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If 4-convex vectors are closed in uniform norms then their second derivatives are also closed in weighted L^2-norm

Muhammad Shoaib SALEEM[1,*], **Josip PEČARIĆ**[2], **Hamood UR REHMAN**[1], **Mobeen MUNIR**[3], **Muhammad WAHAB KHAN**[1]

[1]Department of Mathematics, University of Okara, Okara, Pakistan
[2]Faculty of Textile Technology, University of Zagreb, Zagreb Croatia
[3]Department of Mathematics, University of Education, Lahore, Pakistan

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Abstract: In this paper, we develop the weighted energy estimates for arbitrary 4-convex vectors and the vectors having both 4-convex and 4-concave functions as their arguments. To do this, we first develop these estimates for smooth 4-convex vectors and then, through mollification, extend the results for arbitrary 4-convex vectors. This type of estimates are valuable in problems of financial mathematics for the establishment of optimal investment strategies

Key words: Smooth convex vectors, smooth concave vectors, vector convolution

1. Introduction

The role of convex sets, convex functions, and their generalizations are important in applied mathematics, especially in nonlinear programming and optimization theory [7]. For example, in economics, convexity plays a fundamental role in equilibrium and duality theory. The convexity of sets and functions has been the subject of many studies in recent years. However, in many new problems encountered in applied mathematics the notion of convexity is not enough to produce favorite results and hence it is necessary to extend the notion of convexity to the new generalized notions. Recently, several extensions have been considered for the classical convex functions such that some of these new concepts are based on extension of the domain of a convex function (a convex set) to a generalized form and some of them are new definitions in which there is no generalization on domain but on the form of the definition [3].

A function is convex if and only if it is convex when restricted to any line that intersects its domain. The analysis of convex functions is a well-developed field. Many results for convex functions can be interpreted geometrically using epi graphs and applying results for convex sets [4].

The basic inequality in convex analysis is

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y) \tag{1.1}$$

sometimes called Jensen's inequality [3]. Many famous inequalities can be derived by applying Jensen's inequality to some appropriate convex function. The term convexity is generalized by different mathematicians in many directions such as quasi convex functions, log convex functions, k convex functions, n convex functions,

*Correspondence: shaby455@yahoo.com

and (n-m) convex functions [4]. These are the new generalizations and all of these generalizations have many interesting applications according to their constructions.

One of the important generalizations of the definition of convexity is n-convex functions. The function $f(x)$ is said to be an n-convex function if

$$f^{(n)}(x) \geq 0.$$

The weighted energy estimates for the convex functions and 4-convex functions are established in [1] and [5]. In [6], the author defined the convex vector and established the results for it and the vector having both convex and concave components. It is natural to derive similar results for 4-convex vectors and vectors having both 4-convex and 4-concave components. For convenience we will use the following notations and definitions:

$I = I(x_0, r)$ stands for interval $I(x_0, r)$, where $x_o = \frac{a+b}{2}$ and $r = \frac{a+b}{2} - a$ and $\overline{I}$ is closed interval $[a, b]$.

The n-dimensional vector

$$F(x) = \big(f_1(x), f_2(x), ..., f_n(x)\big) \tag{1.2}$$

is called a smooth convex vector if

$$\frac{d^2}{dx^2} f_i(x) \geq 0 \;\; \forall i = 1, 2, ..., n \tag{1.3}$$

and a smooth 4-convex vector if

$$\frac{d^4}{dx^4} f_i(x) \geq 0 \;\; \forall i = 1, 2, ..., n. \tag{1.4}$$

The vector $F(x)$ in (1.2) is arbitrary 4-convex provided

$$f_i{}^{(2)}(\lambda x + (1-\lambda)y) \leq \lambda f_i{}^{(2)}(x) + (1-\lambda) f_i^{(2)}(y) \;\forall i = 1, 2, \ldots, n \tag{1.5}$$

for each $\lambda \in [0, 1]$ and all x, y belong to $\mathbb{R}$.

Let $\chi_{[1,j]}^{[j+1,n]}[a, b]$ be the class of vectors having convex functions on its first j components and the remaining components are concave functions at interval $[a, b]$ and $\chi_{[1,j]}^{[j+1,n]}[a, b]$ be the class of vectors having concave functions on its first j components and the remaining are convex at the interval $[a, b]$. It is trivial that if $F(x) \in \chi_{[1,j]}^{[j+1,n]}[a, b]$ then $-F(x) \in \chi_{[j+1,n]}^{[1,j]}[a, b]$.

Now we define the class of 4-convex vectors:

Let $\Upsilon_{[1,j]}^{[j+1,n]}[a, b]$ be the class of vectors having 4-convex functions on its first j components and the remaining components are 4-concave functions at interval $[a, b]$ and $\Upsilon_{[1,j]}^{[j+1,n]}[a, b]$ be the class of vectors having 4-concave functions on its first j components and the remaining are 4-convex at the interval $[a, b]$. It is trivial that if $F(x) \in \Upsilon_{[1,j]}^{[j+1,n]}[a, b]$ then $-F(x) \in \Upsilon_{[j+1,n]}^{[1,j]}[a, b]$.

Let $h(x)$ be the weight function, which is a nonnegative 2-concave function in $C^4[a, b]$, i.e.

$$h(a) = h(b) = 0, h^{'}(a) = h^{'}(b) = 0, \;\; h^{''}(a) = h^{''}(b) = 0, \;\; h^{'''}(a) = h^{'''}(b) = 0. \tag{1.6}$$

with $a \leq x \leq b$.

The paper is organized as follows:

In the second section we develop the energy estimates for the smooth vectors, and we define the molification of arbitrary convex vectors. In the last section we develop the estimates for arbitrary convex vectors belonging to $\Upsilon_{[1,j]}^{[j+1,n]}[a,b]$ and also belonging to $\chi_{[1,j]}^{[j+1,n]}[a,b]$.

2. The case of smooth 4-convex vectors

Lemma 2.1 *[1] Let $f(x)$ and $g(x)$ be both smooth 4-convex as well as smooth convex functions. Let $h(x)$ be the nonnegative smooth weight function as defined in (1.6) and satisfying the condition*

$$h^{''}(x) \leq 0 \;\; \forall \;\; x \in I \;\; \text{and}\; h^{'}(x) = h^{''}(x) = h^{'''}(x) = 0 \;\;\; \forall \;\; x \in \partial I. \tag{2.1}$$

Then the following estimate holds:

$$\int_I \left(|f^{''}(x) - g^{''}(x)|\right)^2 h(x)dx \qquad \leq \int_I \Bigg(\frac{\big(f(x)-g(x)\big)^2}{2} - \sup_{x\in I}|f(x)-g(x)| \\ (f(x)+g(x))\Bigg) h^{(iv)}(x)dx. \tag{2.2}$$

Lemma 2.2 *[1] Let $f(x)$ and $g(x)$ be both 4-concave as well as 2-concave functions. Let $h(x)$ be the nonnegative smooth weight function as defined in (1.6) and satisfying the condition $h^{''}(x) \leq 0 \;\; \forall \;\; x \in I$ and*

$$h^{'}(x) = h^{''}(x) = h^{'''}(x) = 0 \;\;\; \forall \;\; x \in \partial I.$$

Then the following estimate holds:

$$\int_I \left(|f^{''}(x) - g^{''}(x)|\right)^2 h(x)dx \qquad \leq \int_I \Bigg(\frac{\big(f(x)-g(x)\big)^2}{2} + \sup_{x\in I}|f(x)-g(x)| \\ (f(x)+g(x))\Bigg) h^{(iv)}(x)dx. \tag{2.3}$$

We will start with the following theorem:

Theorem 2.3 *Let $F(x)$ and $G(x)$ be two smooth 4-convex vectors as well as smooth convex vectors as defined in (1.3) and (1.5). Let $h(x)$ be the smooth nonnegative weight function defined in (1.6) and satisfying (2.1); then the following energy estimate is valid:*

$$\int_I \left|F^{''}(x) - G^{''}(x)\right|^2 h(x)dx \qquad \leq \sum_{i=1}^{n} \int_I \Bigg[\frac{(f_i(x)-g_i(x))^2}{2} - \sup_{x\in I}|f_i(x)-g_i(x)| \\ \times\big(f_i(x)+g_i(x)\big)\Bigg] h^{(iv)}(x)dx. \tag{2.4}$$

Proof Take

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx$$

and we have

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx = \sum_{i=1}^{n} \int_I \left(f_i''(x) - g_i''(x)\right)^2 h(x)dx.$$

Now using lemma (2.1), we have

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx \leq \sum_{i=1}^{n} \int_I \left[\frac{\left(f_i(x) - g_i(x)\right)^2}{2} - \sup_{x\in I}\left|f_i(x) - g_i(x)\right| \left(f_i(x) + g_i(x)\right)\right] h^{(iv)}(x)dx \tag{2.5}$$

□

Remark 2.4 *Taking the supremum of (2.5), we obtain*

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx \leq \sum_{i=1}^{n} \left[\frac{1}{2}\left\|f_i(x) - g_i(x)\right\|_{L^\infty}^2 + \left\|f_i(x) - g_i(x)\right\|_{L^\infty} \times \left(\left\|f_i(x)\right\|_{L^\infty} + \left\|g_i(x)\right\|_{L^\infty}\right)\right] \int_I |h^{(iv)}(x)|dx. \tag{2.6}$$

Remark 2.5 *If $F(x)$ and $G(x)$ are 4-concave vectors then using lemma (2.2) we have*

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx \leq \sum_{i=1}^{n} \int_I \left[\frac{(f_i(x) - g_i(x))^2}{2} + \sup_{x\in I}|f_i(x) - g_i(x)| \times (f_i(x) + g_i(x))\right] h^{(iv)}(x)dx \tag{2.7}$$

Theorem 2.6 *Let $F(x)$ and $G(x)$ be two vectors that belong to $\Upsilon_{[1,j]}^{[j+1,n]}[a,b]$ and also belong to $\chi_{[1,j]}^{[j+1,n]}[a,b]$.*

Let $h(x)$ be the nonnegative weight function satisfying (1.6); then the following inequality is valid:

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx \leq \int_I \Bigg[\sum_{i=1}^{n} \frac{\left(f_i(x) - g_i(x)\right)^2}{2} + \sum_{i=1}^{j} \sup_{x\in I} \left|f_i(x) - g_i(x)\right| \left[f_i(x) + g_i(x)\right] - \sum_{i=j+1}^{n} \sup_{x\in I} \left|f_i(x) - g_i(x)\right| \times \left[f_i(x) + g_i(x)\right]\Bigg] h^{(iv)}(x)dx. \quad (2.8)$$

Proof Take

$$\int_I |F''(x) - G''(x)|^2 h(x)dx,$$

and we have

$$\int_I |F''(x) - G''(x)|^2 h(x)dx = \sum_{i=1}^{n} \int_I \left(f_i''(x) - g_i''(x)\right)^2 h(x)dx \quad (2.9)$$

$$= \sum_{i=1}^{j} \int_I \left(f_i''(x) - g_i''(x)\right)^2 h(x)dx + \sum_{i=j+1}^{n} \int_I \left(f_i''(x) - g_i''(x)\right)^2 h(x)dx. \quad (2.10)$$

Using lemma (2.1) on the first integral, we obtain

$$\int_I \left(f_i''(x) - g_i''(x)\right)^2 h(x)dx \leq \int_I \Bigg[\sum_{i=1}^{n} \frac{\left(f_i(x) - g_i(x)\right)^2}{2} - \sup_{x\in I} \left|f_i(x) - g_i(x)\right| \times \Big(f_i(x) + g_i(x)\Big)\Bigg] h^{(iv)}(x)dx, \quad (2.11)$$

similarly using the lemma (2.2) on the second integral, we have the following inequality (2.8):

$$\int_I \left(f_i''(x) - g_i''(x)\right)^2 h(x)dx \leq \int_I \Bigg[\sum_{i=1}^{n} \frac{\left(f_i(x) - g_i(x)\right)^2}{2} + \sup_{x\in I} \left|f_i(x) - g_i(x)\right| \times \Big(f_i(x) + g_i(x)\Big)\Bigg] h^{(iv)}(x)dx, \quad (2.12)$$

On combining the inequality (2.11) and (2.12), we have the required inequality (2.8). □

Now we define the vector convolution for

$$F(x) \in \Upsilon_{[1,j]}^{[j+1,n]}[a, b]$$

in the following way as

$$F(x) = \big(f_1(x), f_2(x), ..., f_n(x)\big)$$

take

$$\epsilon = \big(\epsilon_1, \epsilon_2, ..., \epsilon_n\big),$$

and

$$\epsilon \rightarrow 0 \text{ means } \{\max\big(\epsilon_1, \epsilon_2, \ldots, \epsilon_n\big)\} \rightarrow 0.$$

Take

$$\theta_\epsilon(x) = (\theta_{\epsilon_1}(x), \theta_{\epsilon_2}(x), ..., \theta_{\epsilon_n}(x))$$

and

$$\theta_{\epsilon_i}(x) = \begin{cases} cexp\frac{1}{\frac{x}{\epsilon_i}-1} & \text{if} \quad |x| < \epsilon_i \\ 0 & \text{if} \quad |x| > \epsilon_i \end{cases}$$

$\forall\ i = 1, 2, \ldots, n.$
We define the convolution as

$$F * \theta_\epsilon(x) = \big(f_1 * \theta_{\epsilon_1}, f_2 * \theta_{\epsilon_2}, ..., f_n * \theta_{\epsilon_n}\big)$$

Let us denote

$$f_{\epsilon_i} = f_i * \theta_{\epsilon_i} = \int\limits_{\Re} f_i(x-y)\theta_{\epsilon_i}(y)dy.$$

If f_i is continuous then f_{ϵ_i} converges uniformly to f_i in any compact subset
$K_i \subseteq I$ i.e.

$$|f_{\epsilon_i} - f_i|\xrightarrow[\epsilon_i \rightarrow 0]{} 0 \text{ this implies that}$$

$$|F_\epsilon - F|^2 = \sum_{i=1}^{n} |f_{\epsilon_i} - f_i|^2\xrightarrow[\epsilon_i \rightarrow 0]{} .0$$

Now we claim that

$$F_\epsilon \in \Upsilon^{[i,j]}_{[j+1,n]}[a,b]$$

i.e. f_{ϵ_i} is a convex function $\forall\ i = 1, 2, ..., j$ and concave for $\forall\ i = j+1, \ldots, n$. It can be seen in the following way:

take

$$f_{\epsilon_i}(\lambda x_1 + (1-\lambda)x_2) = \int\limits_I f_i\big(\lambda x_1 + (1-\lambda)x_2 - y\big)\theta_{\epsilon_i}(y)dy$$

$$= \int\limits_I f_i\big[\lambda(x_1 - y) + (1-\lambda)(x_2 - y)\big]\theta_{\epsilon_i}(y)dy. \tag{2.13}$$

Now for $i = 1, \ldots j$, we have

$$f_{\epsilon_i}(\lambda x_1 + (1-\lambda)x_2) \leq \int\limits_I \big[\lambda f_i(x_1 - y) + (1-\lambda) f_i(x_2 - y)\big]\theta_{\epsilon_i}(y)dy$$

$$= \lambda \int\limits_I f_i(x_1 - y)\theta_{\epsilon_i}(y)dy + (1-\lambda)\int\limits_I f_i(x_2 - y)\theta_{\epsilon_i}(y)dy$$

$$= \lambda f_{\epsilon_i}(x_1) + (1-\lambda) f_{\epsilon_i}(x_2)$$

and for $i = j+1, \dots n$, from (2.13) we have

$$f_{\epsilon_i}(\lambda x_1 + (1-\lambda)x_2) \geq \lambda \int\limits_I f_i(x_1 - y)\theta_{\epsilon_i}(y)dy + (1-\lambda)\int\limits_I f_i(x_2 - y)\theta_{\epsilon_i}(y)dy$$

$$= \lambda f_{\epsilon_i}(x_1) + (1-\lambda) f_{\epsilon_i}(x_2).$$

Thus f_{ϵ_i} is 2-convex. Similarly, the 2-convexity of ${f_i}^2$ yields the 2-convexity of $f_{\epsilon_i}^2$. Therefore, f_{ϵ_i} is 4-convex.

3. The case of arbitrary 4-convex vectors

We will use I_k for the interval $I(x_o, r_k)$, x_0 is the center, and radius r_k is defined as where

$$r_k = r(\frac{k+1}{k+2}).$$

It is trivial that $I_k \subset I_{k+1}$ and $\bigcup\limits_{k=1}^{\infty} I_k = I$

Theorem 3.1 *Let $F(x)$ be the continuous arbitrary 4-convex vector; then it satisfies*

$$\int\limits_I \left|F^{''}(x)\right|^2 h(x)dx < \infty.$$

Proof Take $F_\epsilon(x)$, the mollification of the arbitrary 4-convex vector $F(x)$ as defined in (1.5). Writing the inequality (2.6) for $F = F_m$ and $G = 0$ and for intervals $I_{k+l} \subset I$, we have

$$\int\limits_{I_{k+l}} \left|F_m^{''}(x)\right|^2 h_{k+l}dx \leq \sum_{i=1}^{n}\left[\frac{3}{2}\left\|f_{(m,i)}(x)\right\|^2_{L^\infty_{I_{k+l}}}\right]\int\limits_{I_{k+l}}\left|h_{k+l}^{(iv)}(x)\right| dx. \tag{3.1}$$

Denote

$$\int\limits_{I_{k+l}} \left|h_{k+l}^{(iv)}(x)dx\right| = c_{k+l}$$

and we have

$$\int\limits_{I_{k+l}} \left|F_m^{''}(x)\right|^2 h_{k+l}dx \leq \sum_{i=1}^{n}\left[\frac{3}{2}\left\|f_{(m,i)}(x)\right\|^2_{L^\infty_{I_{k+l}}}\right].(c_{k+l}) \tag{3.2}$$

Applying limit as $m \to \infty$, we have

$$\int\limits_{I_{k+l}} \left|F^{''}(x)\right|^2 h_{k+l}dx \leq \sum_{i=1}^{n}\frac{3}{2}\left\|f_i(x)\right\|^2_{L^\infty_{I_{k+l}}}(c_{k+l}).$$

Since $I_k \subseteq I_{k+l}$ so

$$\int\limits_{I_k} \left|F^{''}(x)\right|^2 h_{k+l}(x)dx \leq \frac{3}{2}\left\|F(x)\right\|^2_{L^{\infty}_{I_{k+l}}}(c_{k+l}).$$

In the above integral make $l \to \infty$, and we have

$$\int\limits_{I} \left|F^{''}(x)\right|^2 h(x)dx \leq \frac{3}{2}\left\|F(x)\right\|^2_{L^{\infty}_{I}} c_{\infty} < \infty.$$

Since the above integral is bounded for each k, so

$$\int\limits_{I} \left|F^{''}(x)\right|^2 h(x)dx < \infty.$$

□

Theorem 3.2 *Let $F(x)$ and $G(x)$ be two arbitrary 4-convex vectors that belong to $\Upsilon^{[i,j]}_{[j+1,n]}[a,b]$ and also belong to $\chi^{[i,j]}_{[j+1,n]}[a,b]$ and let $h(x)$ be the nonnegative weight function defined in (1.6) over the interval I; then the following estimate is valid:*

$$\begin{aligned}\int\limits_{I} \left|F^{''}(x)-G^{''}(x)\right|^2 h(x)dx \quad &\leq \quad \sum_{i=1}^{n}\frac{1}{2}\left\|f_i(x)-g_i(x)\right\|^2_{L^{\infty}}+\left\|f_i(x)-g_i(x)\right\|_{L^{\infty}} \\ &\times \quad \left(\left\|f_i(x)\right\|_{L^{\infty}}+\left\|g_i(x)\right\|_{L^{\infty}}\right)\int\limits_{I} h^{(iv)}(x)dx. \qquad (3.3)\end{aligned}$$

Proof For arbitrary 4-convex vectors $F(x)$ and $G(x)$, which are continuous, take smooth approximation $F_m(x)$ and $G_m(x)$. There exist integer m_{k+l} such that F_m and $G_m(x)$ are smooth over the interval I_{k+l} and $F_m(x)$ and $G_m(x)$ converge uniformly to $F(x)$ and $G(x)$, respectively, for $m \geq m_{k+l}$. Let us write the inequality (2.6) for the functions $F_m(x)$ and $G_m(x)$ on the interval I_{k+l}

$$\begin{aligned}\int\limits_{I_{k+l}} \left|F^{''}_m(x)-G^{''}_m(x)\right|^2 h_{k+l}dx \qquad &\leq \sum_{i=1}^{n}\left[\frac{\left\|f_{(m,i)}(x)-g_{(m,i)}(x)\right\|^2_{L^{\infty}_{I_{k+l}}}}{2}\right. \\ &+\left\|f_{(m,i)(x)}-g_{(m,i)}(x)\right\|_{L^{\infty}_{I_{k+l}}} \\ &\times\Bigg(\left\|f_{(m,i)}(x)\right\|_{L^{\infty}_{I_{k+l}}} \\ &\left.+\left\|g_{(m,i)}(x)\right\|_{L^{\infty}_{I_{k+l}}}\Bigg)\right] \\ &\times\int\limits_{I_{k+l}} h^{(iv)}_{k+l}(x)dx. \qquad (3.4)\end{aligned}$$

Applying limit $m \to \infty$, we get

$$\int_{I_{k+l}} \left|F^{''}(x) - G^{''}(x)\right|^2 h_{k+l} dx \leq \sum_{i=1}^{n} \Bigg[\frac{\|f_i(x) - g_i(x)\|^2_{L^\infty_{I_{k+l}}}}{2} + \|f_i(x) - g_i(x)\|_{L^\infty_{I_{k+l}}} \times \Big(\|f_i(x)\|_{L^\infty_{I_{k+l}}} + \|g_i(x)\|_{L^\infty_{I_{k+l}}} \Big) \Bigg] \int_{I_{k+l}} \left| h^{(iv)}_{k+l}(x) \right| dx. \tag{3.5}$$

Writing the left integral for I_{k+l} smaller interval $I_k \subset I_{k+l}$ and taking the limit as $l \to \infty$, we obtain

$$\int_{I_k} \left|F^{''}(x) - G^{''}(x)\right|^2 h_k(x) dx \leq \Bigg[\sum_{i=1}^{n} \frac{\|f_i(x) - g_i(x)\|^2_{L^\infty_I}}{2} + \|f_i(x) - g_i(x)\|_{L^\infty_I} \left(\|f_i(x)\|_{L^\infty_I} + \|g_i(x)\|_{L^\infty_I} \right) \Bigg] \int_I |h^{(iv)}(x)| dx \tag{3.6}$$

by the last theorem, we have

$$\int_I \left|F^{''}(x) - G^{''}(x)\right|^2 h(x) dx < \infty.$$

Taking the limit as $k \to \infty$, we obtain the result (3.3). □

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