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Normality and quotient in crossed modules over groupoids and double groupoids

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Abstract: We consider the categorical equivalence between crossed modules over groupoids and double groupoids with thin structures, and by this equivalence, we prove how normality and quotient concepts are related in these two categories and give some examples of these objects.

Key words: Quotient crossed module, double groupoid, quotient double groupoid

1. Introduction

The concept of a crossed module over groups introduced by Whitehead in [23, 24] in the investigation of the properties of second relative homotopy groups for topological spaces, which can be viewed as a 2-dimensional group [3], has been widely used in homotopy theory [5], the theory of identities among relations for group presentations [6], algebraic K-theory [16], and homological algebra [15, 17]. See [5, p. 49], for some discussion of the relation of crossed modules to crossed squares and so to homotopy 3-types.

The categorical equivalence of crossed modules over groups and group-groupoids, which are internal groupoids in the category of groups and widely used in the literature under the names *2-groups*[2], $\mathcal{G}$*-groupoids*, or *group objects* in the category of groupoids [8], was proved by Brown and Spencer in [8, Theorem 1], and then some important results were obtained by means of this equivalence. For example, recently normal and quotient objects in these two categories were compared and the corresponding objects in the category of group-groupoids were characterized in [18]. The equivalence of these categories was also generalized by Porter in [22, Section 3] to a more general algebraic category C, whose idea comes from Higgins [12] and Orzech [20, 21] and is called the category of groups with operations. This result was used as a tool in the study of [1]. Applying Porter's result, the study of internal category theory in C was continued in the works of Datuashvili [9, 10].

On the other hand, it was pointed out in [5, Chapter 6] that the structure of a crossed module is inadequate to give a proof of the 2-dimensional Seifert–van-Kampen theorem and hence one needs the idea of a double groupoid, which can be expressed as a groupoid object in the category of groupoids. The categorical equivalence of crossed modules over groupoids and double groupoids with thin structures was proved in that work.

In this paper, parallel to the work presented in [18], we define the normal subcrossed module and quotient crossed modules over groupoids, and then using the latter equivalence of the categories we obtain the normal

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subdouble groupoid and quotient double groupoids. A motivational point for relating these objects in both categories is to produce more examples of normal subdouble and quotient double groupoids.

2. Preliminaries on crossed modules over groupoids and double groupoids

In the notations of [5, Chapter 6.1], a groupoid G has a set G_0 of *objects* and a set G of morphisms, together with maps $\partial^-, \partial^+\colon G \to G_0$ and $\varepsilon\colon G_0 \to G$ such that $\partial^-\varepsilon = \partial^+\varepsilon = 1_{G_0}$. The maps ∂^-, ∂^+ are respectively called *initial* and *final* point maps and the map ε is called *object inclusion*. If $a, b \in G$ and $\partial^+(a) = \partial^-(b)$, then the *composite* $a \circ b$ exists such that $\partial^-(a \circ b) = \partial^-(a)$ and $\partial^+(a \circ b) = \partial^+(b)$. Thus, there exists a partial composition defined by the map $G\,{}_{\partial^+}\!\times_{\partial^-} G \to G, (a, b) \mapsto a \circ b$, where $G\,{}_{\partial^+}\!\times_{\partial^-} G$ is the pullback of ∂^+ and ∂^-. Furthermore, this partial composition is associative; for $x \in G_0$ the element $\varepsilon(x)$ denoted by 1_x acts as the identity, and each morphism a has an inverse a^{-1} such that $\partial^-(a^{-1}) = \partial^+(a)$, $\partial^+(a^{-1}) = \partial^-(a)$, $a \circ a^{-1} = (\varepsilon\partial^-)(a)$, $a^{-1} \circ a = (\varepsilon\partial^+)(a)$. The map $G \to G$, $a \mapsto a^{-1}$, is called the *inversion*. The set of all morphisms from x to x is a group, called the *object group* at x, and denoted by $G(x)$. A groupoid G is *transitive (resp. totally intransitive)* if $G(x, y) \neq \emptyset$ (resp. $G(x, y) = \emptyset$) for all $x, y \in G_0$ such that $x \neq y$.

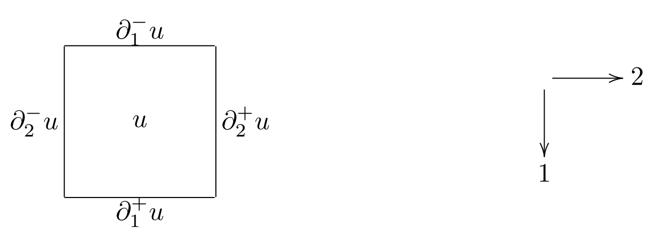
A subgroupoid H of G is called *wide* if $H_0 = G_0$ and a wide subgroupoid H is called *normal* if $a \circ H(y) = H(x) \circ a$ for objects $x, y \in H_0$ and $a \in G(x, y)$. For example, $\operatorname{Ker} f$, the kernel of a groupoid morphism $f\colon G \to K$, is a normal subgroupoid of G.

The quotient groupoid is formed as follows (see Higgins [13, p. 86] and [4, p. 420]). Let H be a normal subgroupoid of the groupoid G. The components of H define a partition on G_0 and we write $[x]$ for the class containing x. Then H also defines an equivalence relation on G by $a \sim b$ for $a, b \in G$ if and only if $a = m \circ b \circ n$ for some $m, n \in H$. A partial composition $[a] \circ [b]$ on the morphisms is defined if and only if there exist $a_1 \in [a]$, $b_1 \in [b]$ such that $a_1 \circ b_1$ is defined in G and then $[a] \circ [b] = [a_1 \circ b_1]$. This partial composition defines a groupoid on classes $[x]$ as objects. The groupoid defined in this manner is called a *quotient groupoid*.

We recall the notion of double groupoids from [5, Chapter 6].

A *double groupoid* denoted by $G = (G_2, G_1, G_0)$ has the sets G_0, G_1, and G_2 of points or vertices, edges, and squares, respectively, and three structures of groupoids. The first one defined on (G_1, G_0) has maps ∂^-, ∂^+, and ε and the composition denoted as multiplication. The other two are defined on (G_2, G_1), a horizontal one with maps ∂_2^-, ∂_2^+, and ε_2, composition denoted by $u +_2 v$, and a vertical one with maps ∂_1^-, ∂_1^+, and ε_1, composition denoted by $u +_1 w$, satisfying some conditions given in detail in [5, Chapter 6].

As a diagram a square $u \in G_2$ has bounding edges as follows.



For an edge $a \in G_1$, the identities $\varepsilon_1(a)$ and $\varepsilon_2(a)$ for vertical and horizontal composites of squares that are called *degeneracies* have respectively the boundaries

$$\begin{array}{c} a \\ \boxed{\;\varepsilon_1(a)\;} \\ a \end{array} \qquad \text{and} \qquad \begin{array}{ccc} a & \boxed{\;\varepsilon_2(a)\;} & a \end{array}$$

where the unlabeled edges of squares are identities.

A *morphism* $f\colon G \to K$ of double groupoids is a triple of functions

$$f_i\colon G_i \to K_i, \qquad (i = 0, 1, 2),$$

which commute with all structural maps of three groupoids (faces, degeneracies, compositions, etc.). Then we have the category DCatG of double groupoids and morphisms between them.

The following example of a double groupoid is from [5, Definition 6.1.10].

Example 2.1 For a groupoid G, a double groupoid of 'squares' or 2-'shells' in G denoted by $\square' G$ is defined as follows:

The groupoid structure of $(\square' G)_1$ is the same as that of G. The squares of $(\square' G)_2$ consist of quadruples $\begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix}$ for $a, b, c, d \in G_1$; the horizontal and vertical compositions are defined by

$$\begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix} +_2 \begin{pmatrix} d & \begin{smallmatrix} g \\ f \end{smallmatrix} & h \end{pmatrix} = \begin{pmatrix} a & \begin{smallmatrix} cg \\ bf \end{smallmatrix} & h \end{pmatrix},$$

$$\begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix} +_1 \begin{pmatrix} f & \begin{smallmatrix} b \\ g \end{smallmatrix} & h \end{pmatrix} = \begin{pmatrix} af & \begin{smallmatrix} c \\ g \end{smallmatrix} & dh \end{pmatrix}.$$

The double groupoid of commutative squares or commutative 2-shells is denoted by $\square G$. The squares of $(\square G)_2$ consist of quadruples $\begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix}$ for $a, b, c, d \in G_1$ such that $ab = cd$.

Definition 2.2 [5, Definition 6.4.1] A *thin structure* for a double groupoid G is a morphism of double groupoids

$$\ominus\colon \square(G_1) \to G,$$

which is identity on G_0 and G_1. A 2-dimensional element $\ominus(\alpha) \in G_2$ for an $\alpha \in (\square G_1)_2$ is called a *thin square*.

In [5, Section 6.4] a criterion for the existence of a thin structure is given in terms of thin squares.

Let DGpds be the category of double groupoids with thin structures and morphisms between them preserving the thin structures.

Example 2.3 For a groupoid G one can easily check that the double groupoid $\square' G$ of 2-shells defined in Example 2.1 has a trivial double groupoid morphism as a thin structure

$$\ominus\colon \square(G_1) \to \square' G$$

mapping the commutative 2-shells to themselves.

We know from [5, Theorem 6.4.11] that in a double groupoid that has a thin structure the horizontal and vertical groupoid structures on squares are isomorphic. Hence, the conditions of being subgroupoid and normal subgroupoid in one direction can be transferred to the other one and therefore we state and prove the theorems in Section 3 for only horizontal composite $+_2$.

In [5, Definition 6.2.1] a *crossed module over a groupoid* $P = (P, P_0)$ is defined as a morphism of groupoids

$$\mu\colon M \to P$$

in which $M = (M, P_0)$ is a totally intransitive groupoid and μ is identity on objects. Here $+$ is used for the composition in groupoid M. The groupoid P operates on the right on M and the action denoted by

$$M(p) \times P(p, q) \to M(q)$$

$$(x, a) \mapsto x^a$$

satisfies the usual axioms of an action (i) $x^1 = x$, $(x^a)^b = x^{ab}$ and (ii) $(x + y)^a = x^a + y^a$, and the following conditions are satisfied:

CM1. μ preserves the action, i.e. $\mu(x^a) = \mu(x)^a = a^{-1}\mu(x)a$.

CM2. For all $c \in M(p)$, $\mu(c)$ acts on M by conjugation:

$$x^{\mu(c)} = -c + x + c,$$

for any $x \in M(p)$.

Example 2.4 If H is a totally intransitive normal subgroupoid of G, then the inclusion $\imath\colon H \to G$ is a crossed module over groupoids with the action of conjugation.

For a topological example called *fundamental crossed module* over groupoids see [5, Example 6.2.2].

A *morphism* $(f_2, f_1)\colon (\mu\colon M \to P) \to (\nu\colon N \to Q)$ of crossed modules over groupoids is a pair of groupoid morphisms $f_2\colon M \to N$ and $f_1\colon P \to Q$ with the same induced map of vertices such that $f_1\mu = \nu f_2$ and $f_2(m^a) = f_2(m)^{f_1(a)}$ for $m \in M$ and $a \in P$. Hence, crossed modules over groupoids and morphisms of them constitute a category denoted by XMod. From now on by a crossed module we mean a crossed module over groupoids.

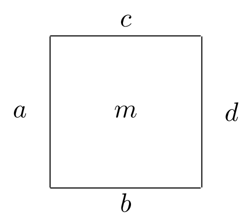
The following result was proved in [5, Section 6.6]. Since we need some details of the proof in the following section, we give a sketch proof.

Theorem 2.5 *The category* XMod *and* DGpds *are equivalent.*

Proof A functor $\lambda\colon \mathrm{XMod} \to \mathrm{DGpds}$ is defined as follows: for a crossed module $\mu\colon M \to P$, there is an associated double groupoid $G = \lambda(\mu\colon M \to P)$ in which $G_0 = P_0$, $G_1 = P$, and the set G_2 of squares consists of the elements

$$\left(m\ ;\ \left(a \begin{array}{c} c \\ b \end{array} d\right)\right)$$

such that $a,b,c,d \in P$ and $\mu(m) = b^{-1}a^{-1}cd$. The elements of G_2 can be represented by the following.



The horizontal and vertical compositions of squares are respectively defined by

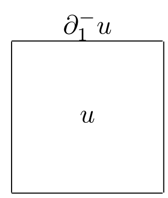
$$\left(m\ ;\ \left(a\ \genfrac{}{}{0pt}{}{c}{b}\ d\right)\right) +_2 \left(n\ ;\ \left(d\ \genfrac{}{}{0pt}{}{g}{f}\ h\right)\right) = \left(m^f + n\ ;\ \left(a\ \genfrac{}{}{0pt}{}{cg}{bf}\ h\right)\right),$$

$$\left(m\ ;\ \left(a\ \genfrac{}{}{0pt}{}{c}{b}\ d\right)\right) +_1 \left(n\ ;\ \left(f\ \genfrac{}{}{0pt}{}{b}{g}\ h\right)\right) = \left(n + m^h\ ;\ \left(af\ \genfrac{}{}{0pt}{}{c}{g}\ dh\right)\right).$$

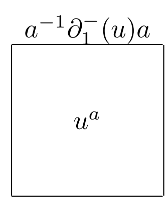
A thin structure $\ominus \colon \square G_1 \to G$ on G is defined by

$$\left(a\ \genfrac{}{}{0pt}{}{c}{b}\ d\right) \mapsto \left(1\ ;\ \left(a\ \genfrac{}{}{0pt}{}{c}{b}\ d\right)\right).$$

Conversely, for a double groupoid G, an associated crossed module $\gamma(G) = (\mu \colon M \to P)$ is given by $P = (G_1, G_0)$ and $M = (M, G_0)$, in which M has the squares $u \in G_2$ in the following form.



Here the boundary morphism μ is ${\partial_1}^- \colon M \to P$ and the action u^a is defined by



for $a \in G_1(x,y)$ and $u \in M(x)$, and the conditions [CM1] and [CM2] are satisfied. Hence, we have a functor $\gamma \colon \mathrm{DGpds} \to \mathrm{XMod}$.

The functors $\gamma\lambda \colon \mathrm{XMod} \to \mathrm{XMod}$ and $\lambda\gamma \colon \mathrm{DGpds} \to \mathrm{DGpds}$ are naturally equivalent to the identity functors 1_{XMod} and 1_{DGpds}, respectively. $\square$

We remark that as related to Theorem 2.5, the categorical equivalence of the crossed modules over groups and special double groupoids with connections was proved in [7], and using this equivalence, in [11] the normal and quotient objects in these categories were characterized. In the next section this is generalized to the equivalence given in Theorem 2.5.

3. Normality and quotient in double groupoids

In this section using the equivalence of the categories as stated in Theorem 2.5, we will determine the normal and quotient objects in the category DGpds of double groupoids with thin structures. Before this determination we initially define subcrossed modules and normal subcrossed modules over groupoids similar to the group cases given in [19].

Definition 3.1 A crossed module $\nu\colon N \to Q$ is called a *subcrossed module* of the crossed module $\mu\colon M \to P$ if the following hold:

SCM1. N is a subgroupoid of M and Q is a subgroupoid of P,

SCM2. ν is a restriction of μ to N, and

SCM3. The action of Q on N is the restriction of the action of P on M.

□

Definition 3.2 A subcrossed module $\nu\colon N \to Q$ of the crossed module $\mu\colon M \to P$ is called *normal* if:

NCM1. Q is a normal subgroupoid of P,

NCM2. $n^a \in N(q)$ for all $a \in P(p,q)$, $n \in N(p)$, and

NCM3. $-m + m^b \in N(p)$ for all $b \in Q(p)$ and $m \in M(p)$.

□

In this definition N also becomes a normal subgroupoid of M. Indeed, for $n \in N(p)$ and $m \in M(p)$, we have $n^{\mu(m)} \in N(p)$ by condition [NCM2] and $n^{\mu(m)} = -m + n + m \in N(p)$ by [CM2].

Example 3.3 The *kernel* of a crossed module morphism $(f_2, f_1)\colon (\mu\colon M \to P) \to (\nu\colon N \to Q)$ defined by

$$\mu^\star\colon \operatorname{Ker} f_2 \to \operatorname{Ker} f_1$$

as a restriction of μ is a normal subcrossed module of the crossed module $\mu\colon M \to P$.

Theorem 3.4 *Let $\nu\colon N \to Q$ be a normal subcrossed module of $\mu\colon M \to P$. If the groupoid Q is totally intransitive, then $\rho\colon M/N \to P/Q$ is a crossed module over groupoids.*

Proof By the construction of the quotient groupoid we know that when Q is totally intransitive, the object set of the quotient groupoid P/Q is P_0. Similarly, since N is totally intransitive, $(M/N)_0 = M_0 = P_0$. Hence, M/N and P/Q have the same objects. Since M and N are totally intransitive, so is M/N. Therefore, the induced morphism $\rho\colon M/N \to P/Q$ is well defined.

An action of P/Q on M/N is defined by

$$(M/N)(p) \times (P/Q)(p,q) \to (M/N)(q)$$

$$(N(p)+m,\ Q(p)a) \mapsto (N(p)+m)^{Q(p)a} = N(q) + m^a$$

for $a \in P(p,q)$ and $m \in M(p)$. We now prove that this action is well defined. If $m_1 \in (N(p)+m)$ and $a_1 \in Q(p)a$, there exist $n \in N(p)$ and $b \in Q(p)$ such that $m_1 = n+m$ and $a_1 = ba$. Here $-m+m^b = n_1 \in N(p)$ by [NCM3] and hence $m^b = m+n_1$. There exists $n_2 \in N(p)$ such that $m+n_1 = n_2+m$ by [NCM2]. By the action of Q on N and [NCM2] we have $n^{ba}+n_2^a \in N(q)$. If we write n_3 for $n^{ba}+n_2^a$, then we get

$$\begin{aligned} m_1^{a_1} &= (n+m)^{ba} = n^{ba} + m^{ba} = n^{ba} + (m^b)^a \\ &= n^{ba} + (m+n_1)^a = n^{ba} + (n_2+m)^a \\ &= n^{ba} + n_2^a + m^a \\ &= n_3 + m^a. \end{aligned}$$

Hence, $n_3 + m^a \in (N(q)+m^a)$, i.e. the action is well defined. It can be easily checked that the conditions [CM1] and [CM2] are satisfied and hence $\rho\colon M/N \to P/Q$ becomes a crossed module over groupoids. □

We call $\rho\colon M/N \to P/Q$ a *quotient crossed module* over groupoids.

Let $f\colon P \to Q$ be a morphism of groupoids. To use it in the following example we recall from [13, Proposition 26] that if f is injective on objects, then $\operatorname{Ker} f$ is totally intransitive.

Example 3.5 Let $\mu^\star\colon \operatorname{Ker} f_2 \to \operatorname{Ker} f_1$ be the kernel of a crossed module morphism $(f_2, f_1)\colon (\mu\colon M \to P) \to (\nu\colon N \to Q)$ as defined in Example 3.3 such that f_1 is injective on objects. Then by Theorem 3.4 the induced morphism $\rho\colon M/\operatorname{Ker} f_2 \to P/\operatorname{Ker} f_1$ is a quotient crossed module.

Since in a double groupoid with a thin structure the horizontal and vertical groupoid structures on squares are isomorphic, in the following theorems we use only horizontal composite $+_2$.

Theorem 3.6 *Let $\nu\colon N \to Q$ be a subcrossed module of a crossed module $\mu\colon M \to P$. Suppose that $H = (H_2, H_1, H_0)$ and $G = (G_2, G_1, G_0)$ are respectively the double groupoids corresponding to these crossed modules. Then the following are satisfied:*

1. *(H_1, H_0) is a subgroupoid of (G_1, G_0).*
2. *$(H_2, H_1, +_2)$ is a subgroupoid of $(G_2, G_1, +_2)$.*
3. *The thin structure $\ominus\colon \square(H_1) \to H$ on H is a restriction of that $\ominus\colon \square(G_1) \to G$ on G.*

Proof

1. By the proof of Theorem 2.5, we know that $(H_1, H_0) = Q$ and $(G_1, G_0) = P$, and (H_1, H_0) becomes a subgroupoid of (G_1, G_0) by [SCM1] of Definition 3.1.

2. H_2 consists of the elements

$$\alpha = \left(m \ ; \ \begin{pmatrix} a & \overset{c}{\underset{b}{}} & d \end{pmatrix} \right)$$

for $m \in N$, $a, b, c, d \in Q$ such that $\nu(m) = b^{-1}a^{-1}cd$, and such a quintuple has an inverse

$$-_2\alpha = \left((-m)^{b^{-1}} \ ; \ d \ \overset{c^{-1}}{\underset{b^{-1}}{}} \ a \right)$$

for horizontial composite. Since N is a groupoid and P acts on N, we have that $(-m)^{b^{-1}} \in N$. Hence, $-_2\alpha \in H_2$ and thus $(H_2, H_1, +_2)$ is a subgroupoid of $(G_2, G_1, +_2)$.

3. This is obvious by the details of the proof of Theorem 2.5.

□

Hence, we can state the definition of a subdouble groupoid as follows.

Definition 3.7 A double groupoid $H = (H_2, H_1, H_0)$ is called *a subdouble groupoid* of $G = (G_2, G_1, G_0)$ if the following are satisfied:

SDG1. (H_1, H_0) is a subgroupoid of (G_1, G_0).

SDG2. $(H_2, H_1, +_2)$ is a subgroupoid of $(G_2, G_1, +_2)$.

SDG3. The thin structure $\ominus\colon \square(H_1) \to H$ on H is a restriction of the thin structure $\ominus\colon \square(G_1) \to G$ on G.

We now prove that the converse of Theorem 3.6 is also satisfied.

Theorem 3.8 *Let $H = (H_2, H_1, H_0)$ be a subdouble groupoid of a double groupoid $G = (G_2, G_1, G_0)$. Assume that $\nu\colon N \to Q$ and $\mu\colon M \to P$ are respectively the crossed modules corresponding to H and G. Then the crossed module $\nu\colon N \to Q$ is a subcrossed module of the crossed module $\mu\colon M \to P$.*

Proof We need to prove that the conditions of Definition 3.1 are satisfied.

SCM1. By Theorem 2.5, $Q = (H_1, H_0)$ and $P = (G_1, G_0)$, and by [SDG1] (H_1, H_0) is subgroupoid of (G_1, G_0). Hence, Q is a subgroupoid of P. Furthermore, N and M consist of respectively specific squares of H_2 and G_2 as stated in the proof of Theorem 2.5 and by [SDG2] $(H_2, H_1, +_2)$ is a subgroupoid of $(G_2, G_1, +_2)$. Hence, N becomes a subgroupoid of M.

SCM2. By the details of the proof of Theorem 2.5, the crossed modules $\nu\colon N \to Q$ and $\mu\colon M \to P$ are defined by

$$\nu\colon N \to Q, n \mapsto \partial_1^- n$$
$$\mu\colon M \to P, m \mapsto \partial_1^- m$$

and hence ν is a restriction of μ.

SCM3. It is clear that the action of P on N is a restriction of the action of P on M.

Hence, the crossed module $\nu\colon N \to Q$ becomes a subcrossed module of the crossed module $\mu\colon M \to P$ as required. □

As a result of Theorem 3.6 and Theorem 3.8 the following corollary can be stated.

Corollary 3.9 *Let G be a double groupoid and $\mu\colon M \to P$ the crossed module corresponding to G. Then the category of the subdouble groupoids of G and the category of subcrossed modules of $\mu\colon M \to P$ are equivalent.*

As we can see from the proof of Theorem 3.11 in the subdouble groupoid corresponding to a normal subcrossed module, the horizontal and the vertical subgroupoids are not necessarily wide. We need the definition of a normal subgroupoid without wideness condition as follows.

Definition 3.10 Let G be a groupoid and H be a subgroupoid of G. Then H is called a *nonwide normal subgroupoid* of G if $gag^{-1} \in H(x)$ when $g \in G(x,y)$ and $a \in H(y)$ for $x, y \in H_0$.

Theorem 3.11 *Let $\nu\colon N \to Q$ be a normal subcrossed module of a crossed module $\mu\colon M \to P$. Suppose that $H = (H_2, H_1, H_0)$ and $G = (G_2, G_1, G_0)$ are respectively the double groupoids corresponding to these crossed modules. Then we have the following.*

1. *(H_1, H_0) is a normal subgroupoid of (G_1, G_0).*
2. *The groupoid $(H_2, H_1, +_2)$ is a nonwide normal subgroupoid of $(G_2, G_1, +_2)$.*
3. *The thin structure $\ominus\colon \square(H_1) \to H$ on H is the restriction of the thin structure $\ominus\colon \square(G_1) \to G$ on G.*

Proof

1. By the equivalence given in Theorem 2.5, $(H_1, H_0) = Q$ and $(G_1, G_0) = P$, and by [NCM1] in Definition 3.2 Q is a normal subgroupoid of P. Hence, (H_1, H_0) is a normal subgroupoid of (G_1, G_0).

2. Let $\alpha = \left(m\ ;\ \left(a \ \begin{smallmatrix} c \\ b \end{smallmatrix} \ d\right)\right) \in G_2$ and $\beta = \left(n\ ;\ \left(d \ \begin{smallmatrix} g \\ f \end{smallmatrix} \ d\right)\right) \in H_2$. Then we have that

$$\begin{aligned} \alpha +_2 \beta -_2 \alpha &= \left(m\ ;\ \left(a \ \begin{smallmatrix} c \\ b \end{smallmatrix} \ d\right)\right) +_2 \left(n\ ;\ \left(d \ \begin{smallmatrix} g \\ f \end{smallmatrix} \ d\right)\right) +_2 \left((-m)^{b^{-1}}\ ;\ \left(d \ \begin{smallmatrix} c^{-1} \\ b^{-1} \end{smallmatrix} \ a\right)\right) \\ &= \left(m^f + n\ ;\ \left(a \ \begin{smallmatrix} cg \\ bf \end{smallmatrix} \ d\right)\right) +_2 \left((-m)^{b^{-1}}\ ;\ \left(d \ \begin{smallmatrix} c^{-1} \\ b^{-1} \end{smallmatrix} \ a\right)\right) \\ &= \left((m^f + n)^{b^{-1}} + (-m)^{b^{-1}}\ ;\ \left(a \ \begin{smallmatrix} cgc^{-1} \\ bfb^{-1} \end{smallmatrix} \ a\right)\right). \end{aligned}$$

By [NCM1] since Q is normal subgroupoid of P we have that $cgc^{-1} \in Q$ and $bfb^{-1} \in Q$ for $f, g \in Q$ and $c, b \in P$. Here we need to prove that $(m^f + n)^{b^{-1}} + (-m)^{b^{-1}} \in N$. If we write n_1 for $-m + m^f \in N$ by [NCM3], then we have that

$$\begin{aligned} (m^f + n)^{b^{-1}} + (-m)^{b^{-1}} &= (m^f + n - m)^{b^{-1}} \\ &= (m - m + m^f - m + m + n - m)^{b^{-1}} \\ &= (m + n_1 - m + m + n - m)^{b^{-1}}. \end{aligned}$$

Here by [NCM2] $m + n_1 - m$, $m + n - m \in N$ and for $b \in Q$, $n, n_1 \in N$ and $m \in M$; $(m^f + n)^{b^{-1}} + (-m)^{b^{-1}} \in N$. We now show that the boundary condition

$$\mu((m^f + n)^{b^{-1}} + (-m)^{b^{-1}}) = bf^{-1}b^{-1}a^{-1}cgc^{-1}a$$

is satisfied. Since $\mu(m) = b^{-1}a^{-1}cd$, $\mu(n) = f^{-1}d^{-1}gd$, we have that

$$\begin{aligned} \mu((m^f + n)^{b^{-1}} + (-m)^{b^{-1}}) &= \mu(m^{(fb^{-1})})\mu(n^{b^{-1}})\mu((-m)^{b^{-1}}) \\ &= (bf^{-1}\mu(m)fb^{-1})(b\mu(n)b^{-1})(b\mu(-m)b^{-1}) && \text{(by [CM1])} \\ &= bf^{-1}b^{-1}a^{-1}cgc^{-1}a. \end{aligned}$$

Hence, we obtain that $\alpha +_2 \beta -_2 \alpha \in H_2$, i.e. $(H_2, H_1, +_2)$ is a nonwide normal subgroupoid of $(G_2, G_1, +_2)$ in the sense of Definition 3.10.

3. This comes from the fact that H is a subdouble groupoid of G.

□

Hence, we can give the definition of a normal subdouble groupoid as follows.

Definition 3.12 A subdouble groupoid $H = (H_2, H_1, H_0)$ of a double groupoid $G = (G_2, G_1, G_0)$ is called a *normal subdouble groupoid* if the following hold:

NDG1. (H_1, H_0) is a normal subgroupoid of (G_1, G_0).

NDG2. The groupoid $(H_2, H_1, +_2)$ is a nonwide normal subgroupoid of $(G_2, G_1, +_2)$.

We use the equivalence of the categories given in Theorem 3.11 to produce the following examples of normal subdouble groupoids.

Example 3.13 Let $(f_2, f_1)\colon (\mu\colon M \to P) \to (\nu\colon N \to Q)$ be a morphism of a crossed module with the kernel $\mu^\star\colon Kerf_2 \to Kerf_1$ as defined in Example 3.3, which is a normal subcrossed module of $\mu\colon M \to P$. Let $H = (H_2, H_1, H_0)$ be the double groupoid corresponding to $\mu^\star\colon Kerf_2 \to Kerf_1$, where $H_0 = P_0$, $H_1 = \operatorname{Ker} f_1$ and H_2 consists of

$$\left(n\ ;\ \begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix} \right)$$

such that $a, b, c, d \in \operatorname{Ker} f_1$ and $n \in \operatorname{Ker} f_2$ with $\mu(n) = b^{-1}a^{-1}cd$. Assume that $G = (G_2, G_1, G_0)$ is the double groupoid corresponding to μ and hence defined by $G_0 = P_0$ and $G_1 = P$, and G_2 consists of

$$\left(m\ ;\ \begin{pmatrix} a & \begin{smallmatrix} c \\ b \end{smallmatrix} & d \end{pmatrix} \right)$$

such that $a, b, c, d \in P$ and $m \in M_1$ with $\mu(m) = b^{-1}a^{-1}cd$. Then by Theorem 3.11 H becomes a normal subdouble groupoid of G.

Example 3.14 Let H be a normal subgroupoid of G. Then 2-shell groupoid $\Box' H$ becomes a normal subdouble groupoid of the double groupoid $\Box' G$.

Theorem 3.15 *Let $H = (H_2, H_1, H_0)$ be a normal subdouble groupoid of a double groupoid $G = (G_2, G_1, G_0)$. Let $\nu\colon N \to Q$ and $\mu\colon M \to P$ be respectively the crossed modules corresponding to H and G. Then the crossed module $\nu\colon N \to Q$ is a normal subcrossed module of the crossed module $\mu\colon M \to P$.*

Proof We must show that the conditions of Definition 3.2 are satisfied:

NCM1. Since $Q = (H_1, H_0)$, $P = (G_1, G_0)$, and by [NDG1] (H_1, H_0) is a normal subgroupoid of (G_1, G_0), Q becomes a normal subgroupoid of P.

NCM2. We need to prove that for $a \in P(p, q)$ and $n \in N(p)$, n^a is in $N(q)$. Here n^a is of the following form.

$$\begin{tikzcd} \cdot \arrow[r, dash, "a^{-1}\nu(n)a"] \arrow[d, dash] \arrow[dr, phantom, "n^a"] & \cdot \arrow[d, dash] \\ \cdot \arrow[r, dash] & \cdot \end{tikzcd}$$

By [NDG1], since (H_1, H_0) is a normal subgroupoid of (G_1, G_0), it implies that $a^{-1}\nu(n)a \in H_1$ and hence $n^a \in N(q)$.

NCM3. We prove that $-m + m^b \in N$ for $b \in Q = H_1$ and $m \in M$. Here the square $-m + m^b$ can be denoted by a square as follows.

$$\begin{tikzcd} \cdot \arrow[r, dash, "\mu(-m)b^{-1}\mu(m)b"] \arrow[d, dash] \arrow[dr, phantom, "-m+m^b"] & \cdot \arrow[d, dash] \\ \cdot \arrow[r, dash] & \cdot \end{tikzcd}$$

Here $b \in Q = H_1$, $\mu(m) \in P = G_1$, and by [NDG1] since (H_1, H_0) is a normal subgroupoid of (G_1, G_0) we have that $\mu(-m)b^{-1}\mu(m) \in Q$ and $\mu(-m)b^{-1}\mu(m)b \in Q$. As a result, we get

$$-m + m^b \in N.$$

Thus, $\nu\colon N \to P$ is a normal subcrossed module of $\mu\colon M \to P$. $\square$

Hence, the following corollary is stated as a result of Theorem 3.11 and Theorem 3.15.

Corollary 3.16 *Let G be a double groupoid and $\mu\colon M \to P$ the crossed module corresponding to G. Then the category of the normal subdouble groupoids of G and the category of normal subcrossed modules of the crossed module $\mu\colon M \to P$ are equivalent.*

Definition 3.17 Let $\nu\colon N \to Q$ be a normal subcrossed module of $\mu\colon M \to P$ such that Q is totally intransitive. Then the double groupoid corresponding to the quotient crossed module $\rho\colon M/N \to P/Q$ by Theorem 3.4 is called a *quotient double groupoid.*

Hence, we can characterize a quotient groupoid as follows: let $H = (H_2, H_1, H_0)$ be a normal subdouble groupoid of $G = (G_2, G_1, G_0)$ such that the groupoid (H_1, H_0) is totally intransitive. Then by Corollary 3.16 we can construct the quotient double groupoid G/H whose points are same as that of G, edges are the morphisms of quotient groupoid G_1/H_1, and squares are in the form of

$$\begin{tikzcd} \cdot \arrow[r, dash, "{[c]}"] \arrow[d, dash, "{[a]}"'] \arrow[dr, phantom, "u"] & \cdot \arrow[d, dash, "{[d]}"] \\ \cdot \arrow[r, dash, "{[b]}"'] & \cdot \end{tikzcd}$$

for $[a], [b], [c], [d] \in G_1/H_1$. The horizontal and vertical compositions are induced by those in G.

The concept of a quotient double groupoid using congruences was also studied in terms of quotient groupoids in [14] (see also [11]).

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