Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2018) 42: 2325 – 2335
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doi:10.3906/mat-1612-114

Congruences modulo 9 for bipartitions with designated summands

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Received: 30.12.2016 • **Accepted/Published Online:** 26.06.2018 • **Final Version:** 27.09.2018

Abstract: Andrews, Lewis, and Lovejoy studied arithmetic properties of partitions with designated summands that are defined on ordinary partitions by tagging exactly one part among parts with equal size. A bipartition of n is an ordered pair of partitions (π_1, π_2) with the sum of all of the parts being n. In this paper, we investigate arithmetic properties of bipartitions with designated summands. Let $PD_{-2}(n)$ denote the number of bipartitions of n with designated summands. We establish several Ramanujan-like congruences and an infinite family of congruences modulo 9 satisfied by $PD_{-2}(n)$.

Key words: Partition with designated summands, bipartition, congruence

1. Introduction

In [1], Andrews et al. investigated the number of partitions with designated summands that are defined on ordinary partitions by designating exactly one part of each part size. Let $PD(n)$ denote the number of partitions of n with designated summands. For instance, there are ten partitions of 4 with designated summands:

$$\begin{array}{ccccc} 4', & 3'+1', & 2'+2, & 2+2', & 2'+1'+1, \\ 2'+1+1', & 1'+1+1+1, & 1+1'+1+1, & 1+1+1'+1, & 1+1+1+1'. \end{array}$$

Thus, $PD(4) = 10$. Andrews et al. [1] obtained the generating function of $PD(n)$ as given by

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty} = \frac{f_6}{f_1 f_2 f_3}, \tag{1.1}$$

where here and throughout this paper $(a;q)_\infty$ stands for the q-shifted factorial

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}), \ |q|<1,$$

and for any positive integer k, f_k is defined by

$$f_k = (q^k;q^k)_\infty.$$

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2010 *AMS Mathematics Subject Classification:* 05A17, 11P83

By using modular forms and q-series identities, Andrews et al. [1] studied arithmetic properties of the partition function $PD(n)$. In particular, they obtained a 2-dissection formula for the generating function of $PD(n)$ and a Ramanujan-type congruence as given by

$$PD(3n+2) \equiv 0 \pmod{3}. \tag{1.2}$$

Later, Chen et al. [8] established a 3-dissection formula for the partition function $PD(n)$ relying on Chan's identity[6] on Ramanujan's cubic continued fraction and some identities on cubic theta functions. The generating function of $PD(3n+2)$ implies the congruence (1.2) of Andrews et al. By introducing a rank for partitions with designated summands, Chen et al. [8] gave a combinatorial interpretation of the congruence (1.2). Recently, Xia[22] also investigated the arithmetic properties of the partition function $PD(n)$. He proved several infinite families of congruences modulo 9 and 27 for $PD(n)$ by utilizing the generating function of $PD(3n)$ and $PD(3n+1)$ derived in [8]. Xia[22] also found some congruences modulo 27 for $PD(n)$ by employing some results due to Newman[19].

A bipartition π of n is an ordered pair of partitions (π_1, π_2) with the sum of all of the parts being n. Let $p_{-2}(n)$ denote the number of bipartitions of n. The generating function of $p_{-2}(n)$ equals

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{(q;q)_\infty^2} = \frac{1}{f_1^2}.$$

There are numerous remarkable results on arithmetic properties for $p_{-2}(n)$ (see [2, 11, 12, 20]). Recently, arithmetic properties for bipartitions with certain restrictions on each partition have drawn a great deal of interest (see [5, 7, 9, 10]).

In this paper, we wish to consider bipartitions with designated summands. More specifically, a bipartition with designated summands is a bipartition $\pi = (\pi_1, \pi_2)$ for which π_1 and π_2 are both partitions with designated summands. Notice that π_1 and π_2 are allowed to have one part of equal size tagged in common. For instance, $\pi = (2', 2')$ is a bipartition of 4 with designated summands. Let $PD_{-2}(n)$ denote the number of bipartitions of n with designated summands. We have the generating function of $PD_{-2}(n)$ as given by

$$\sum_{n=0}^{\infty} PD_{-2}(n)q^n = \frac{(q^6;q^6)_\infty^2}{(q;q)_\infty^2 (q^2;q^2)_\infty^2 (q^3;q^3)_\infty^2} = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}. \tag{1.3}$$

In a very recent work, Naika and Shivashankar[18] investigated the arithmetic properties of the generating function of $PD_{-2}(n)$. They proved that this function satisfied various congruence properties modulo 3 and powers of 2.

In this paper, we proceed to study the congruence properties of bipartitions with designated summands. The main objective of this paper is to prove the following four congruences modulo 9 for the partition function $PD_{-2}(n)$:

$$PD_{-2}(9n+6) \equiv 0 \pmod{9}, \tag{1.4}$$

$$PD_{-2}(12n+6) \equiv 0 \pmod{9}, \tag{1.5}$$

$$PD_{-2}(12n+10) \equiv 0 \pmod{9}, \tag{1.6}$$

$$PD_{-2}(3^\alpha(6n+2)) \equiv 0 \pmod{9}, \tag{1.7}$$

where $n \geq 0$ and $\alpha \geq 1$.

2. Preliminaries

In this section, we present some results that will be used in our proofs.

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2},\ |ab|<1. \tag{2.1}$$

Jacobi triple product identity can be illustrated by Ramanujan's notation as follows:

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

We recall two special cases of $f(a,b)$ [3, Eq. (1.2.2) and Eq. (1.2.3)] as given by

$$\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\frac{f_2^5}{f_1^2f_4^2}, \tag{2.3}$$

$$\psi(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{f_2^2}{f_1}. \tag{2.4}$$

It is easy to check that

$$\varphi(-q)=\frac{f_1^2}{f_2}. \tag{2.5}$$

The following dissection formula proved by Hirschhorn and Sellers[16] is crucial in our proofs.

Lemma 2.1 *We have*

$$\frac{1}{\varphi(-q)}=\frac{\varphi(-q^9)}{\varphi(-q^3)^4}\left(\varphi(-q^9)^2+2q\varphi(-q^9)X(-q^3)+4q^2X(-q^3)^2\right), \tag{2.6}$$

where

$$X(q)=\frac{f_2^2f_3f_{12}}{f_1f_4f_6}.$$

Replacing q by $-q$ in [9, Eq. (2.9)], we obtain the following lemma.

Lemma 2.2 *We have*

$$\frac{1}{\psi(q)}=\frac{\psi(q^9)}{\psi(q^3)^4}\left(A(q^3)^2-qA(q^3)\psi(q^9)+q^2\psi(q^9)^2\right), \tag{2.7}$$

where

$$A(q)=\frac{f_2f_3^2}{f_1f_6}. \tag{2.8}$$

We also need the following five 2-dissection formulae.

Lemma 2.3 *The following* 2*-dissections hold:*

$$\frac{f_1}{f_3^3}=\frac{f_2f_4^2f_{12}^2}{f_6^7}-q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}, \tag{2.9}$$

$$\frac{f_3^3}{f_1}=\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}, \tag{2.10}$$

$$\frac{f_3}{f_1^3}=\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}, \tag{2.11}$$

$$\frac{f_1^3}{f_3}=\frac{f_4^3}{f_{12}}-3q\frac{f_2^2f_{12}^3}{f_4f_6^2}, \tag{2.12}$$

$$\frac{1}{f_1^2f_3^2}=\frac{f_8^4f_{12}^{10}}{f_2^4f_4^2f_6^8f_{24}^4}+2q\frac{f_4^4f_{12}^4}{f_2^6f_6^6}+q^2\frac{f_4^{10}f_{24}^4}{f_2^8f_6^4f_8^4f_{12}^2}. \tag{2.13}$$

(2.9) and (2.10) were proved in [15, Lemma 2.1]. (2.11) and (2.13) were obtained by Xia and Yao [23, Eq. (3.38) and Eq. (3.13)]. (2.12) is due to Hirschhorn et al. [14, Eq. (1.35)].

The cubic theta function $a(q)$ was introduced by Borwein et al. [4] and is defined by

$$a(q)=\sum_{m,n=-\infty}^{\infty}q^{m^2+mn+n^2}.$$

From [13] we find that

$$a(q)=1+6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right).$$

Let

$$P(q)=f_1a(q)=f_1\left(1+6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right)\right). \tag{2.14}$$

Wang proved the following results.

Lemma 2.4 ([21, Lemma 2.3]) *The following* 3*-dissection holds:*

$$f_1^3=P(q^3)-3qf_9^3, \tag{2.15}$$

where

$$P(q)=f_1a(q)=f_1\varphi(q)\varphi(q^3)+4qf_1\psi(q^2)\psi(q^6). \tag{2.16}$$

Lemma 2.5 ([21, Lemma 2.4]) *We have*

$$P(q^3)-27qf_3^9=\frac{f_1^{12}}{f_3^3}, \tag{2.17}$$

and

$$\frac{1}{f_1^3}=\frac{f_9^3}{f_3^{12}}\left(P(q^3)^2+3qP(q^3)f_9^3+9q^2f_9^6\right). \tag{2.18}$$

For a partition π, we can construct its Ferrers–Young diagram. For example, the Ferrers–Young diagram for the partition $\pi = (4,3,3,1)$ is illustrated as below.

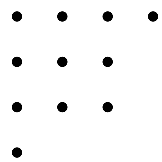

Figure. $\pi = (4,3,3,1)$.

The node in position (i,j) in the Ferrers–Young diagram can be assigned a hook number $h(i,j)$, which is defined as the number of nodes in the hook containing that node. For instance, the hook numbers of the nodes in the first row of $\pi = (4,3,3,1)$ are 7, 5, 4, and 1, respectively. Given a partition π of n, we call π a t-core if it has no hook numbers divisible by t (see [17]). Let $B_3(n)$ be the number of partition triples of n where each partition is a 3-core. The generating function of $B_3(n)$ is given by

$$\sum_{n=0}^{\infty} B_3(n)q^n = \frac{f_3^9}{f_1^3}.$$

Wang[21] derived the following dissection formulae of $B_3(n)$ and a 2-dissection formula of $\frac{1}{f_1^5 f_3}$.

Lemma 2.6 *We have*

$$\sum_{n=0}^{\infty} B_3(3n)q^n = P(q)^2\frac{f_3^3}{f_1^3}, \tag{2.19}$$

$$= \frac{f_2^{10}f_6^{10}}{f_1^5 f_3 f_4^4 f_{12}^4} + 16q^2\frac{f_3^3 f_4^4 f_{12}^4}{f_1 f_2^2 f_6^2} + 8q\frac{f_2^4 f_3 f_6^4}{f_1^3}, \tag{2.20}$$

$$\sum_{n=0}^{\infty} B_3(6n)q^n = \frac{f_2^{10}f_3^9}{f_1^7 f_6^6} + 16q\frac{f_2^7 f_6^3}{f_1^4} + 27q\frac{f_2^2 f_3^5 f_6^2}{f_1^3}. \tag{2.21}$$

(2.19) was proved in [21, Theorem 2.2]. (2.20) and (2.21) were proved in [21, Theorem 2.4].

Lemma 2.7 *We have*

$$\frac{1}{f_1^5 f_3} = \left(\frac{f_4^{14}}{f_2^{17} f_6 f_{12}^2} + 3q^2\frac{f_4^6 f_{12}^6}{f_6^5 f_2^{13}}\right) + q\left(5\frac{f_4^{10} f_{12}^2}{f_2^{15} f_6^3} - 9q^2\frac{f_4^2 f_{12}^{10}}{f_6^7 f_2^{11}}\right). \tag{2.22}$$

Identity (2.22) was obtained by Wang[21, Theorem 2.4].

3. Congruences modulo 9 for $PD_{-2}(n)$

In this section, we begin by proving the first two congruences modulo 9 for the generating function of $PD_{-2}(n)$.

Theorem 3.1 *For each nonnegative integer n, we have*

$$PD_{-2}(12n+6) \equiv 0 \pmod 9, \tag{3.1}$$

$$PD_{-2}(18n+6) \equiv 0 \pmod 9. \tag{3.2}$$

Proof Utilizing (1.3), (2.4), (2.7), and (2.18), we get

$$\begin{aligned}\sum_{n=0}^{\infty} PD_{-2}(n)q^n &= \frac{f_6^2}{f_3^2}\cdot\frac{1}{f_1^3}\cdot\frac{f_1}{f_2^2}\\ &\equiv \frac{f_6^2}{f_3^2}\cdot\frac{f_9^3}{f_3^{12}}\left(P(q^3)^2+3qP(q^3)f_9^3\right)\\ &\quad\times\frac{\psi(q^9)}{\psi(q^3)^4}\left(A(q^3)^2-qA(q^3)\psi(q^9)+q^2\psi(q^9)^2\right) \pmod 9. \end{aligned} \tag{3.3}$$

Choosing the terms on both sides of (3.3) for which the powers of q are of the form $3n$, and replacing q^3 by q, we obtain that

$$\sum_{n=0}^{\infty} PD_{-2}(3n)q^n \equiv H_1(q)+H_2(q) \pmod 9, \tag{3.4}$$

where

$$H_1(q) = P(q)^2A(q)^2\frac{f_2^2f_3^3}{f_1^{14}}\cdot\frac{\psi(q^3)}{\psi(q)^4}, \tag{3.5}$$

$$H_2(q) = 3qP(q)\frac{f_2^2f_3^6}{f_1^{14}}\cdot\frac{\psi(q^3)^3}{\psi(q)^4}. \tag{3.6}$$

For all positive integers m and k, it is easy to see that

$$f_{3k}^{3m} \equiv f_k^{9m} \pmod 9. \tag{3.7}$$

Hence, by (2.4), (2.14), (3.6), and (3.7), we find that

$$H_2(q) \equiv 3qf_1\cdot\frac{f_2^2f_3^6}{f_1^{14}}\cdot\frac{f_1^4}{f_2^8}\cdot\frac{f_6^6}{f_3^3} \equiv 3qf_2^{12} \pmod 9. \tag{3.8}$$

In view of (2.4), (2.8), (2.19), (3.5), and (3.7), we deduce that

$$\begin{aligned}H_1(q) &= P(q)^2\frac{f_2^2f_3^4}{f_1^2f_6^2}\cdot\frac{f_2^2f_3^3}{f_1^{14}}\cdot\frac{f_1^4}{f_2^8}\cdot\frac{f_6^2}{f_3}\\ &\equiv P(q)^2\frac{f_3^3}{f_1^3f_2^4} = \frac{1}{f_2^4}\sum_{n=0}^{\infty}B_3(3n)q^n \pmod 9. \end{aligned} \tag{3.9}$$

Invoking (2.21), (3.4), (3.7), (3.8), and (3.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} PD_{-2}(6n)q^n &\equiv \frac{1}{f_1^4}\sum_{n=0}^{\infty}B_3(6n)q^n\\ &\equiv \frac{1}{f_1^4}\left(\frac{f_2^{10}f_3^9}{f_1^7f_6^6}+16q\frac{f_2^7f_6^3}{f_1^4}\right)\\ &\equiv \frac{f_3^6}{f_6^3}\cdot\frac{1}{\varphi(-q)}+16q\frac{f_6^6}{f_3^3}\cdot\frac{1}{\psi(q)} \pmod 9. \end{aligned} \tag{3.10}$$

By (2.4) and (2.5), we find that (3.10) can be written as

$$\sum_{n=0}^{\infty} PD_{-2}(6n)q^n \equiv \frac{f_2}{f_6^3}\cdot\frac{f_3^6}{f_1^2} + 16q\frac{f_6^6}{f_2^2}\cdot\frac{f_1}{f_3^3} \pmod 9. \tag{3.11}$$

Using (2.9) and (2.10), we choose the coefficients of q^{2n+1} on both sides of (3.11) and obtain that

$$\sum_{n=0}^{\infty} PD_{-2}(12n+6)q^n \equiv 18\frac{f_2^2 f_6^2}{f_1 f_3} \pmod 9. \tag{3.12}$$

As a result, (3.1) is true.

Next, employing (2.6) and (2.7), we consider the coefficients of q^{3n+1} on both sides of (3.10) and observe that

$$\sum_{n=0}^{\infty} PD_{-2}(18n+6)q^n \equiv 2\frac{f_1^6}{f_2^3}\cdot\frac{\varphi(-q^3)^2}{\varphi(-q)^4}X(-q) + 16\frac{f_2^6}{f_1^3}\cdot\frac{\psi(q^3)}{\psi(q)^4}A(q)^2 \pmod 9. \tag{3.13}$$

By (2.4), (2.5), (2.8), and the fact

$$X(-q) = \frac{f_1 f_6^2}{f_2 f_3},$$

we deduce that

$$\sum_{n=0}^{\infty} PD_{-2}(18n+6)q^n \equiv 18\frac{f_3^3}{f_1} \pmod 9. \tag{3.14}$$

Hence, we arrive at (3.2). This completes the proof. □

In order to show (1.4), we need to prove the following theorem.

Theorem 3.2 *For each nonnegative integer n, we have*

$$PD_{-2}(18n+15) \equiv 0 \pmod 9. \tag{3.15}$$

Proof Employing (3.4), (3.8), and (3.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} PD_{-2}(3n)q^n &\equiv H_1(q) + H_2(q)\\ &\equiv \frac{1}{f_2^4}\sum_{n=0}^{\infty} B_3(3n)q^n + 3qf_2^{12} \pmod 9.\end{aligned} \tag{3.16}$$

Invoking (2.10), (2.11), (2.20), (2.22), and (3.7), we extract those terms involving the powers q^{2n+1} of (3.16) and find that

$$\begin{aligned}\sum_{n=0}^{\infty} PD_{-2}(6n+3)q^n &\equiv 13\frac{f_2^6 f_3^7}{f_1^9 f_6^2} + 16q\frac{f_2^3 f_6^7}{f_1^6 f_3^2} + 3f_1^{12}\\ &\equiv 13\frac{f_3^4}{f_6^2}\cdot f_2^9\cdot\frac{1}{f_2^3}\cdot\frac{f_3^3}{f_1^9} + 16q\frac{f_6^7}{f_3^2}\cdot\frac{f_2^3}{f_1^9}\cdot f_1^3 + 3f_3^4\\ &\equiv 13f_3^4 f_6\cdot\frac{1}{f_2^3} + 16q\frac{f_6^7}{f_3^5}\cdot f_2^3 f_1^3 + 3f_3^4 \pmod 9.\end{aligned} \tag{3.17}$$

Applying (2.15) and (2.18), the above identity can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} PD_{-2}(6n+3)q^n \equiv& 13\frac{f_3^4 f_{18}^3}{f_6^{11}}\left(P(q^6)^2+3q^2P(q^6)f_{18}^3+9q^4f_{18}^6\right)\\&+16q\frac{f_6^7}{f_3^5}\left(P(q^6)-3q^2f_{18}^3\right)\left(P(q^3)-3qf_9^3\right)+3f_3^4\\\equiv& 13\frac{f_3^4 f_{18}^3}{f_6^{11}}\left(P(q^6)^2+3q^2P(q^6)f_{18}^3+9q^4f_{18}^6\right)\\&+16q\frac{f_6^7}{f_3^5}\left(P(q^6)P(q^3)-3qf_9^3P(q^6)-3q^2f_{18}^3P(q^3)+9q^3f_9^3f_{18}^3\right)+3f_3^4 \pmod 9.\end{aligned} \tag{3.18}$$

Selecting those terms on both sides of (3.18) whose powers of q are of the form $3n+2$, we obtain that

$$\sum_{n=0}^{\infty} PD_{-2}(18n+15)q^{3n+2} \equiv 13\frac{f_3^4 f_{18}^3}{f_6^{11}}\cdot 3q^2f_{18}^3P(q^6)-16q\frac{f_6^7}{f_3^5}\cdot 3qf_9^3P(q^6) \pmod 9.$$

Dividing the above formula by q^2, and replacing q^3 by q, we derive that

$$\sum_{n=0}^{\infty} PD_{-2}(18n+15)q^n \equiv 39\frac{f_1^4 f_6^6}{f_2^{11}}P(q^2)-48\frac{f_2^7 f_3^3}{f_1^5}P(q^2) \pmod 9. \tag{3.19}$$

By (2.14), (3.7), and (3.19), we find

$$\begin{aligned}\sum_{n=0}^{\infty} PD_{-2}(18n+15)q^n &\equiv 39\frac{f_1^4 f_6^6}{f_2^{10}}-48\frac{f_2^8 f_3^3}{f_1^5}\\&\equiv 39f_1^4f_2^8-48f_1^4f_2^8=-9f_1^4f_2^8 \pmod 9,\end{aligned} \tag{3.20}$$

which yields (3.15). This completes the proof. □

Combining (3.2) and (3.15), it turns out that (1.4) is true.

Next, we present a proof of the third congruence.

Theorem 3.3 *For each nonnegative integer n, we have*

$$PD_{-2}(12n+10) \equiv 0 \pmod 9. \tag{3.21}$$

Proof Selecting those terms on both sides of (3.3) for which the powers of q are of the form $3n+1$, dividing by q, and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{-2}(3n+1)q^n \equiv G_1(q)-G_2(q) \pmod 9, \tag{3.22}$$

where

$$G_1(q)=3P(q)A(q)^2\frac{f_2^2 f_3^6}{f_1^{14}}\cdot\frac{\psi(q^3)}{\psi(q)^4}, \tag{3.23}$$

$$G_2(q)=P(q)^2A(q)\frac{f_2^2 f_3^3}{f_1^{14}}\cdot\frac{\psi(q^3)^2}{\psi(q)^4}. \tag{3.24}$$

By (2.4), (2.8), and (2.14), we deduce that

$$G_1(q) \equiv 3\frac{f_2^2 f_3^6}{f_1^{14}} \cdot f_1 \cdot \frac{1}{\psi(q)} \cdot \frac{f_2^2 f_3^4}{f_1^2 f_6^2} \equiv 3\frac{f_2^2}{f_6^2}\left(\frac{f_3^3}{f_1}\right)^2 \pmod 9, \tag{3.25}$$

$$G_2(q) = P(q)^2 \frac{f_2^2 f_3^3}{f_1^{14}} \cdot \frac{f_1^4 f_6^4}{f_2^8 f_3^2} \cdot \frac{f_2 f_3^2}{f_1 f_6} \equiv P(q)^2 \frac{f_2^4}{f_1^2} \pmod 9. \tag{3.26}$$

Invoking (2.3), (2.14), (2.16), and (3.26), we get

$$\begin{aligned}
G_2(q) &\equiv f_2^4\left(1 + 12\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\right) \\
&= 2f_2^4\left(1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\right) - f_2^4 \\
&\equiv 2f_2^4\left(\varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6)\right) - f_2^4 \\
&\equiv 2\frac{f_2^9 f_6^5}{f_4^2 f_{12}^2} \cdot \frac{1}{f_1^2 f_3^2} + 8q\frac{f_2^3 f_4^2 f_{12}^2}{f_6} - f_2^4 \pmod 9.
\end{aligned} \tag{3.27}$$

Hence, by (3.22), (3.25), and (3.27), we derive that

$$\sum_{n=0}^{\infty} PD_{-2}(3n+1)q^n \equiv 3\frac{f_2^2}{f_6^2}\left(\frac{f_3^3}{f_1}\right)^2 - 2\frac{f_2^9 f_6^5}{f_4^2 f_{12}^2} \cdot \frac{1}{f_1^2 f_3^2} - 8q\frac{f_2^3 f_4^2 f_{12}^2}{f_6} + f_2^4 \pmod 9. \tag{3.28}$$

By (2.10) and (2.13), the above identity can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} PD_{-2}(3n+1)q^n \equiv{}& 3\frac{f_2^2}{f_6^2}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right)^2 - 2\frac{f_2^9 f_6^5}{f_4^2 f_{12}^2}\left(\frac{f_8^4 f_{12}^{10}}{f_2^4 f_4^2 f_6^8 f_{24}^4}\right. \\
&\left. + 2q\frac{f_4^4 f_{12}^4}{f_2^6 f_6^6} + q^2\frac{f_4^{10} f_{24}^4}{f_2^8 f_6^4 f_8^4 f_{12}^2}\right) - 8q\frac{f_2^3 f_4^2 f_{12}^2}{f_6} + f_2^4 \pmod 9.
\end{aligned} \tag{3.29}$$

Choosing the terms on both sides of (3.29) for which the powers of q are of the form $2n+1$, dividing by q, and replacing q^2 by q, we obtain that

$$\sum_{n=0}^{\infty} PD_{-2}(6n+4)q^n \equiv 6f_2^2 f_6^2 - 12f_2^2 f_6^2 \cdot \frac{f_1^3}{f_3} \pmod 9. \tag{3.30}$$

Applying (2.12) to (3.30), we get

$$\sum_{n=0}^{\infty} PD_{-2}(6n+4)q^n \equiv 6f_2^2 f_6^2 - 12f_2^2 f_6^2\left(\frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right) \pmod 9. \tag{3.31}$$

Extracting the terms on both sides of (3.31) for which the powers of q are of the form $2n+1$, dividing by q, and replacing q^2 by q, we are led to

$$\sum_{n=0}^{\infty} PD_{-2}(12n+10)q^n \equiv 36\frac{f_1^4 f_6^3}{f_2} \pmod 9. \tag{3.32}$$

Hence, we arrive at (3.21). This completes the proof. □

Finally, we close this article by proving the last congruence.

Theorem 3.4 *For $n \geq 0$ and $\alpha \geq 1$, we have*

$$PD_{-2}(3^{\alpha}(6n+2)) \equiv 0 \pmod{9}. \tag{3.33}$$

Proof By (2.6), (2.7), and (3.10), we deduce that

$$\sum_{n=0}^{\infty} PD_{-2}(6n)q^n \equiv \frac{f_3^6}{f_6^3} \cdot \frac{\varphi(-q^9)}{\varphi(-q^3)^4} \left(\varphi(-q^9)^2 + 2q\varphi(-q^9)X(-q^3) + 4q^2X(-q^3)^2\right)$$
$$+ 16q\frac{f_6^6}{f_3^3} \cdot \frac{\psi(q^9)}{\psi(q^3)^4} \left(A(q^3)^2 - qA(q^3)\psi(q^9) + q^2\psi(q^9)^2\right) \pmod{9}. \tag{3.34}$$

Extracting those terms associated with powers q^{3n} on both sides of (3.34) and replacing q^3 by q, we observe that

$$\sum_{n=0}^{\infty} PD_{-2}(18n)q^n \equiv \frac{f_1^6}{f_2^3} \cdot \frac{\varphi(-q^3)^3}{\varphi(-q)^4} + 16q\frac{f_2^6}{f_1^3} \cdot \frac{\psi(q^3)^3}{\psi(q)^4} \pmod{9}. \tag{3.35}$$

Applying (2.4) and (2.5) to (3.35), we are led to

$$\sum_{n=0}^{\infty} PD_{-2}(18n)q^n \equiv \frac{f_3^6}{f_6^3} \cdot \frac{1}{\varphi(-q)} + 16q\frac{f_6^6}{f_3^3} \cdot \frac{1}{\psi(q)} \pmod{9}. \tag{3.36}$$

In view of (3.10) and (3.36), we obtain

$$PD_{-2}(6n) \equiv PD_{-2}(18n) \pmod{9}. \tag{3.37}$$

Based on (3.2) and (3.37), by induction on α, it yields that (3.33) is true. This completes the proof. □

Acknowledgments

The authors would like to thank the anonymous referee for carefully reading the manuscript and for giving many helpful comments and suggestions, which substantially helped improve the quality of the paper. The first author was supported by the Scientific Research Foundation of Nanjing Institute of Technology (No. YKJ201627) and the second author was supported by the Natural Science Foundation of Jiangsu Province of China (No. BK20160855).

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