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# Existence and uniqueness of solution for differential equation of fractional order $2 < \alpha \leq 3$ with nonlocal multipoint integral boundary conditions

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**Abstract:** This paper is devoted to the study of nonlinear fractional differential equation involving Caputo fractional derivative of order $2 < \alpha \leq 3$ with nonlocal multipoint integral boundary conditions. Some efficient results about the existence and uniqueness are obtained by applying the Banach fixed point theorem, Schaefer's fixed point theorem, and a nonlinear alternative for single valued maps. Two examples are given to illustrate the results.

**Key words:** Fractional differential equation, Caputo fractional derivative, Riemann–Liouville fractional integral, Banach fixed point theorem, Schaefer's fixed point theorem, nonlinear alternative for single valued maps

## 1. Introduction

The purpose of this paper is to study the existence and uniqueness of solutions for nonlinear fractional differential equations involving a Caputo fractional derivative, given as

$$ {}^cD^{\alpha}x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \tag{1.1}$$

with the following nonlocal multipoint integral boundary conditions:

$$\left.\begin{aligned} &x(0) = 0, \\ &x(1) = \beta \int_0^{\delta} x(s)ds, \qquad 0 < \delta < 1, \\ &[J^q x](\lambda) = x(1), \qquad 0 < \lambda < 1, \end{aligned}\right\} \tag{1.2}$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha$, $J^q$ denotes the Riemann–Liouville fractional integral of order $q$, $f : [0, 1] \times I\!R \to I\!R$ is a continuous function, and $\beta \in I\!R$ with the condition that $\beta \neq \frac{2}{\delta^2}$.

Boundary value problems with integral boundary conditions constitute a very attractive and significant class of problems. These include two-point, three-point, multipoint, and nonlocal boundary value problems as special cases. In addition, integral boundary conditions appear in blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, nonlinear oscillations in earthquakes, etc. For more details and applications as regards the nonlinear integral boundary conditions, see [11, 15, 17, 18, 22, 23, 25].

Fractional-order models are found to be more adequate than integer-order models in some real-world problems. Moreover, for the description of memory and hereditary properties, the fractional derivative is an

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excellent instrument. Fractional differential equations have gained considerable attention in various fields of applied mathematics and engineering such as physics, polymer rheology, regular vibration in thermodynamics, and biophysics. For more information, see [3, 5, 6, 10, 16, 18, 20].

Recently, boundary value problems for nonlinear differential equations have arisen in a variety of areas of mathematics. Benchohra et al. [8] established sufficient conditions for the existence of solutions of boundary value problems with nonlinear fractional differential equations. Akram and Anjum [4] studied fractional boundary value problems using the Mittag–Leffler function. Benchohra and Lazreg worked on nonlinear implicit fractional differential equations with boundary conditions [9].

Agarwal et al. [1] studied the existence and uniqueness of fractional differential equations and inclusions involving Caputo fractional derivatives for initial and boundary value problems. Zhang worked on the existence of a positive solution for a nonlinear fractional differential equation [24]. Yu and Gao [25] obtained a sufficient condition for the existence of solutions of fractional differential equations. Kalamani et al. studied existence results for fractional evolution systems in Banach spaces using Riemann–Liouville fractional derivatives [13]. Khan et al. worked on the existence theorems and Hyers–Ulam stability for a coupled system of fractional differential equations using the p-Laplacian operator [14]. Kalamani et al. [12] obtained sufficient conditions for the existence of solutions for a class of impulsive fractional neutral stochastic integrodifferential systems with nonlocal conditions and state-dependent delay. Xu studied a class of boundary value problems of fractional differential equations with integral and antiperiodic boundary conditions [23].

Motivated by the previous literature, this paper is concerned with the existence and uniqueness of fractional differential equations with nonlocal multipoint integral boundary conditions.

The organization of the rest of the paper consists of four sections. In Section 2, some notations, definitions of Caputo fractional derivative, the Riemann–Liouville fractional integral, and the Riemann–Liouville derivative are presented. In the same section, some lemmas are given, which will be used in Section 4. In Section 3, the solution of the BVP (1.1) with the boundary conditions (1.2) is obtained. Some useful results about the existence and uniqueness of nonlinear fractional differential equations are formulated in Section 4. In Section 5, two appropriate examples are given to substantiate the results.

## 2. Preliminaries

Let $C([0,1],\mathbb{R})$ be a Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm defined by

$$||x|| = \sup\{|x(t)|;\ t \in [0,1]\}.$$

**Definition 1** *[20] The Riemann–Liouville fractional integral of order $\alpha$ for a continuous function $f(t)$ is defined as*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(q)}{(t-s)^{\alpha-1}} f(s)dq, \quad \alpha > 0.$$

**Definition 2** *[21] For a continuous function $f(t)$, the Riemann–Liouville fractional derivative of order $\alpha$ is defined as*

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{\alpha-1} f(s)dq, \quad n = [\alpha]+1.$$

**Definition 3** *[21] For a continuous function* $f : [0,\infty) \to I\!R$, *the Caputo derivative of fractional order* $\alpha$ *is defined as*

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}f^{n}(s)ds, \quad n-1<\alpha<n, \ n=[\alpha]+1,$$

*provided that* $f^{n}(t)$ *exists, where* $[\alpha]$ *denotes the integer part of the real number* $\alpha$.

**Lemma 2.1** *[28] Consider the following differential equation and for* $\alpha > 0$*:*

$$^{c}D^{\alpha}x(t) = 0. \tag{2.1}$$

*The solution of Eq. (2.1) can be written in the following form:*

$$x(t) = a_0 + a_1 t + a_2 t^2 + ... + a_{n-1}t^{n-1},$$

*where* $a_i \in I\!R, \ i = 0,1,2,...,n-1, \quad n = [\alpha]+1.$

**Lemma 2.2** *[23] Assuming* $\alpha > 0$, *then*

$$J^{\alpha}\ {}^{c}D^{\alpha}x(t) = x(t) + a_0 + a_1 t + a_2 t^2 + ... + a_{n-1}t^{n-1},$$

*for some* $a_i \in I\!R, \ i = 0,1,2,...,n-1$ *where* $n$ *is the smallest integer greater than or equal to* $\alpha$.

**Lemma 2.3** *[9] (Banach fixed point theorem) If* $C$ *is a nonempty closed subset of a Banach space* $C([0,1], I\!R)$ *and* $T : R \to R,$ *is a contraction mapping, then* $T$ *has a unique fixed point.*

**Lemma 2.4** *[2] (Schaefer's fixed point theorem)*
*Assume that* $C([0,1], I\!R)$ *is a Banach space of all continuous functions and* $T : C([0,1], I\!R) \to C([0,1], I\!R)$ *is a continuous compact mapping and the set*

$$\bigcup_{0\le\nu\le1}\{x \in C([0,1], I\!R) : x = \nu T(x)\}$$

*is bounded. Then* $T$ *has a fixed point.*

**Lemma 2.5** *[19] (Nonlinear alternative for single valued maps)*
*Let* $H$ *be a Banach space,* $S$ *be a closed and convex subset of* $H$, $N$ *an open subset of* $S$, *and* $0 \in N$. *Assume that* $T : \overline{N} \to S$ *is a continuous, compact (that is,* $T(\overline{N})$ *is a relatively compact subset of* $S$*) map. Then either:*
*(i)* $T$ *has a fixed point in* $\overline{N}$, *or*
*(ii) there is a* $n \in \partial N$ *(the boundary of* $N$ *in* $S$*) and* $\nu \in (0,1)$ *with* $n = \nu T(n)$.

## 3. Unique solution of boundary value problem (1.1)

Consider the boundary value problem as

$$\left.\begin{array}{l} {}^cD^{\alpha}x(t) = h(t),\ 0 < t < 1, \quad 2 < \alpha \leq 3, \\ x(0) = 0, \\ x(1) = \beta \int_0^{\delta} x(s)ds, \qquad 0 < \delta < 1, \\ [J^q x](\lambda) = x(1), \qquad 0 < \lambda < 1, \end{array}\right\} \tag{3.1}$$

where $(h \in C[0,1], \mathbb{R})$.

Define

$$A = \left( \frac{1}{-2\lambda^{q+1}(\beta\delta^2 - 3)(q+2) - 6\lambda^{q+2}(2 - \beta\delta^2) + (q+2)!\beta\delta^2(2\delta - 3)} \right),$$

such that $A \neq 0$.

Take the Riemann–Liouville integral on both sides of BVP (3.1) as

$J^{\alpha}\ {}^cD^{\alpha}x(t) = J^{\alpha}h(t)$.
Using Lemma (2.2), the solution $x(t)$ can be written in the form of an integral equation as

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - a_0 - a_1 t - a_2 t^2, \tag{3.2}$$

for some constants $a_0,\ a_1,\ a_2 \in \mathbb{R}$.
By applying the given boundary condition $x(0) = 0$, it follows that $a_0 = 0$.

Applying the second boundary condition, $x(1) = \beta \int_0^{\delta} x(s)ds$, gives

$$\begin{aligned} \beta \int_0^{\delta} x(s)ds &= \beta \int_0^{\delta} \left( \int_0^{s} \frac{(s-m)^{\alpha-1}}{\Gamma(\alpha)} h(m)dm - a_0 s - a_2 s^2 \right) ds \\ &= \beta \int_0^{\delta} \left( \int_0^{s} \frac{(s-m)^{\alpha-1}}{\Gamma(\alpha)} h(m)dm \right) ds - \beta a_0 \frac{\delta^2}{2} - \beta a_2 \frac{\delta^3}{3} \end{aligned}$$

and

$$x(1) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - a_1 - a_2,$$

which imply that

$$a_1 = \frac{2}{(2-\beta\delta^2)}\left\{\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - \beta\int_0^\delta\left(\int_0^s \frac{(s-m)^{\alpha-1}}{\Gamma(\alpha)}h(m)dm\right)ds + a_2(\frac{\beta\delta^3}{3}-1)\right\}. \tag{3.3}$$

Substituting the above value of $a_1$ in Eq.(3.2) yields

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - t\left[\frac{2}{2-\beta\delta^2}\left\{\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - \beta\int_0^\delta(\int_0^s \frac{(s-m)^{\alpha-1}}{\Gamma(\alpha)}h(m)dm)ds + a_2(\frac{\beta\delta^3}{3}-1)\right\}\right] - a_2t^2. \tag{3.4}$$

Taking the Riemann–Liouville integral of Eq. (3.4) of order $q$, it can be written as

$$J^qx(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[\int_0^s \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}h(r)dr - s(\frac{2}{2-\beta\delta^2})\left\{\int_0^1 \frac{(1-r)^{\alpha-1}}{\Gamma(\alpha)}h(r)dr - \beta\int_0^\delta(\int_0^r \frac{(r-m)^{\alpha-1}}{\Gamma(\alpha)}h(m)dm)dr + a_2(\frac{\beta\delta^3}{3}-1)\right\}\right] - a_2\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}s^2ds. \tag{3.5}$$

Eq.(3.5) can also be written as

$$J^qx(t) = \frac{1}{\Gamma(q)}\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (t-s)^{q-1}(s-r)^{\alpha-1}h(r)drds - \frac{t^{q+1}}{\Gamma(q+2)}\left[(\frac{2}{2-\beta\delta^2})\left\{\frac{1}{\Gamma(q)}\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^1 (t-s)^{q-1}(1-r)^{\alpha-1}h(r)drds - \beta\int_0^\delta\left(\frac{1}{\Gamma(q)}\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^r (t-s)^{q-1}(r-m)^{\alpha-1}h(m)dm\right)drds + a_2(\frac{\beta\delta^3}{3}-1)\right\}\right] - a_2\frac{2}{\Gamma(q+3)}t^{q+2}. \tag{3.6}$$

Using the third boundary condition, $[J^qx](\lambda) = x(1)$, gives

$$
\begin{aligned}
a_2 &= \left(\frac{3(2-\beta\delta^2)(q+2)!}{-2\lambda^{q+1}(\beta\delta^2-3)(q+2)-6\lambda^{q+2}(2-\beta\delta^2)+(q+2)!\beta\delta^2(2\delta-3)}\right) \\
&\quad\times\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds-\left[\left(\frac{2}{2-\beta\delta^2}\right)\left\{\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds-\right.\right.\right. \\
&\quad\left.\left.-\beta\int_0^\delta\int_0^s \frac{(s-m)^{\alpha-1}}{\Gamma(\alpha)}h(m)dmds\right\}\right]-\frac{1}{\Gamma(q)}\frac{1}{\Gamma(\alpha)}\int_0^\lambda\int_0^s(\lambda-s)^{q-1}(s-r)^{\alpha-1}h(r)drds \\
&\quad+\frac{\lambda^{q+1}}{\Gamma(q+2)}\left[\left(\frac{2}{2-\beta\delta^2}\right)\frac{1}{\Gamma(q)}\frac{1}{\Gamma(\alpha)}\left\{\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}h(r)drds\right.\right. \\
&\quad\left.\left.\left.-\beta\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}h(m)dm\right)drds\right\}\right]\right]. \qquad (3.7)
\end{aligned}
$$

After substituting the value of $a_2$ in Eq. (3.4), the unique solution of BVP (3.1) is obtained as

$$
\begin{aligned}
x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds-\left[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)}\right. \\
&\quad\left.+\frac{(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\int_0^1(1-s)^{\alpha-1}h(s)ds \\
&\quad+\left[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)}+\frac{2\beta(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right] \\
&\quad\times\int_0^\delta\left(\int_0^s(s-m)^{\alpha-1}h(m)dm\right)ds-\frac{(q+2)!}{A\Gamma(q)\Gamma(\alpha)} \\
&\quad\times\left(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)\right)\int_0^\lambda\int_0^s(\lambda-s)^{q-1}(s-r)^{\alpha-1}h(r)drds \\
&\quad+\left[\frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right] \\
&\quad\times\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}h(r)drds \\
&\quad-\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right] \\
&\quad\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}h(m)dm\right)drds. \qquad (3.8)
\end{aligned}
$$

For the sake of convenience, set

$$\begin{aligned}\xi &= \left[\frac{1}{\Gamma(\alpha+1)} + \left|\frac{2}{(2-\beta\delta^2)\Gamma(\alpha+1)} + \frac{(q+2)!(((\beta\delta^2)^2)(-2\delta+3))}{A\Gamma(\alpha+1)(2-\beta\delta^2)}\right|\right. \\ &+ \frac{\delta^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\beta}{(2-\beta\delta^2)} + \frac{2\beta(q+2)!((\beta\delta^2)(-2\delta+3))}{A(2-\beta\delta^2)}\right| \\ &+ \frac{\lambda^{q+\alpha}}{\Gamma(q+\alpha+1)}\left|\frac{(q+2)!(\beta\delta^2(-2\delta+3))}{A}\right| \\ &+ \frac{\lambda^q}{\Gamma(q+1)\Gamma(\alpha+1)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right| \\ &\left.+ \frac{1}{\Gamma(q+1)\Gamma(\alpha+2)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|(-(\lambda-\delta)^q+\lambda^q)\right]. \qquad (3.9)\end{aligned}$$

## 4. Existence and uniqueness results

**Theorem 4.1** Suppose that there exists a constant $K>0$ such that:

(P1) $|f(t,x)-f(t,y)| \le K|x-y|$, for each $t\in[0,1]$ and $\forall\ x,y\in I\!R$. Then boundary value problem (1.1) with boundary conditions (1.2) has a unique solution on [0,1] provided $K\xi<1$.

Proof: Consider the operator $T: C([0,1],I\!R)\to C([0,1],I\!R)$, as

$$\begin{aligned}T(x)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds - \left[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)}\right. \\ &\left.+\frac{(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\int_0^1 (1-s)^{\alpha-1}f(s,x(s))ds \\ &+\left[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)} + \frac{2\beta(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right] \\ &\times\int_0^\delta\left(\int_0^s (s-m)^{\alpha-1}f(m,x(m))dm\right)ds - \frac{(q+2)!}{A\Gamma(q)\Gamma(\alpha)} \\ &\times\left(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)\right)\int_0^\lambda\int_0^s (\lambda-s)^{q-1}(s-r)^{\alpha-1}f(r,x(r))drds \\ &+\left[\frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right] \\ &\times\int_0^\lambda\int_0^1 (\lambda-s)^{q-1}(1-r)^{\alpha-1}f(r,x(r))drds\end{aligned}$$

$$-\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]$$
$$\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}f(m,x(m))dm\right)drds. \tag{4.1}$$

As the fixed points of operator $T$ are the solution of BVP (1.1) with boundary conditions (1.2), Lemma (2.3) will be used to show that $T$ has a fixed point.
If $x,y\in C([0,1],I\!R)$, then $\forall\ t\in[0,1]$,

$$\begin{aligned}
|T(x)(t)-T(y)(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds+\left|\left[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)}\right.\right.\\
&\left.\left.+\frac{(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^1(1-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds\\
&+\left|\left[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)}+\frac{2\beta(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^\delta\left(\int_0^s(s-m)^{\alpha-1}|f(m,x(m))-f(m,y(m))|dm\right)ds\\
&+\left|\frac{(q+2)!\left(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)}\right|\\
&\times\int_0^\lambda\int_0^s(\lambda-s)^{q-1}(s-r)^{\alpha-1}|f(r,x(r))-f(r,y(r))|drds\\
&+\left|\left[\frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}|f(r,x(r))-f(r,y(r))|drds\\
&+\left|\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}|f(m,x(m))-f(m,y(m))|dm\right)drds\\
\leq\ & \frac{K\|x-y\|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds+K\|x-y\|\left|\left[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg|\int\limits_0^1(1-s)^{\alpha-1}ds\\
&+K\|x-y\|\Bigg|\Bigg[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)}\\
&+\frac{2\beta(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg|\\
&\times\int\limits_0^\delta\left(\int\limits_0^s(s-m)^{\alpha-1}dm\right)ds\\
&+K\|x-y\|\left|\frac{(q+2)!\left(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)}\right|\\
&\times\int\limits_0^\lambda\int\limits_0^s(\lambda-s)^{q-1}(s-r)^{\alpha-1}drds\\
&+K\|x-y\|\Bigg|\Bigg[\frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg|\\
&\times\int\limits_0^\lambda\int\limits_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}drds\\
&+K\|x-y\|\Bigg|\Bigg[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg|\\
&\times\int\limits_0^\delta\left(\int\limits_0^\lambda\int\limits_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}dm\right)drds\\
=\;&K\|x-y\|\Bigg[\frac{1}{\Gamma(\alpha+1)}+\Bigg|\frac{2}{(2-\beta\delta^2)\Gamma(\alpha+1)}\\
&+\frac{(q+2)!(((\beta\delta^2)^2)(-2\delta+3))}{A\Gamma(\alpha+1)(2-\beta\delta^2)}\Bigg|\\
&+\frac{\delta^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\beta}{(2-\beta\delta^2)}+\frac{2\beta(q+2)!((\beta\delta^2)(-2\delta+3))}{A(2-\beta\delta^2)}\right|\\
&+\frac{\lambda^{q+\alpha}}{\Gamma(q+\alpha+1)}\left|\frac{(q+2)!(\beta\delta^2(-2\delta+3))}{A}\right|\\
&+\frac{\lambda^q}{\Gamma(q+1)\Gamma(\alpha+1)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|\\
&+\frac{1}{\Gamma(q+1)\Gamma(\alpha+2)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|\\
&\times(-(\lambda-\delta)^q+\lambda^q)\Bigg]. \qquad (4.2)
\end{aligned}
$$

Thus,

$$\|T(x) - T(y)\| \leq K\xi\|x - y\|,$$

which shows that $K < \frac{1}{\xi}$ and $\xi$ depends only on the parameters taken in the problem. Therefore, $T$ is a contraction. As a consequence of the Banach fixed point theorem, $T$ has a unique fixed point that is in fact a unique solution of BVP (1.1) with boundary conditions (1.2).

**Theorem 4.2** Suppose that the function $f : [0,1] \times I\!R \to I\!R$ is continuous and the following assumption holds:

(P2) If there exists a constant $L > 0$ such that $|f(t,x)| \leq L \ \forall\, t \in [0,1],\ x \in I\!R$, then BVP (1.1) with boundary conditions (1.2) has at least one solution on $[0,1]$.

Proof: To show that $T$ has a fixed point, Schaefer's fixed point theorem will be used. The proof is given in several claims.

**Claim 1**: $T$ is continuous.
Assume $\{x_n\}$ to be a sequence such that $x_n \to x$ in $C([0,1], I\!R)$. Then $\forall\, t \in [0,1]$,

$$\begin{aligned}
|T(x_n)(t) - T(x)(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x_n(s)) - f(s,x(s))|ds + \left|\left[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)}\right.\right.\\
&\left.\left.+\frac{(q+2)!(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^1 (1-s)^{\alpha-1}|f(s,x_n(s)) - f(s,x(s))|ds\\
&+\left|\left[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)} + \frac{2\beta(q+2)!(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^\delta\left(\int_0^s (s-m)^{\alpha-1}|f(m,x_n(m)) - f(m,x(m))|dm\right)ds\\
&+\left|\frac{(q+2)!\left(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)}\right|\\
&\times\int_0^\lambda\int_0^s (\lambda-s)^{q-1}(s-r)^{\alpha-1}|f(r,x_n(r)) - f(r,x(r))|drds\\
&+\left|\left[\frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3) - 3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^\lambda\int_0^1 (\lambda-s)^{q-1}(1-r)^{\alpha-1}|f(r,x_n(r)) - f(r,x(r))|drds
\end{aligned}$$

$$
\begin{aligned}
&+\left|\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right| \\
&\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}\right. \\
&\times|f(m,x_n(m))-f(m,x(m))|dm)\,drds \\
\leq\ &\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sup_{s\in[0,1]}|f(s,x_n(s))-f(s,x(s))|ds \\
&+\left|\left[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)}+\frac{(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right| \\
&\times\int_0^1(1-s)^{\alpha-1}\sup_{s\in[0,1]}|f(s,x_n(s))-f(s,x(s))|ds \\
&+\left|\left[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)}+\frac{2\beta(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right| \\
&\times\int_0^\delta\left(\int_0^s(s-m)^{\alpha-1}\sup_{m\in[0,1]}|f(m,x_n(m))-f(m,x(m))|dm\right)ds \\
&+\left|\frac{(q+2)!\left(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)}\right| \\
&\times\int_0^\lambda\int_0^s(\lambda-s)^{q-1}(s-r)^{\alpha-1}\sup_{r\in[0,1]}|f(r,x_n(r))-f(r,x(r))|drds \\
&+\left|\left[\frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right| \\
&\times\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}\sup_{r\in[0,1]}|f(r,x_n(r))-f(r,x(r))|drds \\
&+\left|\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right| \\
&\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}\right. \\
&\left.\times\sup_{m\in[0,1]}|f(m,x_n(m))-f(m,x(m))|dm\right)drds.
\end{aligned}
$$

Since $f$ is a continuous function then $\|T(x_n)-T(x)\|\to 0$ as $n\to\infty$. Consequently, $T$ is continuous.

**Claim 2**: $T$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.
Indeed, it is enough to show that for any $\rho > 0$, $\exists$ a positive constant $\gamma$ such that $\forall$ $x \in B_\rho = \{x \in C([0,1],\mathbb{R}) : ||x|| \leq \rho\}$. For any $x \in B_\rho$,

$$\begin{aligned}
|T(x)(t)| \leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s,x(s))|ds + \Bigg|\Bigg[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)} \\
& + \frac{(q+2)!(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg| \\
& \times \int_0^1 (1-s)^{\alpha-1}|f(s,x(s))|ds \\
& + \Bigg|\Bigg[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)} + \frac{2\beta(q+2)!(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg| \\
& \times \int_0^\delta \left(\int_0^s (s-m)^{\alpha-1}|f(m,x(m))|dm\right) ds \\
& + \left|\frac{(q+2)!\left(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)}\right| \\
& \times \int_0^\lambda \int_0^s (\lambda-s)^{q-1}(s-r)^{\alpha-1}|f(r,x(r))|drds \\
& + \Bigg|\Bigg[\frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3) - 3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg| \\
& \times \int_0^\lambda \int_0^1 (\lambda-s)^{q-1}(1-r)^{\alpha-1}|f(r,x(r))|drds \\
& + \Bigg|\Bigg[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg| \\
& \times \int_0^\delta \left(\int_0^\lambda \int_0^r (\lambda-s)^{q-1}(r-m)^{\alpha-1}|f(m,x(m))|dm\right) drds.
\end{aligned}$$

Using (P2) gives

$$\begin{aligned}
\leq\ & \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}ds + L\Bigg|\Bigg[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)} \\
& + \frac{(q+2)!(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg| \int_0^1 (1-s)^{\alpha-1}ds
\end{aligned}$$

$$
\begin{aligned}
&+L\left|\left[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)}\right.\right.\\
&\left.\left.+\frac{2\beta(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^\delta\left(\int_0^s(s-m)^{\alpha-1}dm\right)ds\\
&+L\left|\frac{(q+2)!\left(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)}\right|\\
&\times\int_0^\lambda\int_0^s(\lambda-s)^{q-1}(s-r)^{\alpha-1}drds\\
&+L\left|\left[\frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}drds\\
&+L\left|\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|\\
&\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}dm\right)drds.
\end{aligned}
$$

Hence, it can be deduced that

$$
\begin{aligned}
|T(x)(t)| \quad\leq\quad & L\left[\frac{1}{\Gamma(\alpha+1)}+\left|\frac{2}{(2-\beta\delta^2)\Gamma(\alpha+1)}\right.\right.\\
&\left.+\frac{(q+2)!(((\beta\delta^2)^2)(-2\delta+3))}{A\Gamma(\alpha+1)(2-\beta\delta^2)}\right|\\
&+\frac{\delta^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\beta}{(2-\beta\delta^2)}+\frac{2\beta(q+2)!((\beta\delta^2)(-2\delta+3))}{A(2-\beta\delta^2)}\right|\\
&+\frac{\lambda^{q+\alpha}}{\Gamma(q+\alpha+1)}\left|\frac{(q+2)!(\beta\delta^2(-2\delta+3))}{A}\right|\\
&+\frac{\lambda^q}{\Gamma(q+1)\Gamma(\alpha+1)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|\\
&+\frac{1}{\Gamma(q+1)\Gamma(\alpha+2)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|\\
&\left.\times\left(-(\lambda-\delta)^q+\lambda^q\right)\right].
\end{aligned}
$$

Thus,

$\|T(x)(t)\| \leq L\xi = \gamma$.

**Claim 3**: $T(B_\rho)$ is equicontinuous with $B_\rho$ defined as in Claim 2.
Supposing $t_1^*, t_2^* \in [0,1]$ with $t_1^* < t_2^*$ and $x \in B_\rho$, where $B_\rho$ is a bounded set of $C([0,1], I\!R)$, then

$$
\begin{aligned}
|T(x)(t_2^*) - T(x)(t_1^*)| &= \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{t_2^*} (t-s)^{\alpha-1} f(s,x(s))ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1^*} (t-s)^{\alpha-1} f(s,x(s))ds \\
&- \left[ \frac{2t_2^*}{(2-\beta\delta^2)\Gamma(\alpha)} + \frac{(q+2)!(-2t_2^*(\beta\delta^3-3) - 3t_{2^*}^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)} \right] \\
&\times \int_0^1 (1-s)^{\alpha-1} f(s,x(s))ds \\
&+ \left[ \frac{2t_1^*}{(2-\beta\delta^2)\Gamma(\alpha)} + \frac{(q+2)!(-2t_1^*(\beta\delta^3-3) - 3t_{1^*}^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)} \right] \\
&\times \int_0^1 (1-s)^{\alpha-1} f(s,x(s))ds \\
&+ \left[ \frac{2t_2^*\beta}{(2-\beta\delta^2)\Gamma(\alpha)} + \frac{2\beta(q+2)!(-2t_2^*(\beta\delta^3-3) - 3t_{2^*}^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)} \right] \\
&\times \int_0^\delta \left( \int_0^s (s-m)^{\alpha-1} f(m,x(m))dm \right) ds \\
&- \left[ \frac{2t_1^*\beta}{(2-\beta\delta^2)\Gamma(\alpha)} + \frac{2\beta(q+2)!(-2t_1^*(\beta\delta^3-3) - 3t_{1^*}^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)} \right] \\
&\times \int_0^\delta \left( \int_0^s (s-m)^{\alpha-1} f(m,x(m))dm \right) ds \\
&- \frac{(q+2)!\left(-2t_2^*(\beta\delta^3-3) - 3t_{2^*}^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)} \\
&\times \int_0^\lambda \int_0^s (\lambda-s)^{q-1}(s-r)^{\alpha-1} f(r,x(r))drds \\
&+ \frac{(q+2)!\left(-2t_1^*(\beta\delta^3-3) - 3t_{1^*}^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)} \\
&\times \int_0^\lambda \int_0^s (\lambda-s)^{q-1}(s-r)^{\alpha-1} f(r,x(r))drds
\end{aligned}
$$

$$+\left[\frac{2\lambda^{q+1}(q+2)(-2t_2^*(\beta\delta^3-3)-3t_2^*(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]$$

$$\times\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}f(r,x(r))drds$$

$$-\left[\frac{2\lambda^{q+1}(q+2)(-2t_1^*(\beta\delta^3-3)-3t_1^*(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]$$

$$\times\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}f(r,x(r))drds$$

$$-\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t_2^*(\beta\delta^3-3)-3t_{2^*}^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]$$

$$\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}f(m,x(m))dm\right)drds.$$

$$+\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t_1^*(\beta\delta^3-3)-3t_{1^*}^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]$$

$$\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}f(m,x(m))dm\right)drds\Bigg|$$

$$\leq\quad\frac{1}{\Gamma(\alpha)}\int_0^{t_1^*}[(t_2^*-s)^{\alpha-1}-(t_1^*-s)^{\alpha-1}]|f(s,x(s))|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1^*}^{t_2^*}(t_2^*-s)^{\alpha-1}|f(s,x(s))|ds+\Bigg|\Bigg[\frac{2(t_2^*-t_1^*)}{(2-\beta\delta^2)\Gamma(\alpha)}$$

$$+\frac{(q+2)!(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg|$$

$$\times\int_0^1(1-s)^{\alpha-1}|f(s,x(s))|ds+\Bigg|\Bigg[\frac{2(t_2^*-t_1^*)\beta}{(2-\beta\delta^2)\Gamma(\alpha)}$$

$$+\frac{2\beta(q+2)!(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\Bigg]\Bigg|$$

$$\times\int_0^\delta\left(\int_0^s(s-m)^{\alpha-1}dm\right)|f(m,x(m))|ds$$

$$+\left|\frac{(q+2)!\left(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)}\right|$$

$$\times\int_0^\lambda\int_0^s(\lambda-s)^{q-1}(s-r)^{\alpha-1}|f(r,x(r))|drds$$

$$+\left|\left[\frac{2\lambda^{q+1}(q+2)(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_2^*-t_1^*)(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|$$

$$\times\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}|f(r,x(r))|drds$$

$$+\left|\left[\frac{2\beta\lambda^{q+1}(q+2)(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|$$

$$\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}|f(m,x(m))|dm\right)drds$$

$$\leq\frac{L}{\Gamma(\alpha)}\int_0^{t_1^*}[(t_2^*-s)^{\alpha-1}-(t_1^*-s)^{\alpha-1}]|ds+\frac{L}{\Gamma(\alpha)}\int_{t_1^*}^{t_2^*}(t_2^*-s)^{\alpha-1}ds$$

$$+L\left|\left[\frac{2(t_2^*-t_1^*)}{(2-\beta\delta^2)\Gamma(\alpha)}\right.\right.$$

$$\left.\left.+\frac{(q+2)!(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|$$

$$\times\int_0^1(1-s)^{\alpha-1}ds+L\left|\left[\frac{2(t_2^*-t_1^*)\beta}{(2-\beta\delta^2)\Gamma(\alpha)}\right.\right.$$

$$\left.\left.+\frac{2\beta(q+2)!(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|$$

$$\times\int_0^\delta\left(\int_0^s(s-m)^{\alpha-1}dm\right)ds$$

$$+L\left|\frac{(q+2)!\left(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)}\right|$$

$$\times\int_0^\lambda\int_0^s(\lambda-s)^{q-1}(s-r)^{\alpha-1}drds$$

$$+L\left|\left[\frac{2\lambda^{q+1}(q+2)(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_2^*-t_1^*)(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|$$

$$\times\int_0^\lambda\int_0^1(\lambda-s)^{q-1}(1-r)^{\alpha-1}drds$$

$$+L\left|\left[\frac{2\beta\lambda^{q+1}(q+2)(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|$$

$$\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}dm\right)drds$$

$$
\begin{aligned}
&\leq \frac{L}{\Gamma(\alpha+1)}[(t_2^*-t_1^*)^\alpha+(t_{2^*}^\alpha-t_{1^*}^\alpha)]+\frac{L}{\Gamma(\alpha+1)}(t_2^*-t_1^*)^\alpha \\
&+\frac{L}{\Gamma(\alpha+1)}\left|\left[\frac{2(t_2^*-t_1^*)}{(2-\beta\delta^2)}\right.\right. \\
&\left.\left.+\frac{(q+2)!(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)(\beta\delta^2)}{A(2-\beta\delta^2)}\right]\right| \\
&+\frac{L\delta^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\beta(t_2^*-t_1^*)}{(2-\beta\delta^2)}\right. \\
&\left.+\frac{2\beta(q+2)!(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)}{A(2-\beta\delta^2)}\right| \\
&+\frac{L\lambda^{q+\alpha}}{\Gamma(q+\alpha+1)} \\
&\times\left|\left[\frac{(q+2)!\left(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)\right)}{A}\right]\right| \\
&+\frac{L\lambda^q}{\Gamma(q+1)\Gamma(\alpha+1)} \\
&\times\left|\left[\frac{2\lambda^{q+1}(q+2)\left(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)\right)}{A}\right]\right| \\
&+\frac{L}{\Gamma(q+1)\Gamma(\alpha+2)} \\
&\times\left|\left[\frac{2\lambda^{q+1}(q+2)\left(-2(t_2^*-t_1^*)(\beta\delta^3-3)-3(t_{2^*}^2-t_{1^*}^2)(2-\beta\delta^2)\right)}{(2-\beta\delta^2)A}\right]\right| \\
&\times(-(\lambda-\delta)^q+\lambda^q). \qquad (4.3)
\end{aligned}
$$

As $t_2^* \to t_1^*$, the R.H.S of Eq. (4.3) tends to zero. In view of the Arzela–Ascoli theorem, $T : C([0,1], I\!R) \to C([0,1], I\!R \to)$ is completely continuous.

**Claim 4**: A priori bounds.

To show that the set $U = \{x \in C([0,1], I\!R) : x = \lambda T(x)\ for\ some\ 0 < \lambda < 1\}$ is bounded, if $x \in U$, then $x(t) = \lambda T(x)(t)$ for some $0 < \lambda < 1$.

Thus, $\forall\ t \in [0,1]$,

$$
\begin{aligned}
T(x)(t) &= \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds - \lambda\left[\frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)}\right. \\
&\left.+\frac{(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]\int_0^1 (1-s)^{\alpha-1} f(s,x(s))ds \\
&+\lambda\left[\frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)}+\frac{2\beta(q+2)!(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)}\right]
\end{aligned}
$$

$$\times \int_0^\delta \left( \int_0^s (s-m)^{\alpha-1} f(m,x(m))dm \right) ds - \lambda \frac{(q+2)!}{A\Gamma(q)\Gamma(\alpha)}$$

$$\times \left(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)\right) \int_0^\lambda \int_0^s (\lambda-s)^{q-1}(s-r)^{\alpha-1} f(r,x(r))drds$$

$$+\lambda \left[ \frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3) - 3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)} \right]$$

$$\times \int_0^\lambda \int_0^1 (\lambda-s)^{q-1}(1-r)^{\alpha-1} f(r,x(r))drds$$

$$-\lambda \left[ \frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)} \right]$$

$$\times \int_0^\delta \left( \int_0^\lambda \int_0^r (\lambda-s)^{q-1}(r-m)^{\alpha-1} f(m,x(m))dm \right) drds. \tag{4.4}$$

Using P(2), $\forall\ t \in [0,1]$ Eq. (4.4) implies

$$\begin{aligned} |T(x)(t)| \quad \leq \quad & \frac{\lambda L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \lambda L \left| \left[ \frac{2t}{(2-\beta\delta^2)\Gamma(\alpha)} \right. \right. \\ & \left. \left. + \frac{(q+2)!(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)(\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)} \right] \right| \int_0^1 (1-s)^{\alpha-1} ds \\ & + \lambda L \left| \left[ \frac{2t\beta}{(2-\beta\delta^2)\Gamma(\alpha)} \quad + \frac{2\beta(q+2)!(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)}{A\Gamma(\alpha)(2-\beta\delta^2)} \right] \right| \\ & \times \int_0^\delta \left( \int_0^s (s-m)^{\alpha-1} dm \right) ds \\ & + \lambda L \left| \frac{(q+2)! \left(-2t(\beta\delta^3-3) - 3t^2(2-\beta\delta^2)\right)}{A\Gamma(q)\Gamma(\alpha)} \right| \\ & \times \int_0^\lambda \int_0^s (\lambda-s)^{q-1}(s-r)^{\alpha-1} drds \\ & + \lambda L \left| \left[ \frac{2\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3) - 3t(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)} \right] \right| \\ & \times \int_0^\lambda \int_0^1 (\lambda-s)^{q-1}(1-r)^{\alpha-1} drds \end{aligned}$$

$$+\lambda L\left|\left[\frac{2\beta\lambda^{q+1}(q+2)(-2t(\beta\delta^3-3)-3t^2(2-\beta\delta^2)}{A\Gamma(q)\Gamma(\alpha)(2-\beta\delta^2)}\right]\right|$$

$$\times\int_0^\delta\left(\int_0^\lambda\int_0^r(\lambda-s)^{q-1}(r-m)^{\alpha-1}dm\right)drds$$

$$\begin{aligned}\leq\quad & L\left[\frac{1}{\Gamma(\alpha+1)}+\left|\frac{2}{(2-\beta\delta^2)\Gamma(\alpha+1)}\right.\right.\\
&+\left.\frac{(q+2)!(((\beta\delta^2)^2)(-2\delta+3))}{A\Gamma(\alpha+1)(2-\beta\delta^2)}\right|\\
&+\frac{\delta^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\beta}{(2-\beta\delta^2)}+\frac{2\beta(q+2)!((\beta\delta^2)(-2\delta+3))}{A(2-\beta\delta^2)}\right|\\
&+\frac{\lambda^{q+\alpha}}{\Gamma(q+\alpha+1)}\left|\frac{(q+2)!(\beta\delta^2(-2\delta+3))}{A}\right|\\
&+\frac{\lambda^q}{\Gamma(q+1)\Gamma(\alpha+1)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|\\
&+\left.\frac{1}{\Gamma(q+1)\Gamma(\alpha+2)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|\times(-(\lambda-\delta)^q+\lambda^q)\right].\end{aligned}$$

Hence, $\|T(x)\|\leq\xi L=\sigma$, and this shows that the set $U$ is bounded. Therefore, $T$ has a fixed point that is in fact a solution of problem (1.1) with boundary conditions (1.2) by Schaefer's fixed point theorem.

## 5. Examples

In this section, two examples are given to show the applicability of the results.

**Example 5.1** *Consider the following boundary value problem:*

$$\left.\begin{array}{l}{}^cD^{\frac{5}{2}}x(t)=\frac{6}{(t+9)^3}\frac{\|x\|}{1+\|x\|},\quad 0\leq t\leq 1,\\ x(0)=0,\\ x(1)=\int_0^{\frac{1}{3}}x(s)ds,\\ (J^{\frac{5}{2}}x)(\frac{3}{4})=x(1).\end{array}\right\}\tag{5.1}$$

*Set* $\delta=\frac{1}{3},\ \alpha=\frac{5}{2},\ q=\frac{5}{2},\ \beta=1,\ \lambda=\frac{3}{4},\ \frac{2}{\delta^2}=18.0180\ \Rightarrow\beta\neq\frac{2}{\delta^2},$
*and* $f(t,x)=\frac{6}{(t+9)^3}\frac{\|x\|}{1+\|x\|}$. *Since* $\|f(t,x)-f(t,y)\|\leq\frac{6}{729}\|x-y\|$, *(P1) is satisfied with* $K=\frac{6}{729}$.
*Moreover,*

$$\begin{aligned}K\xi\quad=\quad & K\left[\frac{1}{\Gamma(\alpha+1)}+\left|\frac{2}{(2-\beta\delta^2)\Gamma(\alpha+1)}+\frac{(q+2)!(((\beta\delta^2)^2)(-2\delta+3))}{A\Gamma(\alpha+1)(2-\beta\delta^2)}\right|\right.\\
&+\frac{\delta^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\beta}{(2-\beta\delta^2)}+\frac{2\beta(q+2)!((\beta\delta^2)(-2\delta+3))}{A(2-\beta\delta^2)}\right|\end{aligned}$$

$$+\frac{\lambda^{q+\alpha}}{\Gamma(q+\alpha+1)}\left|\frac{(q+2)!(\beta\delta^2(-2\delta+3))}{A}\right|$$
$$+\frac{\lambda^{q}}{\Gamma(q+1)\Gamma(\alpha+1)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|$$
$$+\frac{1}{\Gamma(q+1)\Gamma(\alpha+2)}\left|\frac{2\lambda^{q+1}(q+2)(\beta\delta^2(-2\delta+3))}{(2-\beta\delta^2)A}\right|(-(\lambda-\delta)^q+\lambda^q)\Bigg],$$

*where*

$$A=\left(\frac{1}{-2\lambda^{q+1}(\beta\delta^2-3)(q+2)-6\lambda^{q+2}(2-\beta\delta^2)+(q+2)!\beta\delta^2(2\delta-3)}\right).$$

*Therefore,* $K\xi = 0.00547 < 1$.
*Thus, boundary value problem (5.1) has a unique solution by Theorem (4.1).*

**Example 5.2** *Consider the following multipoint integral boundary condition:*

$$\left.\begin{array}{l} {}^cD^{\frac{7}{3}}x(t)=\frac{e^{-2t}t^{\frac{1}{3}}\left[\frac{1}{3}t^{-1}-2-\frac{x^2}{(1+tx^2)}\right]}{(1+tx^2)}, \quad 0\le t\le 1,\\ x(0)=0,\\ x(1)=\int\limits_0^{\frac{1}{2}}x(s)ds,\\ (J^{\frac{8}{3}})(\frac{3}{2})=x(1).\end{array}\right\} \qquad (5.2)$$

*Here* $\delta=\frac{1}{2},\ \alpha=\frac{7}{3},\ q=\frac{8}{3},\ \beta=1,\ \lambda=\frac{3}{2},\ \frac{2}{\delta^2}=8\ \Rightarrow \beta\neq\frac{2}{\delta^2},$

*and* $f(t,x)=\frac{e^{-2t}t^{\frac{1}{3}}\left[\frac{1}{3}t^{-1}-2-\frac{x^2}{(1+tx^2)}\right]}{(1+tx^2)}.$

*Obviously,* $f(t,x)=\frac{e^{-2t}t^{\frac{1}{3}}\left[\frac{1}{3}t^{-1}-2-\frac{x^2}{(1+tx^2)}\right]}{(1+tx^2)}\le 1 = L$. *It is noticed that all the conditions of Theorem* (4.2) *are fulfilled, so, as a consequence, BVP (5.2) has at least one solution.*

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