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On the affine-periodic solutions of discrete dynamical systems

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Abstract: Affine periodicity is a generalization of the notion of conventional periodicity and it is a symmetry property for classes of functions. This study is concerned with the existence of (Q, T)-affine periodic solutions of discrete dynamical systems. Sufficient conditions for the main results are proposed due to discrete exponential dichotomy and fixed point theory. Obtained results are also implemented for some economical and biological models. In particular cases, our results cover some existing results in the literature for periodic, antiperiodic, or quasiperiodic solutions of difference equations.

Key words: Affine periodic, affine symmetric, exponential dichotomy, fixed point

1. Introduction

The symmetry property for solutions of dynamical systems has received great interest due to its potential for application in almost all fields of natural and applied sciences, such as physics, engineering sciences, or economics (see [3, 11, 13]). As it is well known, one of the strongest symmetry properties for the solution of a dynamical system is periodicity. By strict periodicity for a solution of a continuous time dynamical system

$$x'(t) = f(t, x(t)), \; t \in \mathbb{R}, \tag{1.1}$$

we mean that solution x satisfies $x(t+T) = x(t)$ for all $t \in \mathbb{R}$, where T is a fixed positive constant. Periodicity is a relaxable and generalizable concept for the classes of functions. As a relaxation of strict periodicity, the theory of almost periodic and almost automorphic functions was introduced in the 20th century by Bohr and Bochner, respectively (see [4, 5]). These relaxed periodicity notions have been studied for several dynamical systems and real-life models by researchers. In addition to relaxation of periodicity, the generalization of periodicity is introduced as a symmetry property called affine-periodicity. Briefly, for a continuous function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, dynamical system (1.1) is called affine periodic if f is affine symmetric, i.e.

$$f(t+T, x) = Qf\left(t, Q^{-1}x\right)$$

for $Q \in GL(\mathbb{R}^n)$, where $GL(\mathbb{R}^n)$ indicates an n-dimensional linear group over $\mathbb{R}$.

Affine periodic continuous time dynamical systems and the existence of their solutions have been studied by several researchers in the existing literature. For a quick literature review, we refer to [6, 14, 25, 29].

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Unlike the literature on affine periodic differential equations, there is limited research on affine periodic discrete dynamical systems. To the best of our knowledge, affine periodic solutions of dynamical systems involving difference equations were first handled in 2015 by means of topological degree theory (see [20, 24]). Motivated by the lack of studies on affine symmetry and affine periodicity in the theory of difference equations, this study focuses on the existence of affine periodic solutions of discrete dynamical systems by employing fixed point theory.

In our analysis, we use discrete exponential dichotomy, which is commonly used in the study of almost periodic and almost automorphic solutions of dynamical systems (see [1, 2] and [18, 19]). We organize the paper as follows: the next section is devoted to preliminaries on affine symmetry, affine periodicity, and exponential dichotomy for difference equations and their solutions. In the third part of the paper, we propose some sufficient conditions for the existence of (Q,T)-affine periodic solutions of discrete dynamical systems by means of contraction mapping principle and Schauder's fixed point theorem. In the final part, we implement our existence results to some real-life models.

2. Preliminaries

Consider the following difference system:

$$x(t+1) = f(t, x(t)), \; t \in \mathbb{Z}, \tag{2.1}$$

where $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is affine symmetric; that is, there are $Q \in GL(\mathbb{R}^n)$ and $T > 0$ such that

$$f(t+T, x) = Qf\left(t, Q^{-1}x\right) \text{ for all } t \in \mathbb{Z} \tag{2.2}$$

(see [20]).

Definition 1 ([20]) *System (2.1) is called affine periodic if (2.2) holds for all $t \in \mathbb{Z}$. Additionally, solution $x(t)$ of system (2.1) is called (Q,T)-affine periodic if*

$$x(t+T) = Qx(t) \text{ for all } t \in \mathbb{Z}. \tag{2.3}$$

Similar to [6], we give the following definition.

Definition 2 *Let A be an $n\times n$ matrix function and consider the following homogeneous time varying difference system:*

$$x(t+1) = A(t)x(t), \; x(0) = x_0. \tag{2.4}$$

If the matrix function A is (Q,T)-affine periodic, i.e. $A(t+T) = QA(t)Q^{-1}$ for $Q \in GL(\mathbb{R}^n)$, then system (2.4) is called (Q,T)-affine periodic.

The following definition is useful for our further analysis.

Definition 3 ([1]) *Let $X(t)$ be the principal fundamental matrix solution of linear homogeneous system (2.4). Then (2.4) is said to admit an exponential dichotomy if there exist a projection P and positive constants $\alpha_1, \alpha_2, \beta_1,$ and β_2 such that*

$$\left\|X(t)PX^{-1}(s)\right\|_{B(\mathcal{X})} \leq \beta_1 (1+\alpha_1)^{s-t}, \; t \geq s, \tag{2.5}$$

$$\left\|X(t)(I-P)X^{-1}(s)\right\|_{B(\mathcal{X})} \leq \beta_2 (1+\alpha_2)^{t-s}, \; s \geq t, \tag{2.6}$$

where $\mathcal{X}$ *is an abstract (real or complex) Banach space,* $\|.\|_{\mathcal{X}}$ *is the norm of* $\mathcal{X}$*, and* $B(\mathcal{X})$ *is a Banach space of all bounded linear operators from* $\mathcal{X}$ *to* $\mathcal{X}$ *endowed by the norm*

$$\|L\|_{B(\mathcal{X})} := \sup\left\{\|Lx\|_{\mathcal{X}} : x \in \mathcal{X} \text{ and } \|x\|_{\mathcal{X}} \leq 1\right\}.$$

Consider the following time-varying nonhomogeneous linear system:

$$x\left(t+1\right) = A\left(t\right)x\left(t\right) + f(t), \ t \in \mathbb{Z}, \tag{2.7}$$

where $A : \mathbb{Z} \to \mathbb{R}^{n\times n}$ and $f : \mathbb{Z} \to \mathbb{R}^n$. We know by [18, Theorem 2.12] that if (2.4) admits an exponential dichotomy with projection P, then (2.7) has a solution of the following form:

$$x(t) = \sum_{j=-\infty}^{t-1} X\left(t\right)PX^{-1}\left(j+1\right)f\left(j\right) - \sum_{j=t}^{\infty} X\left(t\right)\left(I-P\right)X^{-1}\left(j+1\right)f\left(j\right), \tag{2.8}$$

where $X\left(t\right)$ is the principal matrix solution of (2.4).

Similarly, in [18] a solution of the nonhomogeneous functional system

$$x(t+1) = A(t)x(t) + f\left(t, x\left(t\right)\right), \ t \in \mathbb{Z}, \tag{2.9}$$

was given by

$$x(t) = \sum_{j=-\infty}^{t-1} X\left(t\right)PX^{-1}\left(j+1\right)f\left(j, x\left(j\right)\right) - \sum_{j=t}^{\infty} X\left(t\right)\left(I-P\right)X^{-1}\left(j+1\right)f\left(j, x\left(j\right)\right), \tag{2.10}$$

where $X\left(t\right)$ stands for the principal matrix solution of (2.4).

3. Existence results

In preparation for the next results, we list the following conditions:

C1 The homogeneous system (2.4) admits an exponential dichotomy with projection P and positive constants $\alpha_1, \alpha_2, \beta_1$, and β_2.

C2 $A(t)$ is a (Q,T)-affine periodic matrix function; that is, $A\left(t+T\right) = QA\left(t\right)Q^{-1}$ for $Q \in GL\left(\mathbb{R}^n\right)$.

C3 $f(t)$ is (Q,T)-affine periodic function, i.e. $f\left(t+T\right) = Qf\left(t\right)$ for $Q \in GL\left(\mathbb{R}^n\right)$.

Theorem 1 *Assume C1–C3. Then (2.7) has a* (Q,T)*-affine periodic solution.*

Proof By employing (2.8), we get that

$$\begin{aligned} x(t+T) &= \sum_{j=-\infty}^{t+T-1} X\left(t\right)PX^{-1}\left(j+1\right)f\left(j\right) - \sum_{j=t+T}^{\infty} X\left(t\right)\left(I-P\right)X^{-1}\left(j+1\right)f\left(j\right) \\ &= \sum_{j=-\infty}^{t-1} X\left(t+T\right)PX^{-1}\left(j+T+1\right)f\left(j+T\right) \\ &- \sum_{j=t}^{\infty} X\left(t+T\right)\left(I-P\right)X^{-1}\left(j+T+1\right)f\left(j+T\right). \end{aligned} \tag{3.1}$$

If we set $Y(t) := Q^{-1}X\left(t+T\right)X^{-1}(T)Q$, then we obtain

$$\begin{aligned} Y(t+1) &= Q^{-1}X\left(t+T+1\right)X^{-1}(T)Q \\ &= Q^{-1}A\left(t+T\right)X(t+T)X^{-1}(T)Q \\ &= A(t)Y(t) \end{aligned}$$

by employing (Q,T)-affine periodicity of A. That is, Y is a solution of (2.4) satisfying $Y(0)=I$. From the uniqueness of the solutions of the linear homogeneous system, we observe that X and Y are identical. Thus, one may write $X\left(t+T\right)=QX(t)Q^{-1}X(T)$ and (3.1) can be rewritten as

$$\begin{aligned} x(t+T) = &\sum_{j=-\infty}^{t-1} QX(t)Q^{-1}X(T)PX^{-1}(T)QX^{-1}\left(j+1\right)f\left(j\right) \\ &-\sum_{j=t}^{\infty} QX(t)Q^{-1}X(T)\left(I-P\right)X^{-1}(T)QX^{-1}\left(j+1\right)f\left(j\right). \end{aligned}$$

By using the identities $Q^{-1}X(T)P = PQ^{-1}X(T)$ and $Q^{-1}X(T)\left(I-P\right) = \left(I-P\right)Q^{-1}X(T)$ (see [10, Lemma 7.4]), we obtain

$$x(t+T) = Q\left(\sum_{j=-\infty}^{t-1} X\left(t\right)PX^{-1}\left(j+1\right)f\left(j\right) - \sum_{j=t}^{\infty} X\left(t\right)\left(I-P\right)X^{-1}\left(j+1\right)f\left(j\right)\right),$$

which proves the (Q,T)-affine periodicity of the solution. □

Next, we define the set of (Q,T)-affine periodic functions

$$AP_{(Q,T)} := \left\{f:\mathbb{Z}\to\mathbb{R}^n : f\left(t+T\right)=Qf(t) \text{ for all } t\in\mathbb{Z}\right\}.$$

Similar to [6, Lemma 3.2], $AP_{(Q,T)}$ is a Banach space when its endowed by the norm

$$\|f\|_{AP_{(Q,T)}} := \sup_{t\in[0,T]\cap\mathbb{Z}} |f(t)|.$$

Theorem 2 *In addition to C1 and C2, assume the following:*

C4 $f:\mathbb{Z}\times\mathbb{R}^n\to\mathbb{R}^n$ *is affine symmetric; that is, there are* $Q\in GL\left(\mathbb{R}^n\right)$ *and* $T>0$ *such that*

$$f\left(t+T,x\right)=Qf\left(t,Q^{-1}x\right) \text{ for all } t\in\mathbb{Z}.$$

C5 $\left|f\left(t,x\right)-f\left(t,y\right)\right| \le M\left\|x-y\right\|_{AP_{(Q,T)}}$ *for* $x,y\in AP_{(Q,T)}$ *and* $M>0$.

C6 $M\left(\beta_1\frac{1+\alpha_1}{\alpha_1}+\frac{\beta_2}{\alpha_2}\right)<1$, *where* M *is the Lipschitz constant in C5 and* $\alpha_{1,2},\beta_{1,2}$ *are constants of exponential dichotomy of the homogeneous system.*

Then the functional difference system (2.9) has a unique (Q,T)*-affine periodic solution.*

Proof Assume that C1, C2, and C4–C6 are satisfied. From (2.10), we define the mapping

$$(Hx)(t) := \sum_{j=-\infty}^{t-1} X(t) PX^{-1}(j+1) f(j,x(j)) - \sum_{j=t}^{\infty} X(t)(I-P) X^{-1}(j+1) f(j,x(j)). \tag{3.2}$$

By applying a similar procedure in the proof of Theorem 1, one may show that $H : AP_{(Q,T)} \to AP_{(Q,T)}$. Moreover, for $x, y \in AP_{(Q,T)}$ we write

$$\begin{aligned}
&\|(Hx)(t) - (Hy)(t)\|_{AP_{(Q,T)}} \\
&= \sup_{t\in[0,T]\cap\mathbb{Z}} \left| \sum_{j=-\infty}^{t-1} X(t) PX^{-1}(j+1)(f(j,x(j)) - f(j,y(j))) \right. \\
&\left. - \sum_{j=t}^{\infty} X(t)(I-P) X^{-1}(j+1)(f(j,x(j)) - f(j,y(j))) \right| \\
&\leq \sup_{t\in[0,T]\cap\mathbb{Z}} \left\{ \sum_{j=-\infty}^{t-1} \left| X(t) PX^{-1}(j+1) \right| |f(j,x(j)) - f(j,y(j))| \right. \\
&\left. + \sum_{j=t}^{\infty} \left| X(t)(I-P) X^{-1}(j+1) \right| |f(j,x(j)) - f(j,y(j))| \right\} \\
&\leq M \|x-y\|_{AP_{(Q,T)}} \sup_{t\in[0,T]\cap\mathbb{Z}} \left\{ \sum_{j=-\infty}^{t-1} \beta_1 (1+\alpha_1)^{j+1-t} + \sum_{j=t}^{\infty} \beta_2 (1+\alpha_2)^{t-j-1} \right\} \\
&= M \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right) \|x-y\|_{AP_{(Q,T)}} \leq \|x-y\|_{AP_{(Q,T)}}.
\end{aligned}$$

Thus, the mapping H is a contraction and (2.9) has a unique (Q,T)-affine periodic solution. □

Theorem 3 (Schauder) *Let $\mathbb{B}$ be a Banach space. Assume that K is a closed, bounded, and convex subset of $\mathbb{B}$. If $T : K \to K$ is a compact operator, then it has a fixed point in K.*

In preparation for the next result, we introduce the subset Ω_W of $AP_{(Q,T)}$ by

$$\Omega_W := \left\{ x \in AP_{(Q,T)} : \|x\|_{AP_{(Q,T)}} \leq W \right\}$$

for a fixed W. Then Ω_W is a bounded, closed, and convex subset of $AP_{(Q,T)}$.

Theorem 4 *Assume that C1–C2, C4–C5, and the following condition are satisfied:*

C7 $|f(t,x)| \leq k_1 |x| + k_2$, *for* $k_{1,2} > 0$.

Then nonhomogeneous difference system (2.9) has a (Q,T)-affine periodic solution in Ω_W for

$$W := \frac{k_2 \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right)}{1 - k_1 \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right)}.$$

Proof First, we need to show that the mapping H introduced in (3.2) maps Ω_W into Ω_W. By repeating the same procedure in the proof of Theorem 1, we deduce that $(Hx) \in AP_{(Q,T)}$ if $x \in AP_{(Q,T)}$. In addition to this result, let $x \in \Omega_W$ and consider

$$\begin{aligned}\|Hx\|_{AP_{(Q,T)}} &\leq \sup_{t\in[0,T]\cap\mathbb{Z}} \left\{ \sum_{j=-\infty}^{t-1} \left|X(t) PX^{-1}(j+1)\right| |f(t,x)| + \sum_{j=t}^{\infty} \left|X(t)(I-P)X^{-1}(j+1)\right| |f(t,x)| \right\} \\ &\leq \left(k_1 \|x\|_{AP_{(Q,T)}} + k_2\right)\left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right) \\ &\leq (k_1 W + k_2)\left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right) = W.\end{aligned}$$

Thus, $H : \Omega_W \to \Omega_W$. Next, we show that H is continuous. Suppose $x, y \in \Omega_W$ and define the positive number $\delta(\varepsilon)$ as

$$\delta(\varepsilon) := \frac{\varepsilon}{M\left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right)}.$$

If $\|x-y\|_{AP_{(Q,T)}} < \delta$, then we have

$$\begin{aligned}\|H(x)(t) - H(y)(t)\|_{AP_{(Q,T)}} \leq \sup_{t\in[0,T]\cap\mathbb{Z}} &\left\{ \sum_{j=-\infty}^{t-1} \left|X(t) PX^{-1}(j+1)\right| |f(j,x(j)) - f(j,y(j))| \right. \\ &\left. + \sum_{j=t}^{\infty} \left|X(t)(I-P)X^{-1}(j+1)\right| |f(j,x(j)) - f(j,y(j))| \right\} \\ &\leq M\|x-y\|_{AP_{(Q,T)}} \sup_{t\in[0,T]\cap\mathbb{Z}} \left\{ \sum_{j=-\infty}^{t-1} \beta_1 (1+\alpha_1)^{j+1-t} + \sum_{j=t}^{\infty} \beta_2 (1+\alpha_2)^{t-j-1} \right\} \\ &= M\|x-y\|_{AP_{(Q,T)}} \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right) \\ &< \varepsilon,\end{aligned}$$

which means that H is continuous. Additionally, one may obtain the precompactness of $H(\Omega_W)$ due to the Bolzano–Weierstrass theorem and a diagonalization process. That is, for a sequence $\{x^s\}_{s\in\mathbb{Z}_+} \in \Omega_W$ and for each fixed s, one may find a convergent subsequence $\{x^s(t_k)\}$. By repeating the same process for each $s \in \mathbb{Z}_+$, we construct a convergent subsequence $\{x^{s_k}\}$ of $\{x^s\}$ in Ω_W and we deduce that $\{H(x^s)\}$ has a convergent subsequence in $H(\Omega_W)$ by employing the continuity of H. Thus, $H(\Omega_W)$ is precompact and Schauder's fixed point theorem implies the existence of an $x \in \Omega_W$ such that $H(x) = x$. □

3.1. Particular cases

For a particular choice of matrix Q our Theorem 4 covers some existing results for periodic, antiperiodic, and quasiperiodic solutions of difference equations. We address this generalization by providing the following list:

- If $Q = I$ in conditions C2 and C4, then Theorem 4 implies the existence of a periodic solution of (2.9). For related studies, we refer to [7, 15, 21, 26].

- If $Q = -I$ in conditions C2 and C4, then Theorem 4 implies the existence of an antiperiodic solution of (2.9). We refer to [16, 17, 22] as related studies.
- Moreover, a solution of system (2.9) is called harmonic and quasiperiodic if it satisfies

$$x(t+T) = Qx(t) \text{ for all } t \in \mathbb{Z}$$

for $Q^N = I$ and $Q \in SO(n)$, respectively (see the discussion in [6]).

4. Applications in economic, biological, and mathematical models

In this section, we implement our existence results for economic, biological, and mathematical models.

Example 1 *The Keynesian cross economic model with lagged income as a discrete dynamical system given by*

$$D(t) = C(t) + I(t-1) + G(t-1), \tag{4.1}$$

$$C(t) = cx(t), \tag{4.2}$$

$$x(t+1) = \delta D(t+1) + x(t)(1-\delta), \tag{4.3}$$

where c is a nonnegative constant, D is aggregate demand, x is aggregate income, C is aggregate consumption, I is aggregate investment, G is government spending, and $\delta < 1$ is speed of adjustment term (see [9, Page 23] and also for unification of the model see [19, 23]). It should be noted that we assume demand is affected by the previous period's investment and government spending, and also there is no fixed consumption. By substituting (4.1) and (4.2) into (4.3), one may easily obtain the implicit form of the model as

$$x(t+1) = \frac{1-\delta}{1-\delta c}x(t) + \frac{\delta}{1-\delta c}(I(t) + G(t)) \tag{4.4}$$

by assuming $\delta c \neq 1$. When $\delta c > 1$, the homogeneous part of equation (4.4) admits an exponential dichotomy. Under the assumption that I and G are both (Q,T)-affine periodic, we deduce that the Keynesian cross economic model has a (Q,T)-affine periodic solution as a consequence of Theorem 1.

Example 2 *Consider the following systems of difference equations:*

$$\Delta \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^{-t}\sin\left(\frac{\pi}{2}t\right) \\ e^{-t}\cos(\pi t) \end{bmatrix}, \tag{4.5}$$

where Δ is the forward difference operator, i.e. $\Delta x(t) = x(t+1) - x(t)$. Then system (4.5) can be written as a functional difference equation of the form (2.9) by setting

$$A(t) = \frac{1}{2}I_{2\times 2},$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and

$$f(t, x(t)) = \begin{bmatrix} e^{-t}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{4}x_1(t) \\ e^{-t}\cos(\pi t) + \frac{1}{4}x_2(t) \end{bmatrix},$$

where $I_{2\times 2}$ is a 2×2 identity matrix. By employing the Putzer algorithm, one may get P-matrices as $P_0 = I_{2\times 2}$ and $P_1 = (A - \lambda I)\, P_0 = 0_{2\times 2}$, where $\lambda = \frac{1}{2}$ is the only eigenvalue of A. Furthermore, the principal fundamental matrix solution of the homogeneous system $x(t+1) = Ax(t)$ is obtained as $X(t) = 2^{-t} I_{2\times 2}$ and the exponential dichotomy condition

$$\left|X(t)P_0X^{-1}(s)\right| = 2^{s-t} \le \beta_1 (1+\alpha_1)^{s-t},\ t \ge s,$$

is satisfied for $\beta_1 = \alpha_1 = 1$. Notice that the second condition of discrete exponential dichotomy (the case $s \ge t$) is satisfied directly and we choose $\beta_2 = \alpha_2 = 1$ for the sake of clarity. On the other hand, condition C3 is also satisfied with $Q = e^{2\pi} I_{2\times 2}$. To see this, we consider

$$\begin{aligned} f(t+2\pi, x(t)) &= \begin{bmatrix} e^{-t-2\pi}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{4}x_1(t) \\ e^{-t-2\pi}\cos(\pi t) + \frac{1}{4}x_2(t) \end{bmatrix} = e^{2\pi} I \begin{bmatrix} e^{-t}\sin\left(\frac{\pi}{2}t\right) + e^{-2\pi}\frac{1}{4}x_1(t) \\ e^{-t}\cos(\pi t) + e^{-2\pi}\frac{1}{4}x_2(t) \end{bmatrix} \\ &= e^{2\pi} I_{2\times 2} f\left(t, e^{-2\pi} I x(t)\right). \end{aligned}$$

Finally, we observe that condition C6 is fulfilled with the constant $M = \frac{1}{4}$, and

$$M\left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right) = \frac{3}{4} < 1.$$

Then Theorem 2 implies that discrete dynamical system (4.5) has a unique $(Q, 2\pi)$-affine periodic solution, where $Q = e^{2\pi} I_{2\times 2}$.

Example 3 *Consider the following difference equation:*

$$x(t+1) = -a(t)\, x(t) + b(t)\tanh(x(t)) + c(t),\ t \in \mathbb{Z}, \tag{4.6}$$

which is the discrete analogue of the biological equation used for modeling a single artificial effective neuron by dissipation (see [8, 12] and also for the unification see [28]). In (4.6), we assume $a(t) > 0$ for all $t \in \mathbb{Z}$, b and c are bounded functions, and the homogeneous part of (4.6) admits an exponential dichotomy. If we set a, b, c to be $(-1, T)$-affine periodic, i.e. $a(t+T) = -a(t)$, $b(t+T) = -b(t)$ and $c(t+T) = -c(t)$, then the function $f(t, x(t)) := b(t)\tanh(x(t)) + c(t)$ is $(-1, T)$-affine symmetric. Moreover,

$$\begin{aligned} |f(t,x) - f(t,y)| &\le |b(t)(\tanh(x(t)) - \tanh(y(t)))| \\ &\le U_b\,|x-y|, \end{aligned}$$

where U_b stands for the upper bound of the function b. Additionally, condition C7 is satisfied as a result of the above observation. To conclude, we deduce that the discrete dynamical system has an antiperiodic solution as a consequence of Theorem 4.

Example 4 *Consider the following discrete delayed dynamical equation:*

$$x(t+1) = (1-p)\, x(t) - \lambda e^{-\gamma x(t)}, \tag{4.7}$$

which is the discrete analogue of the Wazewska-Czyzewska and Lasota model proposed in [27]. Notice that the above functional equation is used to model the survival of red blood cells in an animal. For sake of brevity, $x(t)$

indicates density of red blood cells, p is the probability of a decease in red blood cells, and $\lambda, \gamma > 0$ stand for production of cells in (4.7). The homogeneous part of (4.7) admits an exponential dichotomy when $p \in (0,1)$. Furthermore, if we set $f(t,x(t)) := -\lambda e^{-\gamma x(t)}$, then it is $(1,T)$-affine symmetric and all conditions of Theorem 4 are satisfied. Thus, (4.7) has a T-periodic solution.

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