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Generating sets of certain finite subsemigroups of monotone partial bijections

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Abstract: Let I_n be the symmetric inverse semigroup, and let $PODI_n$ and POI_n be its subsemigroups of monotone partial bijections and of isotone partial bijections on $X_n = \{1, \ldots, n\}$ under its natural order, respectively. In this paper we characterize the structure of (minimal) generating sets of the subsemigroups $PODI_{n,r} = \{\alpha \in PODI_n : |\mathrm{im}\,(\alpha)| \leq r\}$, $POI_{n,r} = \{\alpha \in POI_n : |\mathrm{im}\,(\alpha)| \leq r\}$, and $E_{n,r} = \{\mathrm{id}_A \in I_n : A \subseteq X_n \text{ and } |A| \leq r\}$ where id_A is the identity map on $A \subseteq X_n$ for $0 \leq r \leq n-1$.

Key words: Partial bijection, isotone/antitone/monotone map, (minimal) generating set

1. Introduction

Let I_X be the semigroup of all partial one-to-one maps on a nonempty set X under usual composition. It is well known that I_X is an inverse semigroup; that is, for each element α there exists a unique element α' such that $\alpha\alpha'\alpha = \alpha$, which is called *symmetric inverse semigroup*. From the Wagner–Preston theorem, as cited in [4], as the analog of Cayley's theorem for finite groups, every inverse semigroup is isomorphic to a subsemigroup of a suitable symmetric inverse semigroup. Hence, the symmetric inverse semigroups and their subsemigroups have certain important roles in inverse semigroup theory like the symmetric groups in group theory. Moreover, the problem of finding (minimal) generating sets of certain finite transformation semigroups is an important problem for finite semigroup theory and has been much studied over the last 50 years. We examine this problem for certain subsemigroups of I_n, the finite symmetric inverse semigroup on $X_n = \{1, \ldots, n\}$ under its natural order. Among recent contributions are [1, 4, 5, 9].

Let α be a partial map on X_n. If $(\forall x \in \mathrm{dom}\,(\alpha))$ $x\alpha = x$ then α is called the *partial identity map* on $A = \mathrm{dom}\,(\alpha) \subseteq X_n$ and denoted by id_A. If $(\forall x, y \in \mathrm{dom}\,(\alpha))$ $x \leq y \Rightarrow x\alpha \leq y\alpha$ $(x \leq y \Rightarrow x\alpha \geq y\alpha)$ then α is called *isotone* (*antitone*), and if α is isotone or antitone then α is called *monotone*. Notice that if $|\mathrm{im}\,(\alpha)| \leq 1$ then α is both isotone and antitone, and so monotone. Let $\{\alpha_1, \ldots, \alpha_k\}$ be some monotone partial maps on X_n for $k \geq 2$. It is easy to see that the product $\alpha_1 \cdots \alpha_k$ of these maps is isotone if the number of antitone maps in $\{\alpha_1, \ldots, \alpha_k\}$ is an even number; otherwise, it is antitone. Then the subsets

$$\begin{aligned} PODI_n &= \{\alpha \in I_n : \alpha \text{ is monotone}\}, \\ POI_n &= \{\alpha \in I_n : \alpha \text{ is isotone,}\} \text{ and} \\ E_n &= \{\mathrm{id}_A : A \subseteq X_n\} \end{aligned}$$

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are subsemigroups of I_n, and $E_n \leq POI_n \leq PODI_n \leq I_n$. For $0 \leq r \leq n$, let

$$\begin{aligned} PODI_{n,r} &= \{\alpha \in PODI_n : |\mathrm{im}\,(\alpha)| \leq r\}, \\ POI_{n,r} &= \{\alpha \in POI_n : |\mathrm{im}\,(\alpha)| \leq r\}, \text{ and} \\ E_{n,r} &= \{\mathrm{id}_A : A \subseteq X_n \text{ and } |A| \leq r\}, \end{aligned}$$

which are clearly subsemigroups of $PODI_n$, POI_n, and E_n, respectively. It follows from [3, Proposition 2.2] and [5, Proposition 3.2] that, for $0 \leq r \leq n$,

$$|PODI_{n,r}| = 1 + n^2 + \sum_{p=2}^{r} 2\binom{n}{p}^2, \quad |POI_{n,r}| = \sum_{p=0}^{r} \binom{n}{p}^2 \text{ and}$$

$$|E_{n,r}| = \sum_{p=0}^{r} \binom{n}{p}.$$

Let S be any semigroup, and let U be any nonempty subset of S. Then the subsemigroup generated by U, the smallest subsemigroup of S containing U, is denoted by $\langle U \rangle$. The *rank* of a finitely generated semigroup S, a semigroup generated by a finite subset, is defined by

$$\mathrm{rank}\,(S) = \min\{\, |U| : \langle U \rangle = S\}.$$

Moreover, the generating set of S with the cardinality $\mathrm{rank}\,(S)$ is called a *minimal* generating set of S.

The generating sets and the ranks of the semigroup POI_n were studied by Fernandes in [3] and [4]. Also, the generating sets and the ranks of the semigroup $PODI_n$ were studied by Fernandes et al. in [5] and by Zhao and Fernandes in [10]. From [4, Proposition 2.8] we have $\mathrm{rank}\,(POI_{n,n-1}) = n$ and furthermore $\mathrm{rank}\,(POI_n) = n$ since $POI_n \setminus POI_{n,n-1} = \{\mathrm{id}_{X_n}\}$. Let $\lfloor x \rfloor$ be the least integer greater than or equal to x for each $x \in \mathbb{R}$. From [5, Theorem 3.6] we have $\mathrm{rank}\,(PODI_n) = \lfloor \frac{n}{2} \rfloor + 1$ for $1 \leq r \leq n-1$, and from [10, Theorem 4.12] we have $\mathrm{rank}\,(POI_{n,r}) = \mathrm{rank}\,(PODI_{n,r}) = \binom{n}{r}$. Although some special generating sets were found in these studies, there is no method for deciding if they are or are not generating sets of the mentioned semigroups. Therefore, we examine the necessary and sufficient conditions for any subset of S to be a (minimal) generating set of S where S is one of the semigroups $PODI_{n,r}, POI_{n,r}$, and $E_{n,r}$ for $0 \leq r \leq n-1$.

2. Preliminaries

Let $A = \{a_1, \ldots, a_r\}$ be a nonempty subset of X_n for $2 \leq r \leq n$. For convenience, we write $A = \{a_1 < \cdots < a_r\}$ if $a_1 < \cdots < a_r$. For $1 \leq p \leq r \leq n-1$, consider any element of $PODI_n$ with domain set $A = \{a_1 < \cdots < a_r\}$ and image set $B = \{b_1 < \cdots < b_r\}$. Then there exist two cases: either

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix}, \text{ or shortly } \alpha = \begin{pmatrix} A \\ B \end{pmatrix},$$

if α is isotone, or

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ b_p & b_{p-1} & \cdots & b_1 \end{pmatrix}, \text{ or shortly } \alpha = \begin{pmatrix} A \\ B^R \end{pmatrix},$$

if α is antitone.

For $\alpha,\beta \in PODI_{n,r}$ or $\alpha,\beta \in POI_{n,r}$ $(1 \leq r \leq n)$, from the definitions of Green's equivalences it is a routine matter to prove that

(i) $\alpha\mathcal{R}\beta \Leftrightarrow \operatorname{dom}(\alpha) = \operatorname{dom}(\beta)$,

(ii) $\alpha\mathcal{L}\beta \Leftrightarrow \operatorname{im}(\alpha) = \operatorname{im}(\beta)$,

(iii) $\alpha\mathcal{H}\beta \Leftrightarrow \operatorname{dom}(\alpha) = \operatorname{dom}(\beta)$ and $\operatorname{im}(\alpha) = \operatorname{im}(\beta)$, and

(iv) $\alpha\mathcal{D}\beta \Leftrightarrow |\operatorname{im}(\alpha)| = |\operatorname{im}(\beta)|$.

For $0 \leq p \leq r \leq n$ we denote Green's $\mathcal{D}$-class of all elements in S of height p by D_p; that is,

$$D_p = \{\alpha \in S : |\operatorname{im}(\alpha)| = p\},$$

where S is the subsemigroup $PODI_{n,r}$ or $POI_{n,r}$. Then it is clear that there exist $r+1$ many $\mathcal{D}$-classes, namely $D_0, D_1, \ldots, D_r$, and S is the disjoint union of $D_0, D_1, \ldots, D_r$. Moreover, there exist $\binom{n}{p}$ $\mathcal{R}$-classes and $\binom{n}{p}$ $\mathcal{L}$-classes in D_p for each $0 \leq p \leq r$. Notice that for $\alpha \in PODI_{n,r}$, if $|\operatorname{im}(\alpha)| \geq 2$ then $|H_\alpha| = 2$; otherwise, $|H_\alpha| = 1$ where H_α is the $\mathcal{H}$-class contains α. Also notice that for each $\alpha \in POI_{n,r}$ we have $|H_\alpha| = 1$. Let $k = \binom{n}{p}$. Then the $\mathcal{D}$-class D_p in $PODI_{n,r}$ has the following egg box form:

$$D_p:\ \begin{array}{c|c|c|c|} & L_1 & & L_k \\ \hline R_1 & \begin{pmatrix}A_1\\A_1\end{pmatrix},\begin{pmatrix}A_1\\A_1^R\end{pmatrix} & \cdots & \begin{pmatrix}A_1\\A_k\end{pmatrix},\begin{pmatrix}A_1\\A_k^R\end{pmatrix} \\ \hline & \vdots & \ddots & \vdots \\ \hline R_k & \begin{pmatrix}A_k\\A_1\end{pmatrix},\begin{pmatrix}A_k\\A_1^R\end{pmatrix} & \cdots & \begin{pmatrix}A_k\\A_k\end{pmatrix},\begin{pmatrix}A_k\\A_k^R\end{pmatrix} \\ \hline \end{array},$$

and the $\mathcal{D}$-class D_p in $POI_{n,r}$ has the following egg box form:

$$D_p:\ \begin{array}{c|c|c|c|} & L_1 & & L_k \\ \hline R_1 & \begin{pmatrix}A_1\\A_1\end{pmatrix} & \cdots & \begin{pmatrix}A_1\\A_k\end{pmatrix} \\ \hline & \vdots & \ddots & \vdots \\ \hline R_k & \begin{pmatrix}A_k\\A_1\end{pmatrix} & \cdots & \begin{pmatrix}A_k\\A_k\end{pmatrix} \\ \hline \end{array},$$

where $A_1, \ldots, A_k$ are all the subsets of X_n with cardinality p. Similarly, it is easy to see that on $E_{n,r}$

$$\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D} = \{(\operatorname{id}_A, \operatorname{id}_A) : \operatorname{id}_A \in E_{n,r}\}$$

for $0 \leq r \leq n$.

For the definitions of Green's equivalences and for the other terms in semigroup theory that are not explained here, we refer to [6, 8].

It is known from [4, Proof of Lemma 2.7] that $D_{p-1} \subseteq \langle D_p \rangle$ in $POI_{n,r}$ for $1 \leq p \leq r \leq n-1$. Now we prove this claim for $PODI_{n,r}$ and notice that the proof is also effective for $POI_{n,r}$.

Lemma 2.1 *For* $1 \leq p \leq r \leq n-1$, $D_{p-1} \subseteq \langle D_p \rangle$ *in* $PODI_{n,r}$.

Proof First of all, notice that the empty map $\emptyset = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Now we consider the case $2 \leq p \leq r$. If $\alpha \in D_{p-1}$ is isotone then the result follows from [7, Lemma 3.4]. Let $\alpha \in D_{p-1}$ be the antitone map with $\mathrm{dom}\,(\alpha) = \{a_1 < \cdots < a_{p-1}\}$ and $\mathrm{im}\,(\alpha) = \{b_1 < \cdots < b_{p-1}\}$; that is, $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} \\ b_{p-1} & b_{p-2} & \cdots & b_1 \end{pmatrix}$. Since $n-(p-1) \geq 2$, there exist $x \in X_n \setminus \mathrm{dom}\,(\alpha)$ and $y \in X_n \setminus \mathrm{im}\,(\alpha)$ such that $a_{i-1} < x < a_i$ and $b_{j-1} < y < b_j$ for $1 \leq i, j \leq p$ where $a_0 = b_0 = 0$ and $a_p = b_p = n+1$. Notice that $2 \leq p-i+2 \leq n$ and that there exist two cases according to $p-i+2 > j$ or $p-i+2 \leq j$.

First suppose that $p-i+2 > j$. Then it is clear that $\beta\gamma = \alpha$, where β is the antitone map with $\mathrm{dom}\,(\beta) = \mathrm{dom}\,(\alpha) \cup \{x\}$ and $\mathrm{im}\,(\beta) = \{1, 2, \ldots, p+1\} \setminus \{j\}$, and γ is the isotone map with $\mathrm{dom}\,(\gamma) = \{1, 2, \ldots, p+1\} \setminus \{p-i+2\}$ and $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\alpha) \cup \{y\}$. Now suppose that $p-i+2 \leq j$. Also notice that if $j = p-i+2$ then $2 \leq p-i+2 = j \leq p$. Similarly, $\beta\gamma = \alpha$, where β is the antitone map with $\mathrm{dom}\,(\beta) = \mathrm{dom}\,(\alpha) \cup \{x\}$ and $\mathrm{im}\,(\beta) = \{1, 2, \ldots, p+1\} \setminus \{j+1\}$, and γ is the isotone map with $\mathrm{dom}\,(\gamma) = \{1, 2, \ldots, p+1\} \setminus \{p-i+1\}$ and $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\alpha) \cup \{y\}$. □

It is known from [10, Corollary 4.2] that the semigroup $POI_{n,r}$ is generated by its elements in D_r. Similarly, from [10, Corollary 4.3] (also from Lemma 2.1), the semigroup $PODI_{n,r}$ is generated by its elements in D_r. Then we conclude that a nonempty subset U of $PODI_{n,r}$ ($POI_{n,r}$) is a generating set of $PODI_{n,r}$ ($POI_{n,r}$) if and only if $D_r \subseteq \langle U \cap D_r \rangle$. Thus, it is enough to consider only the subsets of D_r to examine the structure of any (minimal) generating set of $PODI_{n,r}$ ($POI_{n,r}$).

For any partial maps α and β it is well known that $\mathrm{dom}\,(\alpha\beta) = (\mathrm{im}\,(\alpha) \cap \mathrm{dom}\,(\beta))\alpha^{-1}$, and so it is a routine matter to prove the following lemma.

Lemma 2.2 *Let* $\alpha_1, \ldots, \alpha_k$ *be some elements of* D_p *in* $PODI_{n,r}$ ($POI_{n,r}$) *for* $2 \leq k$ *and* $1 \leq p \leq n-1$. *Then the product* $\alpha_1 \cdots \alpha_k$ *is also an element of* D_p *if and only if* $\alpha_i\alpha_{i+1}$ *is an element of* D_p*; equivalently,* $\mathrm{im}\,(\alpha_i) = \mathrm{dom}\,(\alpha_{i+1})$ *for each* $1 \leq i \leq k-1$.

As a final of this section we give some definitions about digraphs. Let $\Pi = (V(\Pi), \overrightarrow{E}(\Pi))$ be a digraph. For two vertices $u, v \in V(\Pi)$, if either $(u, v) \in \overrightarrow{E}(\Pi)$ or, for $k \geq 1$, there exist $w_1, \ldots, w_k \in V(\Pi)$ (they do not have to be distinct) such that $(u, w_1), \ldots, (w_i, w_{i+1}), \ldots, (w_k, v) \in \overrightarrow{E}(\Pi)$, then $u \to w_1 \to \cdots \to w_k \to v$ is called *a directed path from* u *to* v. If $u = v$ or there exists a directed path from u to v, then the restricted part from u to v of Π is called a *connection* from u to v and we say u is connected to v. In particular, for distinct vertices $u_1, \ldots, u_k \in V(\Pi)$ where $k \geq 1$, the closed directed path $u_1 \to \cdots \to u_k \to u_1$ is called a *cycle*, and a cycle that consists of a unique vertex is called a *loop*. For any directed path $u_1 \to \cdots \to u_k$, the product $u_1u_2 \cdots u_k$ where $2 \leq k$ is called a *consecutive product*. Let U be a nonempty subset of D_r in $ODI_{n,r}$ ($OI_{n,r}$). Then we define the digraph Γ_U as follows:

- the vertex set of Γ_U, denoted by $V = V(\Gamma_U)$, is U; and
- the directed edge set of Γ_U, denoted by $\overrightarrow{E} = \overrightarrow{E}(\Gamma_U)$, is

$$\overrightarrow{E} = \{(\alpha, \beta) \in V \times V : \alpha\beta \in D_r\} = \{(\alpha, \beta) \in V \times V : \mathrm{im}\,(\alpha) = \mathrm{dom}\,(\beta)\}.$$

3. Generating sets of $PODI_{n,r}$

Notice that $PODI_{n,0} = POI_{n,0} = \{\emptyset\}$, where $\emptyset$ is the empty map on X_n, and $PODI_{n,1} = POI_{n,1}$. Therefore, unless otherwise stated, in this section we consider the cases $2 \leq r \leq n-1$.

Lemma 3.1 *For* $2 \leq r \leq n-1$, $PODI_{n,r} = \langle D_r^a \rangle$ *where*

$$D_r^a = \{\alpha \in PODI_{n,r} : \alpha \textit{ is antitone and } |\mathrm{im}\,(\alpha)| = r\}.$$

Proof Let $\alpha \in D_r$ in $PODI_{n,r}$ be an isotone map with $\mathrm{dom}\,(\alpha) = \{a_1 < \cdots < a_r\}$ and $\mathrm{im}\,(\alpha) = \{b_1 < \cdots < b_r\}$; that is, $\alpha = \begin{pmatrix} a_1 & \cdots & a_r \\ b_1 & \cdots & b_r \end{pmatrix}$. Now consider the antitone maps $\beta = \begin{pmatrix} a_1 & \cdots & a_r \\ a_r & \cdots & a_1 \end{pmatrix}$ and $\gamma = \begin{pmatrix} a_1 & \cdots & a_r \\ b_r & \cdots & b_1 \end{pmatrix}$. Then it is clear that $\alpha = \beta\gamma$. Thus, $D_r \subseteq \langle D_r^a \rangle$, and so $PODI_{n,r} = \langle D_r^a \rangle$. □

Corollary 3.2 *For* $2 \leq r \leq n-1$ *a nonempty subset* U *of* D_r *is a generating set of* $PODI_{n,r}$ *if and only if* $D_r^a \subseteq \langle U \rangle$.

Theorem 3.3 *Let* $2 \leq r \leq n-1$, *and let* $\emptyset \neq U \subseteq D_r$ *in* $PODI_{n,r}$. *Then* U *is a generating set of* $PODI_{n,r}$ *if and only if for each pair of subsets* A *and* B *of* X_n *with cardinality* r *there exist* $\alpha, \beta \in U$ *such that*

- (i) $\mathrm{dom}\,(\alpha) = A$,
- (ii) $\mathrm{im}\,(\beta) = B$, *and*
- (iii) α *is connected to* β *in the digraph* Γ_U, *with the property that the number of antitone maps in the connection is an odd number.*

Proof $(\Rightarrow)$ Suppose that $\emptyset \neq U \subseteq D_r$ is a generating set of $PODI_{n,r}$, i.e. $D_r^a \subseteq \langle U \rangle$. Let A and B be any pair of subsets of X_n with cardinality r. Consider the antitone map $\gamma \in D_r$ with domain set A and image set B. Then there exist $\alpha_1, \ldots, \alpha_t \in U$ such that $\alpha_1 \cdots \alpha_t = \gamma$ for $t \geq 1$. From Lemma 2.2 we have $\mathrm{dom}\,(\alpha_1) = \mathrm{dom}\,(\gamma) = A$ and $\mathrm{im}\,(\alpha_t) = \mathrm{im}\,(\gamma) = B$. Moreover, we have $\alpha_i\alpha_{i+1} \in D_r$, for each $1 \leq i \leq t-1$, and so α_1 is connected to α_t in the digraph Γ_U. Then it is easy to see that the number of antitone maps in this connection must be an odd number since γ is antitone.

$(\Leftarrow)$ Conversely, suppose that the conditions are satisfied, and that $\gamma \in D_r$ be any antitone map with domain set A and image set B. Then there exist $\alpha, \beta \in U$ such that $\mathrm{dom}\,(\alpha) = A = \mathrm{dom}\,(\gamma)$ and $\mathrm{im}\,(\beta) = B = \mathrm{im}\,(\gamma)$, and α is connected to β in the digraph Γ_U, with the property that the number of antitone maps in the connection is an odd number. If we denote the consecutive product of all elements in this connection by ξ, then ξ is also an antitone map, and moreover, $\mathrm{dom}\,(\xi) = \mathrm{dom}\,(\alpha) = \mathrm{dom}\,(\gamma)$ and $\mathrm{im}\,(\xi) = \mathrm{im}\,(\beta) = \mathrm{im}\,(\gamma)$ from Lemma 2.2. Hence, $\gamma = \xi \in \langle U \rangle$, and so $D_r^a \subseteq \langle U \rangle$. Thus, the result is clear from Corollary 3.2. □

Lemma 3.4 *Let* $1 \leq r \leq n-1$, *and let* $\emptyset \neq U \subseteq D_r$. *For any subset* A *of* X_n *with cardinality* r, *let* R_A *and* L_A *be the* $\mathcal{R}$*-class and* $\mathcal{L}$*-class, which contain* id_A, *in* D_r, *respectively, and let* $H_A = R_A \cap L_A$.

- (i) *If* $R_A \cap U \subseteq H_A$, *then* $R_A \cap \langle U \rangle \subseteq H_A$.
- (ii) *If* $L_A \cap U \subseteq H_A$, *then* $L_A \cap \langle U \rangle \subseteq H_A$.

Proof First of all recall that $H_A = \{\begin{pmatrix} A \\ A \end{pmatrix}, \begin{pmatrix} A \\ A^R \end{pmatrix}\}$.

(i) If $R_A \cap U = \emptyset$ then $R_A \cap \langle U \rangle = \emptyset$ since $\mathrm{dom}\,(\beta) \neq A$ for each $\beta \in \langle U \rangle$. Now let $\emptyset \neq R_A \cap U \subseteq H_A$, and let $\beta \in R_A \cap \langle U \rangle$. Then there exist $\beta_1, \ldots, \beta_t \in U$ such that $\beta = \beta_1 \cdots \beta_t\,(t \in \mathbb{Z}^+)$. It follows from Lemma 2.2 that $\mathrm{im}\,(\beta_i) = \mathrm{dom}\,(\beta_{i+1})$ for each $1 \leq i \leq t-1$, and so $\mathrm{dom}\,(\beta) = \mathrm{dom}\,(\beta_1)$. Thus, $\beta_1 \in R_A$, and so $\beta_1 \in R_A \cap U$. Then, from the assumption, we have $\beta_1 \in H_A$. Similarly, since $\mathrm{dom}\,(\beta_{i+1}) = \mathrm{im}\,(\beta_i) = A$ for each $1 \leq i \leq t-1$, it follows that $\beta_1, \ldots, \beta_t \in H_A$, and so $\beta \in H_A$, as required.

(ii) It can be proved similarly. □

Recall that for $2 \leq r \leq n-1$, $\mathrm{rank}\,(PODI_{n,r}) = \binom{n}{r}$, and so we have the following theorem.

Theorem 3.5 *For* $2 \leq r \leq n-1$, *let* $U \subseteq D_r$ *with cardinality* $\binom{n}{r}$. *Then* U *is a minimal generating set of* $PODI_{n,r}$ *if and only if*

(i) $|R \cap U| = |L \cap U| = 1$ *for each* $\mathcal{R}$*-class* R *and* $\mathcal{L}$*-class* L *in* D_r,

(ii) *the digraph* Γ_U *is a cycle, and*

(iii) *the number of antitone maps in* U *is an odd number.*

Proof $(\Rightarrow)$ Suppose that $U \subseteq D_r$ is a (minimal) generating set of $PODI_{n,r}$ with cardinality $\binom{n}{r}$.

(i) The claim is clearly provided from Theorem 3.3.

(ii) First notice that $\binom{n}{r} \geq 2$ for $2 \leq r \leq n-1$, and from the first condition and Lemma 3.4, there is no element in U that has either form $\begin{pmatrix} A \\ A \end{pmatrix}$ or form $\begin{pmatrix} A \\ A^R \end{pmatrix}$ for any subset A of X_n with cardinality r. Hence, there is no loop in Γ_U. Now let α and β be two distinct elements of U. Then consider any map $\gamma \in D_r$ such that $\mathrm{dom}\,(\gamma) = \mathrm{dom}\,(\alpha)$ and $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\beta)$. Notice that α and β are not in the same $\mathcal{R}$-class and not in the same $\mathcal{L}$-class in D_r, from the first condition, and so $\alpha \neq \gamma$, $\beta \neq \gamma$, and moreover, $\gamma \notin U$. Since U is a generating set of $PODI_{n,r}$, there exist $\lambda_1, \ldots, \lambda_t \in U$ such that $\lambda_1 \cdots \lambda_t = \gamma$ and $t \geq 2$. Then, from Lemma 2.2, we have $\mathrm{dom}\,(\lambda_1) = \mathrm{dom}\,(\gamma) = \mathrm{dom}\,(\alpha)$ and $\mathrm{im}\,(\lambda_t) = \mathrm{im}\,(\gamma) = \mathrm{im}\,(\beta)$, and so $\lambda_1 \mathcal{R} \alpha$ and $\lambda_t \mathcal{L} \beta$. From the first condition $\lambda_1 = \alpha$ and $\lambda_t = \beta$, and so there exists a directed path from α to β; that is, α is connected to β in the digraph Γ_U. Moreover, for any $\alpha \in U$, there exists a unique element $\lambda \in (U) \backslash \{\alpha\}$ such that $\mathrm{im}\,(\alpha) = \mathrm{dom}\,(\lambda)$ and a unique element $\mu \in U \backslash \{\alpha\}$ such that $\mathrm{dom}\,(\alpha) = \mathrm{im}\,(\mu)$ from the first condition. That is, there exists a unique edge from α and a unique edge to α in digraph Γ_U. Therefore, Γ_U is a cycle.

(iii) Let $U = \{\mu_1, \ldots, \mu_k\}$. Without loss of generality suppose that the cycle Γ_U is $\mu_1 \to \cdots \to \mu_k \to \mu_1$ where $k = \binom{n}{r}$. Since any product of some isotone maps is also an isotone map, U must contain at least one antitone map. Now consider the map

$$\delta = \begin{cases} \begin{pmatrix} A \\ B^R \end{pmatrix} & \text{if } \mu_1 = \begin{pmatrix} A \\ B \end{pmatrix}, \\ \begin{pmatrix} A \\ B \end{pmatrix} & \text{if } \mu_1 = \begin{pmatrix} A \\ B^R \end{pmatrix}, \end{cases}$$

for two different subsets A and B with cardinality r. It is easy to see from Lemma 2.2 that to generate the map δ we have to use the directed path $\mu_1 \to \cdots \to \mu_k \to \mu_1$ in Γ_U, and δ can be written only as the product $(\mu_1 \cdots \mu_k)^t \mu_1$ for some $t \geq 1$. If the number of antitone maps in U is an even number, then the consecutive product $\mu_1 \cdots \mu_k$ is the partial identity map with domain set $\mathrm{dom}\,(\mu_1)$, and so $(\mu_1 \cdots \mu_k)^t \mu_1 = \mu_1$ for each $t \geq 1$. Thus, we have $\delta \notin \langle U \rangle$, which is a contradiction, and so the number of antitone maps in U is an odd number.

($\Leftarrow$) Suppose that the conditions are satisfied, and let $\gamma \in D_r$. Then, from the first condition, there exist a unique $\alpha \in U$ and a unique $\beta \in U$ such that $\mathrm{dom}\,(\gamma) = \mathrm{dom}\,(\alpha)$ and $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\beta)$. Moreover, from the other conditions Γ_U is a cycle and the number of antitone maps in U is an odd number. If $\gamma \in U$ then $\gamma = \alpha = \beta$, as required. If $\gamma \notin U$ and $\alpha = \beta$, then $\mathrm{dom}\,(\gamma) = \mathrm{dom}\,(\alpha)$, $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\alpha)$, and $\gamma \neq \alpha$; that is, $H \setminus \{\alpha\} = \{\gamma\}$, where H is the $\mathcal{H}$-class contains α. Then, without loss of generality, suppose that $U = \{\alpha, \lambda_1, \ldots, \lambda_{k-1}\}$ and that the cycle Γ_U has a form

$$\alpha \to \lambda_1 \to \cdots \to \lambda_{k-1} \to \alpha$$

where $k = \binom{n}{r}$. It is clear that $\alpha\lambda_1 \cdots \lambda_{k-1}$ is an antitone map, and so

$$\gamma = \alpha\lambda_1 \cdots \lambda_{k-1}\alpha \in \langle U \rangle.$$

Finally, if $\gamma \notin U$ and $\alpha \neq \beta$, then, without loss of generality, suppose that $U = \{\alpha, \lambda_1, \ldots, \lambda_t, \beta, \mu_1, \ldots, \mu_l\}$ for $t, l \geq 0$ (notice that $t + l + 2 = k = \binom{n}{r}$), and that the cycle Γ_U has a form

$$\alpha \to \lambda_1 \to \cdots \to \lambda_t \to \beta \to \mu_1 \to \cdots \to \mu_l \to \alpha.$$

If the number of antitone maps in $\{\alpha, \lambda_1, \ldots, \lambda_t, \beta\}$ is even, then

$$\gamma = \begin{cases} \alpha\lambda_1 \cdots \lambda_t\beta & \text{if } \alpha \text{ is an isotone map,} \\ \alpha\lambda_1 \cdots \lambda_t\beta\mu_1 \cdots \mu_l\alpha\lambda_1 \cdots \lambda_t\beta & \text{if } \alpha \text{ is an antitone map,} \end{cases}$$

and so $\gamma \in \langle U \rangle$. If the number of antitone maps in $\{\alpha, \lambda_1, \ldots, \lambda_t, \beta\}$ is odd, then

$$\gamma = \begin{cases} \alpha\lambda_1 \cdots \lambda_t\beta\mu_1 \cdots \mu_l\alpha\lambda_1 \cdots \lambda_t\beta & \text{if } \alpha \text{ is an isotone map,} \\ \alpha\lambda_1 \cdots \lambda_t\beta & \text{if } \alpha \text{ is an antitone map,} \end{cases}$$

and so $\gamma \in \langle U \rangle$. Thus, $D_r \subseteq \langle U \rangle$, and so U is a minimal generating set of $PODI_{n,r}$. □

4. Generating sets of $POI_{n,r}$

First notice that for $PODI_{n,0} = POI_{n,0} = \{\emptyset\}$ there is nothing to prove. Now consider the subsemigroup $PODI_{n,1} = POI_{n,1}$, and notice that there exist only two $\mathcal{D}$-classes, D_0 and D_1, which have the following egg box forms:

$$D_0:\ \begin{array}{cc} & L_1 \\ R_1 & \boxed{\emptyset} \end{array},$$

$$D_1:\ \begin{array}{c|c|c|c|} & L_1 & & L_n \\ \hline R_1 & \begin{pmatrix}1\\1\end{pmatrix} & \cdots & \begin{pmatrix}1\\n\end{pmatrix} \\ \hline & \vdots & \ddots & \vdots \\ \hline R_n & \begin{pmatrix}n\\1\end{pmatrix} & \cdots & \begin{pmatrix}n\\n\end{pmatrix} \\ \hline \end{array},$$

respectively. As a particular case of Theorem 3.3 we have the following lemma.

Lemma 4.1 *Let* $\emptyset \neq U \subseteq D_1$ *in* $PODI_{n,1} = POI_{n,1}$. *Then* U *is a generating set of* $PODI_{n,1}$ *if and only if, for each* $1 \leq i, j \leq n$, *there exist* $\alpha, \beta \in U$ *such that*

(i) $\operatorname{dom}(\alpha) = \{i\}$,

(ii) $\operatorname{im}(\beta) = \{j\}$, *and*

(iii) α *is connected to* β *in the digraph* Γ_U.

Lemma 4.2 *Let* $\emptyset \neq U \subseteq D_1$ *in* $PODI_{n,1} = POI_{n,1}$ *with cardinality* n. *Then* U *is a minimal generating set of* $PODI_{n,1} = POI_{n,1}$ *if and only if*

(i) $|R \cap U| = |L \cap U| = 1$ *for each* $\mathcal{R}$*-class* R *and* $\mathcal{L}$*-class* L *in* D_1, *and*

(ii) *the digraph* Γ_U *is a cycle.*

Proof The proof is similar to the proof of Theorem 3.5. □

Theorem 4.3 *Let* $2 \leq r \leq n-1$, *and let* $U \subseteq D_r$ *in* $POI_{n,r}$. *Then* U *is a generating set of* $POI_{n,r}$ *if and only if, for each pair of subsets* A *and* B *of* X_n *with cardinality* r, *there exist* $\alpha, \beta \in U$ *such that* $\operatorname{dom}(\alpha) = A$, $\operatorname{im}(\beta) = B$, *and* α *is connected to* β *in the digraph* Γ_U.

Proof The proof is similar to the proof of Theorem 3.3. □

Lemma 4.4 *Let* $1 \leq r \leq n-1$, *and let* $\emptyset \neq U \subseteq D_r$. *For any subset* A *of* X_n *with cardinality* r, *let* R_A *and* L_A *be the* $\mathcal{R}$*-class and* $\mathcal{L}$*-class, which contain* id_A, *in* D_r, *respectively. Moreover, let* $H_A = R_A \cap L_A$; *that is,* $H_A = \{\operatorname{id}_A\}$.

(i) *If* $R_A \cap U \subseteq H_A$, *then* $R_A \cap \langle U \rangle \subseteq H_A$.

(ii) *If* $L_A \cap U \subseteq H_A$, *then* $L_A \cap \langle U \rangle \subseteq H_A$.

Proof The proof is similar to the proof of Lemma 3.4. □

Recall that, for $2 \leq r \leq n-1$, $\operatorname{rank}(POI_{n,r}) = \binom{n}{r}$, and so we have the following theorem.

Theorem 4.5 *For* $2 \leq r \leq n-1$, *let* $\emptyset \neq U \subseteq D_r$ *with cardinality* $\binom{n}{r}$. *Then* U *is a minimal generating set of* $POI_{n,r}$ *if and only if*

(i) $|R \cap U| = |L \cap U| = 1$ *for each* $\mathcal{R}$*-class* R *and* $\mathcal{L}$*-class* L *in* D_r, *and*

(ii) *the digraph* Γ_U *is a cycle.*

Proof The proof is similar to the proof of Theorem 3.5 by using the fact that $|H| = 1$ for each $\mathcal{H}$-class H in $POI_{n,r}$. □

5. Generating set of $E_{n,r}$

Let $M_p = \{\alpha \in E_n : |\mathrm{im}\,(\alpha)| = p\}$ for $0 \leq p \leq n$. Then it is clear that $|M_p| = \binom{n}{p}$ and $E_{n,r}$ is a disjoint union of $M_0, M_1, \ldots, M_r$ for $0 \leq r \leq n$.

Lemma 5.1 *Let* $\mathrm{id}_A \in M_{p-1}$ *for* $1 \leq p \leq r \leq n-1$. *Then there exist* $\emptyset \neq B, C \subseteq X_n$ *with cardinality* p *such that* $\mathrm{id}_A = \mathrm{id}_B \mathrm{id}_C$*; that is,* $M_{p-1} \subseteq (M_p)^2$.

Proof Let $A = \{a_1, \ldots, a_{p-1}\}$. Then there exist at least two distinct elements a, b of $X_n \backslash A$, and it is clear that $\mathrm{id}_A = \mathrm{id}_B \mathrm{id}_C$ where $B = \{a_1, \ldots, a_{p-1}\} \cup \{a\}$ and $C = \{a_1, \ldots, a_{p-1}\} \cup \{b\}$. $\square$

As a result of Lemma 5.1, $E_{n,r}$ is generated by elements of M_r for $0 \leq r \leq n-1$. Moreover, for $\mathrm{id}_A, \mathrm{id}_B \in E_n$, from the fact $\mathrm{id}_A \mathrm{id}_B = \mathrm{id}_{A\cap B}$ and that we have $\mathrm{id}_A \mathrm{id}_B \in M_p$ if and only if $A \cap B = A = B$ for $0 \leq p \leq n$. Then we can state the following corollary.

Corollary 5.2 *The set* M_r *is the minimum generating set of* $E_{n,r}$ *for* $0 \leq r \leq n-1$. *Moreover, it follows from the facts* $E_n \setminus E_{n,n-1} = \{\mathrm{id}_{X_n}\}$ *and that* id_{X_n} *is an identity map on* X_n *that* $\mathrm{rank}\,(E_{n,r}) = \binom{n}{r}$ *and* $\mathrm{rank}\,(E_n) = \binom{n}{n-1} + 1 = n + 1$.

Remark The free semilattice $\mathcal{SL}_A$ over a set A is the semigroup of all subsets of A with set-theoretic intersection as multiplication. In particular, if A is a finite set with cardinality n it is more common to use the notation $\mathcal{SL}_n$ instead of $\mathcal{SL}_A$ (see, for example, [2]). Let $A = X_n$ and consider the subsemigroups $\mathcal{SL}_{n,r} = \{Y \subseteq X_n : |Y| \leq r\}$ of $\mathcal{SL}_n$ for $0 \leq r \leq n-1$. It is clear that $\mathcal{SL}_{n,r} \cong E_{n,r}$, and so we also have $\mathrm{rank}\,(\mathcal{SL}_{n,r}) = \binom{n}{r}$ for $0 \leq r \leq n-1$ and $\mathrm{rank}\,(\mathcal{SL}_n) = n + 1$.

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