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Prime-valent arc-transitive basic graphs with order $4p$ or $4p^2$

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Abstract: A graph Γ is called G-basic if G is quasiprimitive or bi-quasiprimitive on the vertex set of Γ, where $G \leq \mathsf{Aut}\Gamma$. In this paper, we complete the classification of r-valent arc-transitive basic graphs with order $4p$ or $4p^2$, where p and r are odd primes.

Key words: Symmetric graph, basic graph, arc-transitive graph

1. Introduction

Throughout the paper, graphs considered are simple, connected, and undirected. For a graph Γ, we denote the valency, vertex set, edge set, arc set, and full automorphism group of Γ by $\mathsf{val}(\Gamma)$, $V\Gamma$, $E\Gamma$, $A\Gamma$, and $\mathsf{Aut}\Gamma$, respectively. Γ is called *G-vertex-transitive*, *G-edge-transitive*, or *G-arc-transitive* if $G \leq \mathsf{Aut}\Gamma$ is transitive on $V\Gamma$, $E\Gamma$, or $A\Gamma$; in particular, if $G = \mathsf{Aut}\Gamma$, then Γ is simply called vertex-transitive, edge-transitive, or arc-transitive. As we all know, a graph Γ is G-arc-transitive if and only if G is vertex-transitive and the vertex stabilizer G_v of $v \in V\Gamma$ in G is transitive on the neighborhood $\Gamma(v)$ of v. A permutation group G on a set Ω is called quasiprimitive if each nontrivial normal subgroup of G is transitive on Ω; G is called bi-quasiprimitive if each nontrivial normal subgroup of G has at most two orbits, and there is at least one normal subgroup of G that has exactly two orbits. A graph Γ is called G-basic if G is quasiprimitive or bi-quasiprimitive on $V\Gamma$ for some $G \leq \mathsf{Aut}\Gamma$.

For a group G and a subgroup H of G, we use $Z(G)$, $\mathsf{soc}(G)$, $C_G(H)$, and $N_G(H)$ to denote the center, the socle of G, the centralizer, and the normalizer of H in G, respectively. For two groups M and N, we use $M{:}N$ and $M \times N$ to denote the semidirect product and direct product of M by N. For a positive integer n and a prime divisor $r \mid n$, we denote the largest r-power that divides n by n_r, i.e. the r-part of n. We denote the dihedral group of order $2n$ by D_{2n}, the cyclic group of order n by $\mathbb{Z}_n$, and the alternating group and the symmetric group of degree n by A_n and S_n, respectively.

In the literature, the classification of arc-transitive graphs of small valency has been extensively studied; refer to [4–7, 9, 10, 15, 19–23, 27–29] and references therein. In particular, cubic and pentavalent arc-transitive graphs of order $4p$ or $4p^2$ are classified in [13, 16], and heptavalent arc-transitive graphs of order $4p$ are classified in [8], where p is a prime. The purpose of this paper is to characterize prime-valent arc-transitive basic graphs with order four times a prime or a prime square. The main result of this paper is the following theorem.

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Theorem 1.1 *Let Γ be a connected r-valent G-arc-transitive and G-basic graph with order $4p$ or $4p^2$, where $G \leq \mathsf{Aut}\Gamma$, and p, r are odd primes. Then G is almost simple and Γ is T-edge-transitive, where $T = \mathsf{soc}(G)$. Furthermore, $(\Gamma, |V\Gamma|, \mathsf{val}(\Gamma), \mathsf{Aut}\Gamma, (\mathsf{Aut}\Gamma)_\alpha)$ lies in Table 1, where $\alpha \in V\Gamma$.*

Table 1. Prime-valent arc-transitive basic graphs of order $4p$ or $4p^2$.

Row	Γ	$\lvert V\Gamma\rvert$	$\mathsf{val}(\Gamma)$	$\mathsf{Aut}\Gamma$	$(\mathsf{Aut}\Gamma)_\alpha$	*Remark*
1	$\mathcal{C}_{20}$	20	3	$\mathrm{A}_5 \times \mathbb{Z}_2$	S_3	*Example* 2.4 (1)
2	$\mathcal{C}_{12}$	12	5	$\mathrm{A}_5 \times \mathbb{Z}_2$	D_{10}	*Example* 2.4 (2)
3	$\mathcal{G}_{36}$	36	5	A_6	D_{10}	*Example* 2.1
4	K_{4p}	$4p$	$4p-1$	S_{4p}	S_{4p-1}	
5	C_{28}	28	3	$\mathrm{PGL}(2,7)$	D_{12}	*Example* 2.2
6	$P(10,7)$	20	3	$\mathrm{S}_5 \times \mathbb{Z}_2$	D_{12}	*Example* 2.3
7	$\mathcal{C}_{36}$	36	7	$\mathrm{PSL}(2,8)$	D_{14}	*Example* 2.4 (5)
8	$K_{2p^n,2p^n} - 2p^nK_2$	$4p^n$	$2p^n-1$	$\mathrm{S}_{2p^n} \times \mathbb{Z}_2$	S_{2p^n-1}	$n = 1, 2$

2. Examples

In this section, we give some examples of connected r-valent arc-transitive graphs of order $4p$ or $4p^2$ that appear in Theorem 1.1, where p and r are odd primes.

Example 2.1 *Let $G = \mathrm{A}_6$. Take a Sylow 5-subgroup, say P, of G, and set $H = N_G(P)$. From Conway et al. [3] it is known that $H \cong \mathrm{D}_{10}$. Note that all involutions in G are conjugates of each other. For any involution $x \in H$, we have $C_G(x) \cong \mathrm{D}_8$. Take an element g of order 4 in $C_G(x)$. Denote $\mathcal{G}_{36} = \mathsf{Cos}(G, H, HgH)$.*

By [14], we know that $\mathcal{G}_{36}$ is a pentavalent symmetric graph of order 36 and $\mathsf{Aut}\mathcal{G}_{36} \cong \mathrm{A}_6$.

Example 2.2 *We introduce a graph of order 28 that was discovered by Coxeter and investigated by Tutte [25]. Denote this graph by C_{28}. For its construction, see Biggs [1, Fig. 2(ii)]. By Biggs [1], C_{28} is 3-regular and $\mathsf{Aut}\mathrm{C}_{28} \cong \mathrm{PGL}(2,7)$.*

Example 2.3 *([8]) Let $n = 10$ and $3 \in \mathbb{Z}_{10} \setminus \{0\}$. The generalized Petersen graph $P(10,3)$ is the graph with vertex-set $\{x_i, y_i | i \in \mathbb{Z}_{10}\}$ and edge set $\{\{x_i, x_{i+1}\}, \{x_i, y_i\}, \{y_i, y_{i+k}\} | i \in \mathbb{Z}_{10}\}$.*

By [8], we know that $P(10,7)$ is a cubic symmetric graph of order 20.

By using the Magma program [2], we have the following examples.

Example 2.4 *(1) There is a unique connected cubic graph of order 20 that admits A_5 as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}_{20}$. Moreover, $\mathsf{Aut}\mathcal{C}_{20} \cong \mathrm{A}_5 \times \mathbb{Z}_2$.*

(2) There is a unique connected pentavalent graph of order 12 that admits A_5 as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}_{12}$. Moreover, $\mathsf{Aut}\mathcal{C}_{12} \cong \mathrm{A}_5 \times \mathbb{Z}_2$.

(3) *There exist no connected heptavalent graphs of order* 36 *that admit* $\mathrm{PSU}(3,3)$ *as an arc-transitive automorphism group.*

(4) *There is a unique connected cubic graph of order* 20 *that admits* S_5 *as an arc-transitive automorphism group. By Example 2.2 and Lemma 3.4, we know that the graph is isomorphic to* $P(10,7)$*. Moreover,* $\mathsf{Aut}P(10,7)\cong \mathrm{S}_5\times\mathbb{Z}_2$*.*

(5) *There is a unique connected heptavalent graph of order* 36 *that admits* $\mathrm{PSL}(2,8)$ *as an arc-transitive automorphism group. This graph is denoted by* $\mathcal{C}_{36}$*. Moreover,* $\mathsf{Aut}\mathcal{C}_{36}\cong\mathrm{PSL}(2,8)$*.*

(6) *There exist no connected heptavalent graphs of order* 36 *that admit* A_9 *as an arc-transitive automorphism group.*

3. Preliminary results

In this section, we give some necessary preliminary results.

We now give a result that will be useful.

Lemma 3.1 *Let* r *and* p *be odd primes, and let* Γ *be an* r*-valent* G*-arc-transitive graph of order* $4p$ *or* $4p^2$ *for some* $G\leq\mathsf{Aut}\Gamma$*. Let* N *be an insoluble normal subgroup of* G*. Then* $r\bigm| |N_v^{\Gamma(v)}|$ *for each* $v\in V\Gamma$*.*

Proof For each $v\in V\Gamma$, since $1\neq N_v\lhd G_v$ and G is transitive on $V\Gamma$, we have $N_v^{\Gamma(v)}\neq 1$ by connectivity of Γ. Since $G_v^{\Gamma(v)}$ acts primitively on $\Gamma(v)$ and $N_v^{\Gamma(v)}\unlhd G_v^{\Gamma(v)}$, it follows that $r\bigm| |N_v^{\Gamma(v)}|$. □

The following two lemmas may be deduced from the classification of permutation groups of the degree of a product of two prime powers (refer to [18]).

Lemma 3.2 *Let* T *be a nonabelian simple group that has a subgroup* H *of index* $2p$ *or* $2p^2$ *with* p *a prime. Then* $T,H,|H|$*, and* $|T:H|$ *are as in Table 2.*

Table 2. Nonabelian simple groups with a subgroup of index $2p$ or $2p^2$.

T	H	$\|H\|$	$\|T:H\|$	*Remark*
A_5	S_3	$2\cdot 3$	10	
A_{2p^n}	A_{2p^n-1}	$\frac{1}{2}(2p^n-1)!$	$2p^n$	$n=1,2$
$\mathrm{PSL}(d,s^f)$	P_1		$\frac{s^{fd}-1}{s^f-1}=2p^n$	$n=1,2$
$\mathrm{PSL}(d,2^f)$	H_1		$2\frac{2^{fd}-1}{2^f-1}=2p^n$	$n=1,2$
M_{11}	A_6	$2^3\cdot 3^2\cdot 5$	22	
M_{22}	$\mathrm{PSL}(3,4)$	$2^6\cdot 3^2\cdot 5\cdot 7$	22	
$\mathrm{PSU}(3,5)$	A_7	$2^3\cdot 3^2\cdot 5\cdot 7$	50	

Remark 1 $\mathrm{P}_1=[s^{f(d-1)}].(\mathbb{Z}_{\frac{s^f-1}{(d,s^f-1)}}.\mathrm{PSL}(d-1,s^f).\mathbb{Z}_{(d-1,s^f-1)})\cong\mathrm{H}_1.\mathbb{Z}_2$.

Lemma 3.3 *Let T be a nonabelian simple group with a subgroup X of index $4p$ or $4p^2$, where p is a prime. Then X is isomorphic to H or K, and $T, K, |K|, H, |H|$, and $|T{:}X|$ are as in Table 3, where K is a maximal subgroup of T but H is not a maximal subgroup of T.*

Table 3. Nonabelian simple groups with a subgroup of index $4p$ or $4p^2$.

T	K	$\|K\|$	H	$\|H\|$	$\|T:X\|$
A_5			$\mathbb{Z}_3$	3	$4\cdot 5$
			$\mathbb{Z}_5$	5	$4\cdot 3$
A_6	D_{10}	$2\cdot 5$			$4\cdot 3^2$
			$3^2:2$	$3^2\cdot 2$	$4\cdot 5$
A_8	S_6	$2^4\cdot 3^2\cdot 5$			$4\cdot 7$
A_9			S_7	$2^4\cdot 3^2\cdot 5\cdot 7$	$4\cdot 3^2$
A_{4p^n}	A_{4p^n-1}	$\frac{1}{2}(4p^n-1)!$			$4p^n, n=1,2$
M_{11}	$\mathrm{PSL}(2,11)$	$2^2\cdot 3\cdot 5\cdot 11$			$4\cdot 3$
M_{12}	M_{11}	$2^4\cdot 3^2\cdot 5\cdot 11$			$4\cdot 3$
J_2	$\mathrm{PSU}(3,3)$	$2^5\cdot 3^3\cdot 7$			$4\cdot 5^2$
HS	M_{22}	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11$			$4\cdot 5^2$
$\mathrm{PSL}(2,8)$	D_{18}	$2\cdot 3^2$			$4\cdot 7$
$\mathrm{PSL}(2,16)$	A_5	$2^2\cdot 3\cdot 5$			$4\cdot 17$
$\mathrm{PSL}(d,s^f)$	P_1				$\frac{s^{fd}-1}{s^f-1}=4p^n, n=1,2$
$\mathrm{PSL}(d,2^f)$			H_1		$4\frac{2^{fd}-1}{2^f-1}=4p^n, n=1,2$
$\mathrm{PSU}(3,2)$	$\mathrm{GU}(2,2)$	$2\cdot 3^2$			$4\cdot 3$
$\mathrm{PSU}(3,3)$	$\mathrm{PSL}(2,7)$	$2^3\cdot 3\cdot 7$			$4\cdot 3^2$
$\mathrm{PSp}(4,3)$	S_6	$2^4\cdot 3^2\cdot 5$			$4\cdot 3^2$
$\mathrm{PSp}(6,2)$	$\mathrm{PSU}(4,2):2$	$2^7\cdot 3^4\cdot 5$			$4\cdot 7$
	S_8	$2^7\cdot 3^2\cdot 5\cdot 7$			$4\cdot 3^2$
	$\mathrm{P\Omega}^-(6,2)$	$2^7\cdot 3^4\cdot 5$			$4\cdot 7$
$\mathrm{P\Omega}(5,3)$	$\mathbb{Z}_2\times \mathrm{P\Omega}^-(4,3)$	$2^4\cdot 3^2\cdot 5$			$4\cdot 3^2$

Remark 2 $\mathrm{P}_1=[s^{f(d-1)}].(\mathbb{Z}_{\frac{s^f-1}{(d,s^f-1)}}.\mathrm{PSL}(d-1,s^f).\mathbb{Z}_{(d-1,s^f-1)})\cong \mathrm{H}_1.\mathbb{Z}_4$.

The following lemma gives a classification of cubic symmetric graphs of order $4p$ or $4p^2$ for a prime p.

Lemma 3.4 ([8, Theorem 6.2]) *Let Γ be a connected cubic symmetric graph of order $4p$ or $4p^2$ for a prime p. Then Γ is isomorphic to the 2-regular hypercube Q_3 of order 8, the 2-regular generalized Petersen graphs $P(8,3)$ or $P(10,7)$ of order 16 or 20 respectively, the 3-regular dodecahedron of order 20, or the 3-regular Coxeter graph C_{28} of order 28.*

The next two lemmas give the classification of arc-transitive pentavalent graphs of order $4p$ or $4p^2$ for a prime p; see [16, Theorem 1.1] and [16, Corollary 1.2].

Lemma 3.5 ([16, Theorem 1.1]) *There exist no connected arc-transitive pentavalent graphs of order* $4p$ *or* $4p^2$ *for each prime* $p \geq 5$*.*

Lemma 3.6 ([16, Corollary 1.2]) *Let* p *be an odd prime. Then* $\mathcal{G}_{36}$ *is the only connected arc-transitive pentavalent graph of order* $4p^2$*.*

The following three lemmas determine the stabilizers of cubic, pentavalent, and heptavalent arc-transitive graphs, respectively.

Lemma 3.7 [24, 26] *Let* Γ *be a connected* (G,s)*-transitive cubic graph, where* $s \geq 1$*. Then* $s \leq 5$ *and the stabilizer* G_α *and* $|G_\alpha|$ *satisfy Table 4, where* $\alpha \in V\Gamma$*.*

Table 4. The stabilizers of cubic arc-transitive graphs.

s	1	2	3	4	5
G_α	$\mathbb{Z}_3$	S_3	D_{12}	S_4	$\mathrm{S}_4 \times \mathrm{S}_2$
$\lvert G_\alpha \rvert$	3	$2 \cdot 3$	$2^2 \cdot 3$	$2^3 \cdot 3$	$2^4 \cdot 3$

Lemma 3.8 ([11, 30]) *Let* Γ *be a pentavalent* (G,s)*-transitive graph for some* $G \leq \mathsf{Aut}\Gamma$ *and* $s \geq 1$*. Let* $v \in V\Gamma$*. If* G_v *is soluble, then* $|G_v| \bigm| 80$ *and* $s \leq 3$*. If* G_v *is insoluble, then* $|G_v| \bigm| 2^9 \cdot 3^2 \cdot 5$ *and* $2 \leq s \leq 5$*. Furthermore, one of the following holds:*

(1) $s = 1$, $G_v \cong \mathbb{Z}_5$, D_{10} *or* D_{20}*;*

(2) $s = 2$, $G_v \cong \mathrm{F}_{20}$, $\mathrm{F}_{20} \times \mathbb{Z}_2$, A_5 *or* S_5*;*

(3) $s = 3$, $G_v \cong \mathrm{F}_{20} \times \mathbb{Z}_4$, $\mathrm{A}_4 \times \mathrm{A}_5$, $(\mathrm{A}_4 \times \mathrm{A}_5){:}\mathbb{Z}_2$ *or* $\mathrm{S}_4 \times \mathrm{S}_5$*;*

(4) $s = 4$, $G_v \cong \mathrm{ASL}(2,4)$, $\mathrm{AGL}(2,4)$, $\mathrm{A\ L}(2,4)$ *or* $\mathrm{A\ L}(2,4)$*;*

(5) $s = 5$, $G_v \cong \mathbb{Z}_2^6{:}\ \mathrm{L}(2,4)$*.*

Lemma 3.9 ([12, Theorem 1.1]) *Let* X *be a connected* (G,s)*-transitive graph of valency* 7 *for some* $G \leq \mathsf{Aut}X$ *and* $s \geq 1$*. Let* $v \in VX$*. Then* $s \leq 3$ *and one of the following statements holds:*

(1) For $s = 1$, $G_v \cong \mathbb{Z}_7, \mathrm{D}_{14}, F_{21}, \mathrm{D}_{28}$ *or* $F_{21} \times \mathbb{Z}_3$*;*

(2) For $s = 2$, $G_v \cong F_{42}, F_{42} \times \mathbb{Z}_2, F_{42} \times \mathbb{Z}_3, \mathrm{PSL}(3,2), \mathrm{A}_7, \mathrm{S}_7, \mathbb{Z}_2^3 \rtimes \mathrm{SL}(3,2)$ *or* $\mathbb{Z}_2^4 \rtimes \mathrm{SL}(3,2)$*;*

(3) For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6, \mathrm{PSL}(3,2) \times \mathrm{S}_4, \mathrm{A}_7 \times \mathrm{A}_6, \mathrm{S}_7 \times \mathrm{S}_6, (\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2, \mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ *or* $[2^{20}] \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$*.*

Let a and d be positive integers. A prime r is called a primitive prime divisor of $a^d - 1$ if r divides $a^d - 1$ but does not divide $a^i - 1$ for $1 \leq i < d$. The following lemma is a well-known result of Zsigmondy.

Lemma 3.10 ([17, p.508]) *For any positive integers* a *and* d*, either* $a^d - 1$ *has a primitive prime divisor, or* $(d,a) = (6,2)$ *or* $(2, 2^m - 1)$*, where* $m \geq 2$*.*

4. Proof of Theorem 1.1

Lemma 4.1 *Let Γ be a connected r-valent G-arc-transitive and G-basic graph of order $4p$ or $4p^2$, where $G \leq \mathsf{Aut}\Gamma$, and p, r are odd primes. Then G is almost simple and Γ is T-edge-transitive, where $T = \mathsf{soc}(G)$. Furthermore, $(\Gamma, |V\Gamma|, \mathsf{val}\Gamma, T, T_\alpha)$ lies in Table 5, where $\alpha \in V\Gamma$.*

Table 5. Prime-valent G-arc-transitive and G-basic graphs of order $4p$ or $4p^2$.

Γ	$\|V\Gamma\|$	$\mathsf{val}\Gamma$	T	T_α	*Remark*
$\mathcal{C}_{20}$	20	3	A_5	$\mathbb{Z}_3$	
$\mathcal{C}_{12}$	12	5	A_5	$\mathbb{Z}_5$	
$\mathcal{G}_{36}$	36	5	A_6	D_{10}	
K_{12}	12	11	M_{11}	$PSL(2,11)$	
K_{12}	12	11	M_{12}	M_{11}	
K_{4p}	$4p$	$4p-1$	A_{4p}	A_{4p-1}	
K_{4p}	$4p$	$4p-1$	$PSL(d,s^f)$	P_1	
C_{28}	28	3	$PSL(2,7)$	D_6	
$\mathcal{C}_{36}$	36	7	$PSL(2,8)$	D_{14}	
$P(10,7)$	20	3	A_5	S_3	
$K_{2p^n,2p^n} - 2p^nK_2$	$4p^n$	$2p^n-1$	$PSL(d,s^f)$	P_1	$n=1,2$
$K_{2p^n,2p^n} - 2p^nK_2$	$4p^n$	$2p^n-1$	A_{2p^n}	A_{2p^n-1}	$n=1,2$
C_{28}	28	3	$PSL(2,7)$	D_{12}	

Proof Let N be a minimal normal subgroup of G. Then N has at most two orbits on $V\Gamma$. Note that the length of an N-orbit is not a prime power and the order of a nonabelian simple group has at least three distinct prime divisors. Then $N \cong T^k$ with T a nonabelian simple group and integer $k \geq 1$. By Lemma 3.1, $r \mid |N_\alpha|$ for each $\alpha \in V\Gamma$. Furthermore, it is easy to obtain that $|N_\alpha|_r = r$. Suppose that $k \geq 2$. Then, for each $i \in \{1, 2, \dots, k\}$, we have $|T_i{:}(T_i)_\alpha| \mid |N{:}N_\alpha| = 2p, 2p^2, 4p$, or $4p^2$. Since $(T_i)_\alpha \neq 1$, $r \mid |(T_i)_\alpha|_r$. It follows that $r^k \mid |(T_1)_\alpha \times \cdots \times (T_k)_\alpha|$, which is a contradiction as $(T_1)_\alpha \times \cdots \times (T_k)_\alpha \leq N_\alpha$ and $|N_\alpha|_r = r$. Thus, $d = 1$ and $N = T \trianglelefteq G$. It follows that G is almost simple and Γ is T-edge-transitive. Thus, $T \leq G \leq T.O$, where $O \cong \mathsf{Out}(T)$. Note that T has a subgroup of index $2p, 2p^2, 4p$, or $4p^2$, so T is known by Lemmas 3.2 and 3.3.

First, we analyze all the candidates in Table 3. In this case, Γ is T-arc-transitive. It follows that T_α has a subgroup $T_{\alpha\beta}$ with prime index r, where $\beta \in \Gamma(\alpha)$.

Suppose that $(T, T_\alpha) \cong (J_2, PSU(3,3))$. Since $PSU(3,3)$ has no subgroups with a prime index by Conway et al. [3], there exist no graphs Γ in this subcase. Similarly, we can exclude the subcase that $(T, T_\alpha) \cong (HS, M_{22})$. Suppose that $(T, T_\alpha) \cong (A_8, S_6)$ or $(PSp(4,3), S_6)$. Since $T_\alpha \cong S_6$ has no subgroups with an odd prime index by Magma [2], there exist no graphs Γ in this subcase. Similarly, we can exclude the subcase that $(T, T_\alpha) \cong (PSp(6,2), S_8)$.

Suppose that $(T, T_\alpha) \cong (A_9, S_7)$. Then $|V\Gamma| = |T{:}T_\alpha| = 36$ and $\mathsf{val}(\Gamma) = 7$, but by Example 2.4, there exist no graphs Γ in this subcase.

Suppose that $(T, T_\alpha) \cong (A_5, \mathbb{Z}_3)$. Then $|V\Gamma| = |T{:}T_\alpha| = 20$ and $\mathsf{val}(\Gamma) = 3$. By Example 2.4, $\Gamma \cong \mathcal{C}_{20}$. Suppose that $(T, T_\alpha) \cong (A_5, \mathbb{Z}_5)$. Then $|V\Gamma| = 12$ and $\mathsf{val}(\Gamma) = 5$. By Example 2.4, $\Gamma \cong \mathcal{C}_{12}$.

Suppose that $(T,T_\alpha)\cong(\mathrm{A}_6,\mathrm{D}_{10})$. Then $|V\Gamma|=36$, and $\mathsf{val}(\Gamma)=5$. By Lemma 3.6, $\Gamma\cong\mathcal{G}_{36}$. Suppose that $(T,T_\alpha)\cong(\mathrm{PSL}(2,8),\mathrm{D}_{18})$. Then $|V\Gamma|=28$, and $\mathsf{val}(\Gamma)=3$, which is not possible as $|T_\alpha|\bigm| 48$ for arc-transitive cubic graphs.

Suppose that $(T,T_\alpha)\cong(\mathrm{M}_{11},\mathrm{PSL}(2,11))$ or $(\mathrm{M}_{12},\mathrm{M}_{11})$. Then $|V\Gamma|=12$ and $\mathsf{val}(\Gamma)=11$. It follows that $\Gamma\cong K_{12}$.

Suppose that $(T,T_\alpha)\cong(\mathrm{PSU}(3,2),\mathrm{GU}(2,2))$. Then $|V\Gamma|=12$ and $\mathsf{val}(\Gamma)=3$. By Lemma 3.4, there exist no graphs Γ in this subcase. Suppose that $(T,T_\alpha)\cong(\mathrm{PSU}(3,3),$
$\mathrm{PSL}(2,7))$. Then $|V\Gamma|=36$ and $\mathsf{val}(\Gamma)=7$. By Example 2.4, there exist no graphs Γ in this subcase.

Suppose that $(T,T_\alpha)\cong(\mathrm{P}\Omega(5,3),\mathbb{Z}_2\times\mathrm{P}\Omega^-(4,3))$. Then $|V\Gamma|=|T{:}T_\alpha|=36$ and $\mathsf{val}(\Gamma)=5$, but by Lemma 3.8, arc-transitive pentavalent graphs have no such stabilizers, so there exist no graphs Γ in this subcase.

Suppose that $(T,T_\alpha)\cong(\mathrm{PSL}(2,16),\mathrm{A}_5)$. Then $|V\Gamma|=68$ and $\mathsf{val}(\Gamma)=5$. By Lemma 3.5, there exist no graphs Γ in this subcase. Suppose that $(T,T_\alpha)\cong(\mathrm{A}_{4p},\mathrm{A}_{4p-1})$. Since A_{4p} is 2-transitive on $V\Gamma$, $\Gamma\cong K_{4p}$ is the complete graph with $\mathsf{val}(\Gamma)=4p-1$ a prime. Note that $4p^2-1$ is not a prime, so we can exclude the subcase that $(T,T_\alpha)\cong(\mathrm{A}_{4p^2},\mathrm{A}_{4p^2-1})$.

Suppose that $(T,T_\alpha)\cong(\mathrm{A}_6,3^2{:}2)$. Then $|V\Gamma|=20$, and $\mathsf{val}(\Gamma)=3$, which is not possible as $|T_\alpha|\bigm| 48$ for arc-transitive cubic graphs.

Suppose that $(T,T_\alpha)\cong(\mathrm{PSp}(6,2),\mathrm{P}\Omega^-(6,2))$ or $(\mathrm{PSp}(6,2),\mathrm{PSU}(4,2):2)$. Then $|V\Gamma|=28$ and $\mathsf{val}(\Gamma)=3$ or 5, which is not possible as $|T_\alpha|\bigm| 48$ for arc-transitive cubic graphs, and arc-transitive pentavalent graphs have no such stabilizers.

Suppose that $(T,T_\alpha)\cong(\mathrm{PSL}(d,s^f),\mathrm{P}_1)$. Since T is 2-transitive on $V\Gamma$, $\Gamma\cong K_{4p}$ is the complete graph with $r=4p-1$ a odd prime.

Finally, we consider the case where $(T,T_\alpha)\cong(\mathrm{PSL}(d,2^f),\mathrm{H}_1)$. Note that $\frac{2^{fd}-1}{2^f-1}=p$ or p^2. By easy calculation, d is a prime. Assume $d\geq 3$. If $(d,2^f)=(3,2)$, then $T=\mathrm{PSL}(3,2)\cong\mathrm{PSL}(2,7)$, $T_\alpha=\mathrm{D}_6$, and $|V\Gamma|=28$. Since r divides $|T_\alpha|$, $r=3$. By Lemma 3.4 and Example 2.2, $\Gamma\cong\mathrm{C}_{28}$.

Suppose that $d=3$ and $f\geq 2$. Let t be an odd prime divisor of 2^f+1. As $(2^f-1,2^f+1)=1$, $(t,2^f(2^f-1))=1$. It follows that $(t,|\mathrm{P}_1|/|\mathrm{PSL}(2,2^f)|)=1$. Since 2^f+1 divides $|\mathrm{H}_1|$, and H_1 and $\mathrm{H}_1^{\Gamma(\alpha)}$ have the same prime divisors, $t\bigm| |\mathrm{H}_1^{\Gamma(\alpha)}|$. It follows that $\mathrm{PSL}(2,2^f)$ is a nonabelian simple compositor factor of $\mathrm{H}_1^{\Gamma(\alpha)}$. As $\mathrm{H}_1^{\Gamma(\alpha)}=T_\alpha^{\Gamma(\alpha)}$ is a transitive permutation group of prime degree r, either $\mathrm{H}_1^{\Gamma(\alpha)}\leq\mathbb{Z}_r{:}\mathbb{Z}_{r-1}$ is affine or $H^{\Gamma(\alpha)}$ is almost simple, and we further conclude that $\mathsf{soc}(\mathrm{H}_1^{\Gamma(\alpha)})=\mathrm{PSL}(2,2^f)$ is transitive. By checking the index of maximal subgroups of $\mathrm{PSL}(2,2^f)$ and [3], either $2^f=11$ and $r=11$, or $2^f+1=r$. The former case is impossible as $2^f\neq 11$. For the latter case, since 2^f+1 is a prime, $2^f=2^{2^m}$ for some positive integer m. Then $p^n=2^{2f}+2^f+1=(2^f+1)^2-2^f=(2^{2^m}-2^{2^{m-1}}+1)(2^{2^m}+2^{2^{m-1}}+1)$, where $n=1$ or 2, but by easy calculation, there exist no p satisfying the above condition.

Suppose $d\geq 4$. Note that $(2^f)^{d-1}-1$ divides $|\mathrm{H}_1|$. If $(d,2^f)=(7,2)$, then 7 divides $|\mathrm{H}_1|$ and does not divide $|\mathrm{P}_1|/|\mathrm{PSL}(d-1,2^f)|$; if $(d,2^f)\neq(7,2)$, then $(2^f)^{d-1}-1$ has a primitive prime divisor s by Lemma 3.10, and s does not divide $|\mathrm{P}_1|/|\mathrm{PSL}(d-1,2^f)|$. Since H_1 and $\mathrm{H}_1^{\Gamma(\alpha)}$ have the same prime divisors, we conclude that $\mathrm{PSL}(d-1,r)$ is a nonabelian simple compositor factor of $\mathrm{H}_1^{\Gamma(\alpha)}$, and hence $\mathrm{H}_1^{\Gamma(\alpha)}$ is an almost simple group with socle $\mathrm{PSL}(d-1,2^f)$. Thus, $\frac{(2^f)^{d-1}-1}{2^f-1}=r$. It follows that $d-1$ is a prime. Now both d and $d-1$ are primes, implying $d=3$, also yielding a contradiction.

Suppose $d=2$. Note that $2^f+1=p$ or p^2. If $2^f=4$, then $\mathrm{PSL}(2,2^f)=\mathrm{PSL}(2,4)\cong \mathrm{A}_5$, $\mathrm{P}_1\cong \mathbb{Z}_2^2:\mathbb{Z}_3$, and $\mathrm{H}_1\cong \mathbb{Z}_3$. By previous discussion, $\Gamma\cong \mathcal{C}_{20}$. If $2^f=8$, then $\mathrm{PSL}(2,2^f)=\mathrm{PSL}(2,8)\cong \mathrm{A}_5$, $\mathrm{P}_1\cong \mathbb{Z}_2^3:\mathbb{Z}_7$ and $\mathrm{H}_1\cong \mathrm{D}_{14}$. It follows that $|V\Gamma|=36$ and $\mathsf{val}(\Gamma)=7$. By Example 2.4, $\Gamma\cong \mathcal{C}_{36}$. If $2^f\geq 16$, then $\mathrm{P}_1\cong \mathbb{Z}_2^f{:}\mathbb{Z}_{2^f-1}$ has no subgroups with index 4, leading to a contradiction.

Next, we analyze all the candidates in Table 2. In this case, Γ is T-edge-transitive but not T-arc-transitive.

Suppose that $(T,T_\alpha)\cong(\mathrm{A}_5,\mathrm{S}_3)$. Then $G\cong \mathrm{S}_5$, $|V\Gamma|=2|T{:}T_\alpha|=20$, and $\mathsf{val}(\Gamma)=3$. It follows that $G_\alpha\cong \mathrm{S}_3$. By Lemma 3.4 and Example 2.4, $\Gamma\cong P(10,7)$. Suppose that $(T,T_\alpha)\cong(\mathrm{M}_{11},\mathrm{A}_6)$. Then $G\cong \mathrm{M}_{11}$, a contradiction. Suppose that $(T,T_\alpha)\cong(\mathrm{M}_{22},\mathrm{PSL}(3,4))$. Then $G\cong \mathrm{M}_{22}.\mathbb{Z}_2$, $|V\Gamma|=2|T{:}T_\alpha|=44$, and $\mathsf{val}(\Gamma)=3,5$, or 7. It follows that $|\mathrm{PSL}(3,4)|\bigm| |G_\alpha|$. However, cubic, pentavalent, and heptavalent arc-transitive graphs have no stabilizers of order divided by $|\mathrm{PSL}(3,4)|=20160$ (see Lemmas 3.7–3.9), a contradiction. Similarly, we can exclude the case where $(T,T_\alpha)\cong(\mathrm{PSU}(3,5),\mathrm{A}_7)$.

Suppose that $(T,T_\alpha)\cong(\mathrm{PSL}(d,s^f),\mathrm{P}_1)$, or $(\mathrm{A}_{2p^n},\mathrm{A}_{2p^n-1})$. Then $\Gamma\cong K_{2p^n,2p^n}-2p^nK_2$ with $r=2p^n-1$ an odd prime, where $n=1$ or 2.

Suppose that $(T,T_\alpha)\cong(\mathrm{PSL}(d,2^f),\mathrm{H}_1)$. Then $\frac{2^{fd}-1}{2^f-1}=p$ or p^2. By easy calculation, d is a prime. If $d=2$, then $\mathrm{P}_1\cong \mathbb{Z}_2^f{:}\mathbb{Z}_{2^f-1}$ has no subgroups with index 2, leading to a contradiction. If $d=3$ and $2^f=2$, then $G=\mathrm{PGL}(2,7)$, $|V\Gamma|=28$, and $G_\alpha\cong \mathrm{D}_{12}$. By Example 2.2, $\Gamma\cong \mathrm{C}_{28}$. For the remaining subcases, note that $T<G\leq T.\mathsf{Out}(T)$, so we easily know that there exist no Γ in these subcases by the discussion of the previous paragraph. □

Combining Lemma 4.1, and Examples 2.1–2.4, we complete the proof of Theorem 1.1.

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