(1)}{A(1)}\right) f'(x) + \frac{1}{2}\left(x^2 l_2(x) + x(1 + H^{(2)}(1)\right) f^{(2)}(x),$$

uniformly with respect to $x \in [0,a]$, $a > 0$, *where* $l_i(x)$, $i = 1,2$, *are defined in (3.1) and (3.2).*

Proof Using the classical Taylor expansion of f yields

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f^{(2)}(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2, \tag{3.5}$$

where $\varepsilon(t,x) \in C_E(\mathbb{R}_0^+)$ and $\lim_{t\to x} \varepsilon(t,x) = 0$.

Applying the operator $B_n^*(\cdot, x)$ on both sides of (3.5), we have

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{b_n}\left[B_n^*(f;x) - f(x)\right] &= \lim_{n\to\infty} \frac{n}{b_n} B_n^*(e_1 - x; x) f'(x) + \lim_{n\to\infty} \frac{n}{b_n} B_n^*((e_1 - x)^2; x)\frac{f^{(2)}(x)}{2} \\ &+ \lim_{n\to\infty} \frac{n}{b_n} B_n^*(\varepsilon(t,x)(t-x)^2; x).\end{aligned} \tag{3.6}$$

Using the Cauchy–Schwarz inequality in the last term of the right side of (3.6), we get

$$\frac{n}{b_n} B_n^*(\varepsilon(t,x)(t-x)^2; x) \le \sqrt{B_n^*(\varepsilon^2(t,x);x)}\sqrt{\left(\frac{n}{b_n}\right)^2 B_n((e_1 - x)^4; x)}.$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, it follows from Theorem 2.1 that

$$\lim_{n\to\infty} B_n^*\left(\varepsilon^2(t,x); x\right) = \varepsilon^2(x,x) = 0,$$

uniformly with respect to $x \in [0,a]$. Now, from (3.6) and Lemma 3.1, we have the required result. □

4. Approximation properties in weighted spaces

Since the uniform norm is not valid to compute the rate of convergence in the case of a boundless function defined on the noncompact interval $[0,\infty)$, in this section we give approximation properties of the operators B_n^* on the weighted spaces of continuous functions with exponential growth on $[0,\infty)$. In order to obtain this result, we consider the following weighted spaces of functions that are defined on $[0,\infty)$. Let $\rho(x)$ be the weighted function and M_f be a positive constant. Considering the following sets of functions,

$$B_\rho(\mathbb{R}_0^+) = \{f : \mathbb{R}_0^+ \to \mathbb{R} : |f(x)| \le M_f \rho(x)\},$$

$$C_\rho(\mathbb{R}_0^+) = \{f \in B_\rho(\mathbb{R}_0^+) : f \text{ is continuous}\},$$

$$C_\rho^*(\mathbb{R}_0^+) = \left\{f \in C_\rho(\mathbb{R}_0^+) : \lim_{n\to\infty} \frac{f(x)}{\rho(x)} = K_f < \infty\right\},$$

it is obvious that $C_\rho^*(\mathbb{R}_0^+) \subset C_\rho(\mathbb{R}_0^+) \subset B_\rho(\mathbb{R}_0^+)$. The space $B_\rho(\mathbb{R}_0^+)$ is a normed linear space with the following norm:

$$\|f\|_\rho = \sup_{x\in\mathbb{R}_0^+} \frac{|f(x)|}{\rho(x)}. \tag{4.1}$$

Lemma 4.1 *([11, 12]) The sequence of positive linear operators $(B_n)_{n\geq 1}$ act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ if and only if there exists a positive constant k such that*

$$B_n(\rho;x) \leq k\rho(x), \quad i.e.$$

$$\|B_n(\rho;x)\|_\rho \leq k.$$

Theorem 4.1 *([11, 12]) Let $(B_n)_{n\geq 1}$ be the sequence of positive linear operators that act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ such that*

$$\lim_{n\to\infty} \|B_n(t^i;x) - x^i\|_\rho = 0, \quad i \in \{0,1,2\}.$$

Then for any function $f \in C_\rho^k(\mathbb{R}_0^+)$,

$$\lim_{n\to\infty} \|B_n f - f\|_\rho = 0.$$

Lemma 4.2 *Let $\rho(x) = 1 + x^2$ be a weight function. If $f \in C_\rho(\mathbb{R}_0^+)$, then*

$$\|B_n^*(\rho;x)\|_\rho \leq 1 + M,$$

under the equality (1.8) for $k = 1,2$.

Proof Using Lemma 2.1, one has

$$B_n^*(\rho;x) = 1 + \frac{B^{(2)}\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)}x^2 + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)+H^{(2)}(1)A(1)\right)B'\left(\frac{n}{b_n}xH(1)\right)}{A(1)B\left(\frac{n}{b_n}xH(1)\right)}x + \frac{b_n^2}{n^2}\frac{A'(1)+A^{(2)}(1)}{A(1)}$$

$$\begin{aligned}\|B_n^*(\rho;x)\|_\rho &= \sup_{x\geq 0}\left\{\frac{1}{1+x^2}\left(1 + \frac{B^{(2)}\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)}x^2 + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)+H^{(2)}(1)A(1)\right)B'\left(\frac{n}{b_n}xH(1)\right)}{A(1)B\left(\frac{n}{b_n}xH(1)\right)}x\right.\right.\\
&\quad\left.\left. + \frac{b_n^2}{n^2}\frac{A'(1)+A^{(2)}(1)}{A(1)}\right)\right\}\\
&\leq 1 + \frac{B^{(2)}\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)} + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)+H^{(2)}(1)A(1)\right)B'\left(\frac{n}{b_n}xH(1)\right)}{2A(1)B\left(\frac{n}{b_n}xH(1)\right)} + \frac{b_n^2}{n^2}\frac{A'(1)+A^{(2)}(1)}{A(1)}.\end{aligned}$$

Since $\lim_{n\to\infty}\frac{b_n}{n} = 0$ and using condition (1.8), there exists a positive M such that

$$\|B_n^*(\rho;x)\|_\rho \leq 1 + M.$$

This completes the proof. □

By using Lemma 4.2, we can easily see that the operators B_n^* defined by (1.7) act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$.

Theorem 4.2 *Let B_n^* be the sequence of positive linear operators defined by (1.7) that verify the conditions (1.8) for $k = 1,2$, and $\rho(x) = 1 + x^2$. Then, for each $f \in C_\rho^k(\mathbb{R}_0^+)$,*

$$\lim_{n\to\infty} \|B_n^*(f;x) - f(x)\|_\rho = 0.$$

Proof It is enough to prove that the conditions of the weighted Korovkin-type theorem given by Theorem 4.1 are satisfied. From Lemma 2.1, it follows immediately that

$$\lim_{n\to\infty} \|B_n^*(e_0;x) - e_0(x)\|_\rho = 0. \tag{4.2}$$

Using Lemma 2.1 and condition (1.8), we have

$$\begin{aligned}\|B_n^*(e_1;x) - e_1(x)\|_\rho &\le \sup_{x\ge 0} \frac{1}{1+x^2}\left\{\left|\frac{B'\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)} - 1\right| x + \frac{b_n}{n}\left|\frac{A'(1)}{A(1)}\right|\right\} \\ &= \frac{1}{2}\left|\frac{B'\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)} - 1\right| + \frac{b_n}{n}\left|\frac{A'(1)}{A(1)}\right|,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|B_n^*(e_1;x) - e_1(x)\|_\rho = 0. \tag{4.3}$$

By means of Lemma 2.1, we get

$$\begin{aligned}&\|B_n^*(e_2;x) - e_2(x)\|_\rho \\ &\le \sup_{x\ge 0}\left\{\left|\frac{B^{(2)}\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)} - 1\right|\frac{x^2}{1+x^2} + \frac{b_n}{n}\left|\frac{\left(A(1)+2A'(1)+H^{(2)}(1)A(1)\right)B'\left(\frac{n}{b_n}xH(1)\right)}{A(1)B\left(\frac{n}{b_n}xH(1)\right)}\right|\frac{x}{1+x^2}\right. \\ &\left. + \frac{b_n^2}{n^2}\left|\frac{A'(1)+A^{(2)}(1)}{A(1)}\right|\frac{1}{1+x^2}\right\} \\ &= \left|\frac{B^{(2)}\left(\frac{n}{b_n}xH(1)\right)}{B\left(\frac{n}{b_n}xH(1)\right)} - 1\right| + \frac{b_n}{2n}\left|\frac{\left(A(1)+2A'(1)+H^{(2)}(1)A(1)\right)B'\left(\frac{n}{b_n}xH(1)\right)}{A(1)B\left(\frac{n}{b_n}xH(1)\right)}\right| + \frac{b_n^2}{n^2}\left|\frac{A'(1)+A^{(2)}(1)}{A(1)}\right|.\end{aligned}$$

Using conditions (1.8), it follows that

$$\lim_{n\to\infty} \|B_n^*(e_2;x) - e_2(x)\|_\rho = 0. \tag{4.4}$$

From (4.2), (4.3), and (4.4), for $i \in \{0,1,2\}$, we have

$$\lim_{n\to\infty} \|B_n^*(t^i;x) - x^i\|_\rho = 0.$$

Applying Theorem 4.1, we obtain the desired result. □

5. Numerical examples

5.1. Appell polynomials

Appell [4] introduced the sequences of n-degree polynomials $\mathcal{R}_n, n = 0,1,\ldots$ satisfying the recursive relation

$$\mathcal{R}_n'(x) = n\mathcal{R}_{n-1}(x), n = 1,2,\ldots.$$

There exists a power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ $(a_0 \neq 0)$ such that

$$A(t)e^{tx} = \sum_{n=0}^{\infty} \mathcal{R}_n(x)t^n.$$

The Appell polynomials are Boas–Buck-type polynomials with $B(t) = e^t$ and $H(t) = t$. Denote by $B_n^{\mathcal{R}}$ the Chlodowsky variant of Szász-type operators involving the Appell polynomials.

Lemma 5.1 *For all $x \in [0, \infty)$, we have:*

i) $B_n^{\mathcal{R}}(e_0; x) = 1$;

ii) $B_n^{\mathcal{R}}(e_1; x) = x + \dfrac{b_n}{n} \dfrac{A'(1)}{A(1)}$;

ii) $B_n^{\mathcal{R}}(e_1; x) = x^2 + \dfrac{b_n}{n} \dfrac{1}{A(1)} \left(A(1) + 2A'(1)\right) x + \dfrac{b_n^2}{n^2} \dfrac{A'(1) + A^{(2)}(1)}{A(1)}$.

Lemma 5.2 *The operators $B_n^{\mathcal{R}}$ verify:*

i) $B_n^{\mathcal{R}}(e_1 - x; x) = \dfrac{b_n}{n} \dfrac{A'(1)}{A(1)}$;

ii) $B_n^{\mathcal{R}}((e_1 - x)^2; x) = \dfrac{b_n}{n} x + \dfrac{b_n^2}{n^2} \dfrac{A'(1) + A^{(2)}(1)}{A(1)}$;

ii) $$B_n^{\mathcal{R}}((e_1 - x)^4; x) = 3x^2 \frac{b_n^2}{n^2} + x \frac{b_n^3}{n^3 A(1)} \left[6A^{(2)}(1) + 10A'(1) + A(1)\right] + \frac{b_n^4}{n^4 A(1)} \left[A^{(4)}(1) + 6A^{(3)}(1) + 7A^{(2)}(1) + A'(1)\right].$$

Lemma 5.3 *The operators $B_n^{\mathcal{R}}$ verify:*

i) $\lim\limits_{n\to\infty} \dfrac{n}{b_n} B_n^{\mathcal{R}}(e_1 - x; x) = \dfrac{A'(1)}{A(1)}$;

ii) $\lim\limits_{n\to\infty} \dfrac{n}{b_n} B_n^{\mathcal{R}}((e_1 - x)^2; x) = x$;

ii) $\lim\limits_{n\to\infty} \dfrac{n^2}{b_n^2} B_n^{\mathcal{R}}((e_1 - x)^4; x) = 3x^2$.

Theorem 5.1 *(Voronovskaja-type theorem) For every $f \in C_E(\mathbb{R}_0^+)$ such that $f', f^{(2)} \in C_E(\mathbb{R}_0^+)$, we have*

$$\lim_{n\to\infty} \frac{n}{b_n} \left[B_n^{\mathcal{R}}(f; x) - f(x)\right] = \frac{A'(1)}{A(1)} f'(1) + \frac{1}{2} x f^{(2)}(x)$$

uniformly with respect to $x \in [0, a]$, $a > 0$.

Since the classical modulus of continuity of first order $\omega(f; \delta)$, $\delta > 0$ does not tend to zero, as $\delta \to 0$ on $[0; \infty)$, we consider the weighted modulus of continuity defined by Yüksel and Ispir (see [28]). Let $\rho(x) = 1 + x^2$ be the weighted function. For $f \in C_\rho^*(\mathbb{R}_0^+)$, the weighted modulus of continuity (see [28]) is given by

$$\Omega(f, \delta) = \sup_{x\geq 0} \sup_{0<|h|\leq\delta} \frac{|f(x+h) - f(x)|}{1 + (x+h)^2}.$$

The weighted modulus of continuity $\Omega(f,\delta)$ satisfies the following properties:

Lemma 5.4 *[28] If $f \in C^*_\rho(\mathbb{R}^+_0)$, then:*

1. *$\Omega(f,\delta)$ is a monotone increasing function of δ;*
2. *$\lim_{\delta\to 0^+} \Omega(f,\delta) = 0$;*
3. *for any $\lambda \in [0,\infty), \Omega(f,\lambda\delta) \le (1+\lambda)\Omega(f,\delta)$.*

Theorem 5.2 *Let $f \in C^*_\rho(\mathbb{R}^+_0)$. Then,*

$$\sup_{x\in[0,\infty)} \frac{\left|B_n^{\mathcal{R}}(f;x) - f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2 + K_0^{\mathcal{R}}(n) + \sqrt{K_1^{\mathcal{R}}(n)}\right)\Omega\left(f;\delta_n\right), \tag{5.1}$$

where $\delta_n = \sqrt{K_0^{\mathcal{R}}(n)}$ and

$$K_0^{\mathcal{R}}(n) = \frac{b_n}{2n} + \frac{b_n^2}{n^2}\frac{A'(1) + A^{(2)}(1)}{A(1)};$$

$$K_1^{\mathcal{R}}(n) = \frac{3}{4}\frac{b_n^2}{n^2} + \frac{3\sqrt{3}}{16}\frac{b_n^3}{n^3 A(1)}\left[6A^{(2)}(1) + 10A'(1) + A(1)\right] + \frac{b_n^4}{n^4 A(1)}\left[A^{(4)}(1) + 6A^{(3)}(1) + 7A^{(2)}(1) + A'(1)\right].$$

Proof For $x \in (0,\infty)$ and $\delta > 0$, by the definition of the weighted modulus of continuity and Lemma 5.4, we get

$$\begin{aligned}|f(t) - f(x)| &\le \left(1 + (x + |x-t|)^2\right)\Omega(f;|t-x|)\\ &\le 2(1+x^2)(1+(t-x)^2)\left(1 + \frac{|t-x|}{\delta}\right)\Omega(f;\delta).\end{aligned}$$

Applying $B_n^{\mathcal{R}}(\cdot;x)$ for both sides, we can write

$$\begin{aligned}|B_n^{\mathcal{R}}(f;x) - f(x)| &\le 2(1+x^2)\Omega(f;\delta)\\ &\times\left(1 + B_n^{\mathcal{R}}((t-x)^2;x) + B_n^{\mathcal{R}}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\right).\end{aligned} \tag{5.2}$$

From Lemma 5.2, it follows that

$$B_n^{\mathcal{R}}((e_1-x)^2;x) \le K_0^{\mathcal{R}}(n)(1+x^2) \text{ and } B_n^{\mathcal{R}}((e_1-x)^4;x) \le K_1^{\mathcal{R}}(n)(1+x^2)^2. \tag{5.3}$$

Applying the Cauchy–Schwarz inequality in the last term of (5.2), we get

$$\begin{aligned}B_n^{\mathcal{R}}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right) &\le \frac{1}{\delta}\left(B_n^{\mathcal{R}}((e_1-x)^2;x)\right)^{1/2}\\ &+ \frac{1}{\delta}\left(B_n^{\mathcal{R}}((e_1-x)^4;x)\right)^{1/2}\left(B_n^{\mathcal{R}}((e_1-x)^2(x);x)\right)^{1/2}.\end{aligned} \tag{5.4}$$

Combining the estimates (5.2)–(5.4) and taking $\delta := \delta_n = \sqrt{K_0^{\mathcal{R}}(n)}$, we reach the required result. □

The special choices of $A(t)$ give the following well-known polynomials.

5.1.1. Hermite polynomials of variance ν

If $A(t) = e^{-\frac{\nu t^2}{2}}$, then $\mathcal{R}_n(x) = H_n^{(\nu)}(x)$ is the Hermite polynomials of variance ν (see [22]), which have the explicit representation

$$H_n^{(\nu)}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \left(-\frac{\nu}{2}\right)^k \frac{1}{k!(n-2k)!} x^{n-2k},$$

where $[\cdot]$ denotes the integer part. Denote by B_n^H the Chlodowsky variant of Szász-type operators involving the Hermite polynomials. Then,

$$B_n^H(f;x) = e^{\frac{\nu}{2} - \frac{n}{b_n}x} \sum_{k=0}^{\infty} H_n^{(\nu)}(x) f\left(\frac{k}{n} b_n\right).$$

Under the assumption $\nu \leq 0$, restrictions i)–iii) and assumptions (1.8) for the operators $B_n^{(H)}$ are verified.

Theorem 5.3 *For every $f \in C_E(\mathbb{R}_0^+)$ such that $f', f^{(2)} \in C_E[0,\infty)$, we have*

$$\lim_{n\to\infty} \frac{n}{b_n} \left[B_n^H(f;x) - f(x)\right] = -\nu f'(1) + \frac{1}{2} x f^{(2)}(x)$$

uniformly with respect to $x \in [0,a]$, $a > 0$.

Theorem 5.4 *Let $f \in C_\rho^*(\mathbb{R}_0^+)$. Then,*

$$\sup_{x\in[0,\infty)} \frac{\left|B_n^H(f;x) - f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \leq 2\left(2 + K_0^H(n) + \sqrt{K_1^H(n)}\right) \Omega\left(f;\delta_n\right), \tag{5.5}$$

where $\delta_n = \sqrt{K_0^H(n)}$ and

$$K_0^H(n) = \frac{b_n}{2n} + \frac{b_n^2}{n^2}\nu(\nu-2);$$

$$K_1^H(n) = \frac{3}{4}\frac{b_n^2}{n^2} + \frac{3\sqrt{3}}{16}\frac{b_n^3}{n^3}\left(6\nu^2 - 16\nu + 1\right) + \frac{b_n^4}{n^4}\nu\left(\nu^3 - 12\nu^2 + 28\nu - 8\right).$$

Example 5.1 *Let $b_n = n^{\frac{1}{4}}$ and $f(x) = \frac{x}{x^2+1}$. Table 5.1 contains the error estimations of the functions f by the Chlodowsky variant of Szász-type operators involving the Hermite polynomials given in (5.5).*

5.1.2. Gould–Hopper polynomials

Putting $A(t) = e^{ht^m}$, then $R_n(x) = g_n^m(x,h)$ is the Gould–Hopper polynomial (see [13]). The explicit representation of these polynomials is given as:

$$g_k^m(x,h) = \sum_{j=0}^{\left[\frac{k}{m}\right]} \frac{1}{j!(k-mj)!} h^j x^{k-mj},$$

where $[\cdot]$ denotes the integer part.

Table 1. Error of approximation for B_n^H using weighted modulus of continuity.

n	$\nu=-0.0002$	$\nu=-2$	$\nu=-3$
10	1.299038994	1.299038994	1.299038994
10^2	1.299038994	1.299038994	1.299038994
10^3	0.811535257	1.129470981	1.242260837
10^4	0.280046538	0.301127918	0.318338677
10^5	0.089353761	0.090063954	0.090681082
10^6	0.028281473	0.028304073	0.028323844
10^7	0.008944188	0.008944904	0.008945530
10^8	0.002828426	0.002828448	0.002828468

Considering $h \geq 0$, then restrictions i)–iii) and assumptions (1.8) are verified. Let B_n^g be the Chlodowsky variant of Szász-type operators involving the Gould–Hopper polynomials. Then,

$$B_n^g(f;x) = e^{-h-\frac{n}{b_n}x} \sum_{k=0}^{\infty} g_k^m\left(\frac{n}{b_n}x, h\right) f\left(\frac{k}{n}\right).$$

Theorem 5.5 *For every $f \in C_E(\mathbb{R}_0^+)$ such that $f', f^{(2)} \in C_E[0,\infty)$, we have*

$$\lim_{n\to\infty} \frac{n}{b_n}\left[B_n^g(f;x) - f(x)\right] = hmf'(1) + \frac{1}{2}xf^{(2)}(x)$$

uniformly with respect to $x \in [0,a]$, $a > 0$.

Theorem 5.6 *Let $f \in C_\rho^*(\mathbb{R}_0^+)$. Then,*

$$\sup_{x\in[0,\infty)} \frac{|B_n^g(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq 2\left(2 + K_0^g(n) + \sqrt{K_1^g(n)}\right)\Omega\left(f;\delta_n\right), \tag{5.6}$$

where $\delta_n = \sqrt{K_0^g(n)}$ and

$$K_0^g(n) = \frac{b_n}{2n} + \frac{b_n^2}{n^2}m^2h(h+1);$$

$$K_1^g(n) = \frac{3}{4}\frac{b_n^2}{n^2} + \frac{3\sqrt{3}}{16}\frac{b_n^3}{n^3}\left(6h^2m^2 + 6hm^2 + 4hm + 1\right) + \frac{b_n^4}{n^4}hm^4\left(h^3 + 6h^2 + 7h + 1\right).$$

Example 5.2 *Let $b_n = n^{\frac{1}{4}}$, $m = 1.5$ and $f(x) = \dfrac{x}{x^2+1}$. Table 5.2 contains the error estimations of the functions f by the Chlodowsky variant of Szász-type operators involving the Gould–Hopper polynomials given in (5.6).*

Table 2. Error of approximation for B_n^g using weighted modulus of continuity.

n	$h1 = 0.002$	$h2 = 1.2$	$h3 = 3$
10	1.299038994	1.299038994	1.299038994
10^2	1.299038994	1.299038994	1.299038994
10^3	0.811568399	1.074960796	1.298635930
10^4	0.280047924	0.295853132	0.345655230
10^5	0.089353806	0.089881529	0.091729276
10^6	0.028281475	0.028298253	0.028357704
10^7	0.008944188	0.008944719	0.008946603
10^8	0.002828426	0.002828443	0.002828502

5.2. Laguerre polynomials

These polynomials are one of the most important classical orthogonal polynomials. The Laguerre polynomials are Boas–Buck-type polynomials with $A(t) = \dfrac{1}{(1-t)^{\alpha+1}}$, $H(t) = -\dfrac{t}{1-t}$, $B(t) = e^t$, where $0 \le t < 1$, $x < 0$ and α is a nonnegative integer. The Laguerre polynomial of degree k is defined as

$$L_k^{(\alpha)}(x) = \sum_{j=0}^{k} (-1)^j \binom{k+\alpha}{k-j} \frac{x^j}{j!}.$$

In order to ensure restrictions i)–iii) and assumptions (1.8), we modify the generating function of Laguerre polynomials as follows:

$$\frac{1}{(1-\frac{t}{2})^{\alpha+1}} e^{\frac{xt}{2(2-t)}} = \sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}\left(-\frac{x}{2}\right)}{2^k} t^k, 0 \le t < 2.$$

Denote by B_n^L the Chlodowsky variant of Szász-type operators involving the Laguerre polynomials. Then,

$$B_n^L(f;x) = \frac{1}{2^{\alpha+1} e^{\frac{nx}{2b_n}}} \sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}\left(-\frac{x}{2}\right)}{2^k} f\left(\frac{k}{n} b_n\right).$$

Theorem 5.7 *For every $f \in C_E(\mathbb{R}_0^+)$ such that $f', f^{(2)} \in C_E[0,\infty)$, we have*

$$\lim_{n\to\infty} \frac{n}{b_n}\left[B_n^L(f;x) - f(x)\right] = 2^{2\alpha+1}(\alpha+1)f'(1) + \frac{3}{2} x f^{(2)}(x)$$

uniformly with respect to $x \in [0,a]$, $a > 0$.

Theorem 5.8 *Let $f \in C_\rho^*(\mathbb{R}_0^+)$. Then,*

$$\sup_{x\in[0,\infty)} \frac{\left|B_n^L(f;x) - f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2 + K_0^L(n) + \sqrt{K_1^L(n)}\right) \Omega\left(f;\delta_n\right), \tag{5.7}$$

where $\delta_n = \sqrt{K_0^L(n)}$ *and*

$$K_0^L(n) = \frac{3b_n}{2n} + 2^{2\alpha+2}(\alpha+1)(\alpha+3)\frac{b_n^2}{n^2};$$

$$K_1^L(n) = \frac{21}{4}\frac{b_n^2}{n^2} + \frac{3\sqrt{3}}{16}\frac{b_n^3}{n^3}\left(72\cdot 4^\alpha\alpha^2 + 496\cdot 4^\alpha\alpha + 424\cdot 4^\alpha + 75\right) + 2^{2\alpha+2}(\alpha+1)\left(\alpha^3+15\alpha^2+63\alpha+75\right)\frac{b_n^4}{n^4}.$$

Example 5.3 *Let* $b_n = n^{\frac{1}{4}}$ *and* $f(x) = \dfrac{x}{x^2+1}$. *Table 5.3 contains the error estimations of the functions* f *by the Chlodowsky variant of Szász-type operators involving the Laguerre polynomials given in (5.7).*

Table 3. Error of approximation for B_n^L using weighted modulus of continuity.

n	$\alpha 1 = 0.002$	$\alpha 2 = 2$	$\alpha 3 = 3$
10	1.299040568	1.299040570	1.299040580
10^2	1.299040568	1.299040570	1.299040580
10^3	1.288882866	1.299040570	1.299040580
10^4	0.493107972	1.079677838	1.299040580
10^5	0.155072123	0.197421416	0.344434710
10^6	0.048994875	0.050517126	0.058139428
10^7	0.015492121	0.015540988	0.015805560
10^8	0.004898993	0.004900541	0.004908996

5.3. Charlier polynomials

These polynomials have generating functions of the form (see [15])

$$e^t\left(1-\frac{t}{a}\right)^x = \sum_{k=0}^{\infty} C_k^{(a)}(x)t^k,\ |t|<a,$$

where

$$C_k^{(a)}(x) = {}_2F_0\left(-k,-x;-\frac{1}{a}\right) = \sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}(-x)_j\left(\frac{1}{a}\right)^j$$

and $a>1$, $x\in[0,\infty)$, $(x)_0 = 1$, $(x)_j = x(x+1)\dots(x+j-1)$ for $j\geq 1$.

Therefore, the Charlier polynomials are the Boas–Buck-type polynomials with $A(t) = e^t$, $B(t) = e^t$, $H(t) = \ln\left(1-\dfrac{t}{a}\right)$. In order to ensure restrictions i)–iii) and assumptions (1.8), we modify the generating function of Charlier polynomials as follows:

$$e^t e^{-(a-1)x\ln\left(1-\frac{t}{a}\right)} = \sum_{k=0}^{\infty} C_k^{(a)}(-(a-1)x)t^k,\ |t|<a.$$

Let B_n^C be the Chlodowsky variant of Szász-type operators involving the Charlier polynomials. Then,

$$B_n^C(f;x) = \frac{1}{e}\left(1-\frac{1}{a}\right)^{\frac{n(a-1)x}{b_n}}\sum_{k=0}^{\infty} C_k^{(a)}(-(a-1)x)\, f\left(\frac{k}{n}\right).$$

Theorem 5.9 *For every* $f \in C_E(\mathbb{R}_0^+)$ *such that* $f', f^{(2)} \in C_E[0,\infty)$, *we have*

$$\lim_{n\to\infty} \frac{n}{b_n}\left[B_n^C(f;x) - f(x)\right] = f'(1) + \frac{a}{2(a-1)} x f^{(2)}(x)$$

uniformly with respect to $x \in [0,a]$, $a > 0$.

Theorem 5.10 *Let* $f \in C_\rho^*(\mathbb{R}_0^+)$. *Then,*

$$\sup_{x\in[0,\infty)} \frac{\left|B_n^C(f;x) - f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2 + K_0^C(n) + \sqrt{K_1^C(n)}\right)\Omega\left(f;\delta_n\right), \tag{5.8}$$

where $\delta_n = \sqrt{K_0^C(n)}$ *and*

$$K_0^C(n) = \frac{ab_n}{2(a-1)n} + \frac{b_n^2}{n^2};$$

$$K_1^C(n) = \frac{3(a+2)}{4(a-1)}\frac{b_n^2}{n^2} + \frac{3\sqrt{3}}{16}\frac{b_n^3}{n^3}\frac{a(17a^2-20a+9)}{(a-1)^3} + 15\frac{b_n^4}{n^4}.$$

Example 5.4 *Let* $b_n = n^{\frac{1}{4}}$ *and* $f(x) = \dfrac{x}{x^2+1}$. *Table 5.4 contains the error estimations of the functions* f *by the Chlodowsky variant of Szász-type operators involving the Charlier polynomials given in (5.8).*

Table 4. Error of approximation for B_n^C using weighted modulus of continuity.

n	$a1 = 1.5$	$a2 = 2$	$a3 = 3$
10	1.299040568	1.299039881	1.299039482
10^2	1.299040568	1.299039881	1.299039482
10^3	1.189062896	1.076740303	0.990627632
10^4	0.477014557	0.393996650	0.343471676
10^5	0.154507080	0.126301526	0.109453172
10^6	0.048976825	0.039994054	0.034638165
10^7	0.015491549	0.012648938	0.010954371
10^8	0.004898975	0.003999999	0.003464102

Acknowledgment

The work of the third author was financed by Lucian Blaga University of Sibiu research grants LBUS-IRG-2017-03.

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