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Certain strongly clean matrices over local rings

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Abstract: We are concerned about various strongly clean properties of a kind of matrix subrings $L_{(s)}(R)$ over a local ring R. Let R be a local ring, and let $s \in C(R)$. We prove that $A \in L_{(s)}(R)$ is strongly clean if and only if A or $I_2 - A$ is invertible, or A is similar to a diagonal matrix in $L_{(s)}(R)$. Furthermore, we prove that $A \in L_{(s)}(R)$ is quasipolar if and only if $A \in GL_2(R)$ or $A \in L_{(s)}(R)^{qnil}$, or A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ in $L_{(s)}(R)$, where $\lambda \in J(R)$, $\mu \in U(R)$ or $\lambda \in U(R)$, $\mu \in J(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective. Pseudopolarity of such matrix subrings is also obtained.

Key words: Matrix ring, strongly clean matrix, quasipolar matrix

1. Introduction

Throughout, all rings are associative with an identity. An element a in a ring R is *strongly clean* provided that it is the sum of an idempotent and a unit that commute. The *commutant* of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. Set $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. We say $a \in R$ is *quasinilpotent* if $a \in R^{qnil}$. The *double commutant* of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$.

In [6], an element a in a ring R is called *quasipolar* if for any $a \in R$ there exists $e^2 = e \in comm^2(a)$ such that $a + e \in U(R)$ and $ae \in R^{qnil}$. As is well known, an element $a \in R$ is quasipolar if and only if it has generalized Drazin inverse, i.e. there exists $b \in comm^2(a)$ such that $b = b^2a, a - a^2b \in R^{qnil}$ (see [6]).

Following [8], an element a in a ring R has a pseudo Drazin inverse if and only if there exists $b \in comm^2(a)$ such that $b = bab, a^k - a^{k+1}b \in J(R)$ for some $k \geq 1$. In a ring R, evidently, { elements having pseudo Drazin inverses } $\subseteq$ { quasipolar elements } $\subseteq$ { strongly clean elements }. The subjects of strongly clean rings, quasipolar rings, and pseudo Drazin inverses were extensively studied in [1–5, 7] and [10–12].

Evidently, the clean property for a ring does not transfer to its subrings. For example, $\mathbb{Z}_{(2)}$ is strongly clean, but the subring $\mathbb{Z}$ of $\mathbb{Z}_{(2)}$ is not strongly clean where $\mathbb{Z}_{(2)}$ is the localization of $\mathbb{Z}$ at the prime ideal (2). For instance, $4 \in \mathbb{Z}$ cannot be written as the sum of an idempotent and a unit that commute.

The motivation of this paper is to investigate the behave of subrings of a strongly clean ring. This enables us to construct related counterexamples as well. For this purpose, we introduce a kind of matrix subrings over

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a local ring R. Here, a ring R is *local* if it has only one maximal right ideal. Let R be a local ring, and let $s \in C(R)$. Let

$$L_{(s)}(R) = \{\begin{pmatrix} a & b \\ sc & d \end{pmatrix} \in M_2(R) \mid a,b,c,d \in R\},$$

where the operations are defined as those in $M_2(R)$. Then $L_{(s)}(R)$ forms a subring of $M_2(R)$. We prove that $A \in L_{(s)}(R)$ is strongly clean if and only if A or $I_2 - A$ is invertible, or A is similar to a diagonal matrix in $L_{(s)}(R)$. Moreover, we prove that $A \in L_{(s)}(R)$ is quasipolar if and only if $A \in GL_2(R)$ or $A \in L_{(s)}(R)^{qnil}$, or A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ in $L_{(s)}(R)$, where $\lambda \in J(R)$, $\mu \in U(R)$ or $\lambda \in U(R)$, $\mu \in J(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective. Note that for $a \in R$, l_a and r_a denote the abelian group endomorphisms of R given by left and right multiplication by a, respectively. Pseudopolarity of such matrix subrings is also obtained.

We use $J(R)$ to denote the Jacobson radical of R and $U(R)$ to denote the group of units of R. Furthermore, $C(R)$ is the center of a ring R and $\mathbb{N}$ stands for the set of all natural numbers.

2. Strongly clean matrices

The goal of this section is to investigate strong cleanness of $L_{(s)}(R)$. We begin with the following.

Proposition 2.1 *Let R be a ring, and let $s \in C(R) \bigcap U(R)$. Then $L_{(s)}(R) \cong M_2(R)$.*

Proof Let $\varphi : M_2(R) \to L_{(s)}(R)$, defined by

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ s(s^{-1}c) & d \end{pmatrix}.$$

As $s \in C(R) \bigcap U(R)$, one directly checks that φ is a ring isomorphism, as asserted. □

Lemma 2.2 *Let R be a ring, and let $s \in C(R) \bigcap J(R)$. Then $U(L_{(s)}(R)) = \{\begin{pmatrix} c & x \\ sy & d \end{pmatrix} \in M_2(R) \mid c,d \in U(R), x,y \in R\}$.*

Proof As $s \in J(R)$, we easily see that $U(L_{(s)}(R)) \subseteq \{\begin{pmatrix} c & x \\ sy & d \end{pmatrix} \in M_2(R) \mid c,d \in U(R), x,y \in R\}$. To see the converse inclusion, suppose that $c,d \in U(R)$. Then

$$\begin{pmatrix} c & x \\ sy & d \end{pmatrix}^{-1} = \begin{pmatrix} c^{-1} & -c^{-1}xd^{-1} \\ sd^{-1}yc^{-1} & d^{-1} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where $a = (1 - sxd^{-1}yc^{-1})^{-1}$ and $b = (1 - syc^{-1}xd^{-1})^{-1}$, as desired. □

Lemma 2.3 *Let R be a ring, and let $s \in C(R) \bigcap J(R)$. Then $J(L_{(s)}(R)) = \{\begin{pmatrix} c & x \\ sy & d \end{pmatrix} \in M_2(R) \mid c,d \in J(R), x,y \in R\}$.*

Proof As is well known, $r \in J(R)$ if and only if $1 - rx \in U(R)$ for any $x \in R$. We easily obtain the result by Lemma 2.2. □

Lemma 2.4 *Let R be a ring, and let $e, f \in R$ be idempotents. Then the following are equivalent:*

(1) *There exist $u, v \in U(R)$ such that $uev = f$.*

(2) *e is similar to f, i.e. $w^{-1}ew = f$ for some $w \in U(R)$.*

Proof (1) $\Rightarrow$ (2) By hypothesis, there exist $u, v \in U(R)$ such that $uev = f$. Let $w = u^{-1}(-1 + f + ueu^{-1})$. Then $w^{-1} = (-1 + f + ueu^{-1})u$. Therefore, $fw^{-1} = w^{-1}e$, and so $w^{-1}ew = f$.

(2) $\Leftarrow$ (1) This is trivial. □

Lemma 2.5 *Let R be a local ring, let $s \in C(R) \bigcap J(R)$, and let $E^2 = E \in L_{(s)}(R)$. Then $E = 0$, $E = I_2$, or E is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.*

Proof Assume that $E = \begin{pmatrix} c & x \\ sy & d \end{pmatrix} \neq 0$ and $E \neq I_2$. Then c or d is invertible.

Case I. $c \in U(R)$. Then

$$\begin{pmatrix} 1 & 0 \\ -syc^{-1} & 1 \end{pmatrix} E \begin{pmatrix} c^{-1} & -c^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix}.$$

This implies that $\begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix} \in L_{(s)}(R)$ is regular, and then so is $d - syc^{-1}x \in R$. As R is local, we easily check that $d - syc^{-1}x$ is zero or invertible. If $d - syc^{-1}x = 0$, then P_1EQ_1 is an idempotent diagonal matrix where $P_1 = \begin{pmatrix} 1 & 0 \\ -syc^{-1} & 1 \end{pmatrix} \in U(L_{(s)}(R))$ and $Q_1 = \begin{pmatrix} c^{-1} & -c^{-1}x \\ 0 & 1 \end{pmatrix} \in U(L_{(s)}(R))$. If $d - syc^{-1}x$ is invertible, then there exist

$$P_2 = \begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -syc^{-1} & 1 \end{pmatrix} \in U(L_{(s)}(R))$$

and

$$Q_2 = \begin{pmatrix} c^{-1} & -c^{-1}x \\ 0 & 1 \end{pmatrix} \in U(L_{(s)}(R))$$

such that P_2EQ_2 is an idempotent diagonal matrix. In light of Lemma 2.4, E is similar to a diagonal matrix.

Case II. $d \in U(R)$. Then

$$\begin{pmatrix} 1 & -xd^{-1} \\ 0 & 1 \end{pmatrix} E \begin{pmatrix} 1 & 0 \\ -sd^{-1}y & d^{-1} \end{pmatrix} = \begin{pmatrix} c - sxd^{-1}y & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that $\begin{pmatrix} c - sxd^{-1}y & 0 \\ 0 & 1 \end{pmatrix} \in L_{(s)}(R)$ is regular, and then so is $c - sxd^{-1}y \in R$. Hence, $c - sxd^{-1}y$ is zero or invertible. Thus, similar to Case I, we have $P, Q \in U(L_{(s)}(R))$ such that PEQ is an idempotent diagonal matrix. In light of Lemma 2.4, E is similar to a diagonal matrix.

Write $P^{-1}EP = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ for some $P \in U(L_{(s)}(R))$. We may assume that $e = 1$, $f = 0$ or $e = 0$, $f = 1$. This completes the proof. □

Theorem 2.6 *Let R be a local ring, and let $s \in C(R)$. Then $A \in L_{(s)}(R)$ is strongly clean if and only if*

(1) A or $I_2 - A$ is invertible; or

(2) A is similar to a diagonal matrix in $L_{(s)}(R)$.

Proof Since R is local, we see that $s \in U(R)$ or $s \in J(R)$.

Case I. $s \in U(R)$. By virtue of Proposition 2.1, $L_{(s)}(R) \cong M_2(R)$. Then the result follows from [9].

Case II. $s \in J(R)$.

$\Longleftarrow$ If A or $I_2 - A$ is invertible, then $A \in L_{(s)}(R)$ is strongly clean. Suppose that A is similar to a diagonal matrix in $L_{(s)}(R)$. Then there exists $U \in U(L_{(s)}(R))$ such that $U^{-1}AU = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. As R is local, it is strongly clean. Hence, we can find idempotents $e, f \in R$ such that $a - e, b - f \in U(R)$, $ea = ae$, and $bf = fb$. Set $E = U \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} U^{-1}$. Then $E^2 = E \in L_{(s)}(R)$ and $EA = AE$. Furthermore, $A - E \in U(L_{(s)}(R))$, as desired.

$\Longrightarrow$ Suppose that A and $I_2 - A$ are not invertible. Write $A = E + U$ with $EA = AE$, $E^2 = E$, $U \in U(L_{(s)}(R))$. Set $E = \begin{pmatrix} c & x \\ sy & d \end{pmatrix}$. In light of Lemma 2.5, E is similar to a diagonal matrix.

Write $P^{-1}EP = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ for some $P \in U(L_{(s)}(R))$. We may assume that $e = 1$, $f = 0$ or $e = 0$, $f = 1$. If $e = 1$, $f = 0$, then

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + P^{-1}UP$$

and

$$P^{-1}AP \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}AP.$$

This forces that $P^{-1}AP$ is diagonal. If $e = 0$, $f = 1$, we prove that A is similar to a diagonal matrix in a similar way. This completes the proof. □

3. Quasipolar and pseudopolar matrices

The aim of this section is to extend Theorem 2.6 to quasipolar and pseudopolar matrices over a local ring. The following lemma is crucial.

Lemma 3.1 *Let R be a local ring, and $s \in C(R)$, and let $a, b \in R$. Then all matrices that commute with $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$ must be diagonal in $L_{(s)}(R)$ if and only if both $l_a - r_b$ and $r_a - l_b$ are injective.*

Proof $\Longrightarrow$ Assume that $ax = xb$. Then

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in comm\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right).$$

Hence, $x = 0$. Thus, $l_a - r_b : R \to R$ is injective. Assume that $bx = xa$. Then $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in comm\left(\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}\right)$; hence, $x = 0$. Thus, $l_b - r_a : R \to R$ is injective.

$\Longleftarrow$ This is clear by [3, Lemma 3.4], as $L_{(s)}(R)$ is a subring of $M_2(R)$. □

Theorem 3.2 *Let R be a local ring, and let $s \in C(R)$. Then $A \in L_{(s)}(R)$ is quasipolar if and only if*

(1) *$A \in GL_2(R)$; or*

(2) *$A \in L_{(s)}(R)^{qnil}$; or*

(3) *A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ in $L_{(s)}(R)$, where $\lambda \in J(R)$, $\mu \in U(R)$ or $\lambda \in U(R)$, $\mu \in J(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective.*

Proof We may assume that $s \in U(R)$ or $s \in J(R)$.

Case I. $s \in U(R)$. By virtue of Proposition 2.1, $L_{(s)}(R) \cong M_2(R)$. Hence, the result follows from [4].

Case II. $s \in J(R)$.

$\Longrightarrow$ Since $A \in L_{(s)}(R)$ is quasipolar. Write $A + E = U$ with $E^2 = E \in comm^2(A)$, $U \in U(L_{(s)}(R))$ and $AE \in L_{(s)}(R)^{qnil}$. In view of Lemma 2.5, $E = 0$, $E = I_2$, or E is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Case 1. $E = 0$. Then $A \in GL_2(R)$.

Case 2. $E = I_2$. Then $A \in L_{(s)}(R)^{qnil}$.

Case 3. E is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Write $P^{-1}EP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ where $P \in U(L_{(s)}(R))$. Then

$$P^{-1}AP + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P^{-1}UP$$

and

$$P^{-1}AP\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P^{-1}AP.$$

As in the proof of Theorem 2.6, $P^{-1}AP$ is a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $\lambda \in J(R)$, $\mu \in U(R)$. Let $x \in R$ such that $\lambda x = x\mu$. Then $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in comm(P^{-1}AP)$, and so $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in comm\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$. Therefore, $x = 0$, and so $l_\lambda - r_\mu$ is injective. Likewise, $l_\mu - r_\lambda$ is injective.

Case 4. E is similar to $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then similar to the proof of Case 3, A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ in $L_{(s)}(R)$ where $\lambda \in U(R)$, $\mu \in J(R)$. The rest is also similar to the Case 3.

$\Longleftarrow$ Case 1. $A \in GL_2(R)$. Then A is quasipolar.

Case 2. $A \in L_{(s)}(R)^{qnil}$. Then A is quasipolar with the spectral idempotent I_2.

Case 3. A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in J(R)$, $\mu \in U(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective. Then

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda+1 & 0 \\ 0 & \mu \end{pmatrix} \in U(L_{(s)}(R)).$$

Let $\begin{pmatrix} x & q \\ sp & y \end{pmatrix} \in comm\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right)$. In view of Lemma 3.1, $sp = q = 0$. Then $\begin{pmatrix} x & q \\ sp & y \end{pmatrix} \in comm\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$. Therefore,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in comm^2\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right).$$

Furthermore, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in L_{(s)}(R)^{qnil}$ and so A is quasipolar.

Case 4. A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in U(R)$, $\mu \in J(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective. Then similar to Case 3, A is quasipolar with the spectral idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This completes the proof. □

Corollary 3.3 *Let R be a commutative local ring, and let $s \in R$. Then $A \in L_{(s)}(R)$ is quasipolar if and only if*

(1) *$A \in GL_2(R)$; or*

(2) *$A^2 \in J(L_{(s)}(R))$; or*

(3) *A is similar to a diagonal matrix in $L_{(s)}(R)$.*

Proof $\Longrightarrow$ This is obvious by Theorem 3.2.

$\Longleftarrow$ If $A \in GL_2(R)$ or $A^2 \in J(L_{(s)}(R))$, then A is quasipolar by Theorem 3.2. Suppose that A is similar to a diagonal matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. If $\alpha, \beta \in U(R)$, then $A \in U(L_{(s)}(R))$ and so A is quasipolar. If $\alpha, \beta \in J(R)$, then $A \in J(L_{(s)}(R))$ and so A is quasipolar. Otherwise, A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in U(R)$, $\beta \in J(R)$ or $\alpha \in J(R)$, $\beta \in U(R)$. According to Theorem 3.2, $A \in L_{(s)}(R)$ is quasipolar, as asserted. □

Theorem 3.4 *Let R be a local ring, and let $s \in C(R)$. Then $A \in L_{(s)}(R)$ is pseudopolar if and only if*

(1) $A \in GL_2(R)$ *or* $A^2 \in J(L_{(s)}(R))$*; or*

(2) A *is similar to a diagonal matrix* $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ *in* $L_{(s)}(R)$*, where* $\lambda \in J(R)$*,* $\mu \in U(R)$ *or* $\lambda \in U(R)$*,* $\mu \in J(R)$*, and* $l_\mu - r_\lambda$*,* $l_\lambda - r_\mu$ *are injective.*

Proof Let A be pseudopolar. It is clear that $A \in GL_2(R)$ if and only if A is pseudopolar with an idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Also, $A^2 \in J(L_{(s)}(R))$ if and only if A is pseudopolar with an idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, we may assume $A \notin GL_2(R)$ or $A^2 \notin J(L_{(s)}(R))$. Since A is pseudopolar, there exists $P^2 = P \in L_{(s)}(R)$ such that $P \in comm^2(A)$, $A+P \in U(L_{(s)}(R))$, and for some $k \geq 1$, $A^kP \in J(L_{(s)}(R))$. Then by Lemma 2.5, there exists $V \in U(L_{(s)}(R))$ such that $V^{-1}PV = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. We may take $e = 1$, $f = 0$ or $e = 0$, $f = 1$. If $e = 1$ and $f = 0$, since $AP = PA$, A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $\lambda \in J(R)$, $\mu \in U(R)$. Thus, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ is pseudopolar since A is pseudopolar. Hence, the strongly spectral idempotent of $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ is $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. To see that $l_\lambda - r_\mu$ is injective, let $(l_\lambda - r_\mu)(x) = 0$. Then

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Since $Q \in comm^2\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right)$, we have $x = 0$ as asserted. If $(l_\mu - r_\lambda)(y) = 0$, then for $B = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$, $B\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}B$. Hence, similarly, $l_\mu - r_\lambda$ is injective. If $e = 0$ and $f = 1$, then A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $\lambda \in U(R)$ and $\mu \in J(R)$. Furthermore, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ is pseudopolar with the strongly spectral idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, $l_\mu - r_\lambda$ and $l_\lambda - r_\mu$ are injective. By Lemma 3.1, the converse is obvious. □

Corollary 3.5 *Let* R *be a commutative local ring, and let* $s \in R$*. Then* $A \in L_{(s)}(R)$ *is quasipolar if and only if it is pseudopolar.*

Proof $\Longleftarrow$ This is obvious by Theorem 3.2 and Theorem 3.4, as $A^2 \in J(L_{(s)}(R))$ implies that $A \in L_{(s)}(R)^{qnil}$.

$\Longrightarrow$ If $s \in U(R)$, we obtain the result by [3] and Proposition 2.1. We may assume that $s \in J(R)$. Write $A = \begin{pmatrix} x & q \\ sp & y \end{pmatrix} \in L_{(s)}(R)^{qnil}$. Then $xy = det(A) + spq \in J(R)$. Suppose $x + y \in U(R)$. Choose $Y = \begin{pmatrix} -(x+y)^{-1} & 0 \\ 0 & -(x+y)^{-1} \end{pmatrix}$. Then $Y \in comm(A)$. Hence, $I_2 + AY \in U(L_{(s)}(R))$. This shows that $1 - x(x+y)^{-1}, 1 - y(x+y)^{-1} \in U(R)$. Thus, $x, y \in U(R)$, a contradiction. Therefore, $x + y \in J(R)$. Regarding A as a matrix in $M_2(R)$, by the Cayley–Hamilton theorem, $A^2 = tr(A)A - det(A)I_2 \in M_2(J(R))$. Moreover, we have $A^2 \in J(L_{(s)}(R))$. Therefore, we complete the proof by Theorem 3.4 and Theorem 3.2. □

Let R be a commutative local ring, let $s \in R$, and let $A \in L_{(s)}(R)$. Evidently, $A \in L_{(s)}(R)^{qnil}$ if and only if $A \in M_2(R)^{qnil}$ if and only if $A^2 \in M_2(J(R))$.

Example 3.6 Let $R = \mathbb{Z}_4$. Then we have

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, $\begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix}$ is isomorphic to the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in $M_2(\mathbb{Z}_4)$. However, $\begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix}$ is not isomorphic to the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in $L_{(2)}(\mathbb{Z}_4)$. Otherwise, we can find some $p, q \in \mathbb{Z}_4$ such that

$$\begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x & q \\ 2p & y \end{pmatrix} = \begin{pmatrix} x & q \\ 2p & y \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

where x, y are -1 or 1. Thus, $2x = 0$, and so $2 = 0$, which is absurd. In this case, $l_2 - r_1$ and $l_1 - r_2 : \mathbb{Z}_4 \to \mathbb{Z}_4$ are injective.

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