Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 2348 – 2370
© TÜBİTAK
doi:10.3906/mat-1702-79

Olsen-type inequalities for the generalized commutator of multilinear fractional integrals

Xiao YU[1,*], **Shanzhen LU**[2]
[1]Department of Mathematics, Shangrao Normal University, Shangrao, Jiangxi, P.R. China
[2]School of Mathematical Science, Beijing Normal University, Beijing, P.R. China

Received: 17.02.2017 • **Accepted/Published Online:** 28.06.2018 • **Final Version:** 27.09.2018

Abstract: In this paper, we study certain multilinear operators of fractional integral type defined by

$$I_\alpha^{\vec{A}}\vec{f}(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha+\sum_{i=1}^{m}(N_i-1)}}\prod_{i=1}^{m}R_{N_i}(A_i;x,y_i)d\vec{y},$$

where $0<\alpha<mn$ and $R_{N_i}(A_i;x,y_i)=A_i(x)-\sum_{|\gamma|<N_i}\frac{1}{\gamma!}D^\gamma A_i(y_i)(x-y_i)^\gamma$.

For the operator $I_\alpha^{\vec{A}}$, we obtain an Olsen-type inequality on the Morrey space. Our proof is based on a complicated multiple dyadic decomposition.

Key words: Olsen-type inequality, multilinear fractional integral, generalized commutator, dyadic grids, Morrey spaces

1. Introduction

The classical fractional integral

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy\quad(0<\alpha<n)$$

plays important roles in many fields of mathematics. Its most significant feature is that I_α maps $L^p(\mathbb{R}^n)$ continuously into $L^q(\mathbb{R}^n)$, with $1/p-1/q=\frac{\alpha}{n}$ and $1<p<n/\alpha$, through the well known Hardy–Littlewood–Sobolev embedding theorem (see [39]). The commutator I_α^b of I_α is defined in the integral form by

$$I_\alpha^b(f)(x)=[b,I_\alpha](f)(x)=\int_{\mathbb{R}^n}\frac{(b(x)-b(y))}{|x-y|^{n-\alpha}}f(y)dy\quad(0<\alpha<n),$$

where b is a function in the space BMO of bounded mean oscillation. The commutator I_α^b inherits from I_α the embedding relation $I_\alpha^b(L^q(\mathbb{R}^n))\subset L^p(\mathbb{R}^n)$ with $1/p-1/q=\frac{\alpha}{n}$ and $1<p<n/\alpha$. This property was proved by Chanillo [4] in 1982.

*Correspondence: yx2000s@163.com
2010 *AMS Mathematics Subject Classification:* 42B20; 42B25

In another direction, in 1965, Calderón [3] introduced the famous Calderón commutator defined by

$$S_{\Omega}^{A,1}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \frac{A(x)-A(y)}{|x-y|} f(y)dy.$$

Later, Bajsanski and Coifman [2] studied the generalized Calderón commutator as follows:

$$S_{\Omega}^{A,m}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \frac{R_m(A;x,y)}{|x-y|^m} f(y)dy,$$

where $R_m(A,x,y)$ is the difference between a function $A(x)$ and its Taylor polynomial of degree $m-1$ with center y :

$$R_m(A,x,y) = A(x) - \sum_{|\nu|\le m-1} \frac{1}{\nu!} D^{\nu} A(y)(x-y)^{\nu}.$$

Inspired by the above works, Cohen and Gosselin [8] introduced the following generalized commutator:

$$T_{\Omega}^{A}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A,x,y) f(y)dy,$$

where Ω is a homogeneous function of degree 0 and satisfies the integral zero property over the unit sphere $\mathbb{S}^{n-1}$.

Hence, if $m=1$, T_{Ω}^{A} reduces to the commutator of T_{Ω} :

$$[A,T_{\Omega}](f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \left(A(x)-A(y)\right) f(y)dy.$$

When $m \ge 2$, Cohen and Gosselin in [8] showed that T_{Ω}^{A} is a bounded operator on spaces $L^p(\mathbb{R}^n)$ for all $1<p<\infty$, provided that $\Omega \in Lip_1(\mathbb{S}^{n-1})$ and A has derivatives of order $m-1$ in the space $\mathrm{BMO}(\mathbb{R}^n)$. The operator T_{Ω}^{A} later plays an important role in the study of PDEs. For instance, by using the $W^{1,p}$ estimate for the elliptic equation of divergence form with partial BMO coefficients and the L^p boundedness for a generalized commutator of Cohen–Gosselin type (see [27, 45]), Wang and Zhang in [41] obtained a simple proof for Wu's theorem (the reader may see [43] for Wu's theorem).

Motivated by the above background, in [11], Ding introduced the generalized commutator of fractional integral

$$T_{\Omega,\alpha}^{A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+N-1-\alpha}} R_m(A,x,y) f(y)dy,$$

where $0<\alpha<n$. As an expectation from the easy case $N=1$, Ding proved in [11] that if A has derivatives of order $N-1$ in $L^r(\mathbb{R}^n)$, $1<r\le\infty$, and $\Omega \in L^s(\mathbb{S}^{n-1})$ with $s>n/(n-\alpha)$, then $T_{\Omega,\alpha}^{A}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1/p-1/q-1/r=\frac{\alpha}{n}$ and $1<p<n/\alpha$. As a consequence, in [42], Wu and Yang extended Ding's result at the end point $q=\infty$ by replacing $D^{\nu}A \in L^{\infty}(\mathbb{R}^n)$ by $D^{\nu}A \in \mathrm{BMO}(\mathbb{R}^n)$ for all multiindices ν satisfying $|\nu| = m-1$. Moreover, Gürbüz [17] established the BMO estimates for the generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces, respectively.

On the other hand, the theory of multilinear analysis has received extensive studies in the last 3 decades. Among numerous references, in the following we list a few of them about the multilinear fractional integral that are related to the study in this article.

In 1992, Grafakos [16] studied the multilinear fractional integral

$$I_{\alpha,\vec{\theta}}\vec{f}(x)=\int_{\mathbb{R}^n}\frac{1}{|y|^{n-\alpha}}\prod_{i=1}^{m}f_i(x-\theta_i y)dy,$$

where

$$\vec{f}=(f_1,\cdots,f_m)$$

and

$$\vec{\theta}=(\theta_1,\theta_2,...,\theta_m)$$

is a fixed vector with distinct nonzero components.

Another multilinear fractional integral is

$$I_{\alpha,m}\vec{f}(x)=\int_{(\mathbb{R}^n)^m}\frac{1}{|(y_1,...,y_m)|^{mn-\alpha}}\prod_{i=1}^{m}f_i(x-y_i)dy_1...dy_m,$$

where $0<\alpha<nm$. Assume $1/s=-\alpha/mn+\sum\limits_{j=1}^{m}1/t_j>0$ and $1\le t_j<\infty$. Kenig and Stein in [24] proved that if $t_j=1$ for some j then there exists a positive constant C independent of f_i, such that

$$\left\|I_{\alpha,m}\vec{f}\right\|_{L^{s,\infty}(\mathbb{R}^n)}\le C\prod_{i=1}^{m}\|f_i\|_{L^{t_i}(\mathbb{R}^n)},$$

and that if each $t_j>1$ then

$$\left\|I_{\alpha,m}\vec{f}\right\|_{L^{s}(\mathbb{R}^n)}\le C\prod_{i=1}^{m}\|f_i\|_{L^{t_i}(\mathbb{R}^n)}.$$

The commutator theory for the multilinear fractional integral operators can be found in [5, 44], among others. Recently, Mo et al. [28] studied the following generalized commutator of the multilinear fractional integral defined by

$$I_{\alpha}^{\vec{A}}\vec{f}(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha+\sum\limits_{i=1}^{m}(N_i-1)}}\prod_{i=1}^{m}R_{N_i}(A_i;x,y_i)d\vec{y},$$

where $0<\alpha<mn$ and $R_{N_i}(A_i;x,y_i)=A_i(x)-\sum\limits_{|\gamma|<N_i}\frac{1}{\gamma!}D^{\gamma}A(y_i)(x-y_i)^{\gamma}$ with $N_i\in Z^+$.

Mo et al. [28] proved the boundedness of $I_{\alpha}^{\vec{A}}$ with $D^{\gamma_i}A_i\in\dot{\Lambda}_{\beta_i}\,(|\gamma_i|=N_i-1,\ \ 0<\beta_i<1)$ where $\dot{\Lambda}_{\beta_i}$ is the homogeneous Lipschitz space. However, for the case where A_i has derivatives of order N_i-1 in BMO$(\mathbb{R}^n)$, the boundedness of $I_{\alpha}^{\vec{A}}$ is still unknown. In this paper, we will try to study this question in some sense. Before giving the main results of this paper, we introduce another space that plays important roles in PDEs.

Besides the Lebesgue space L^p, the Morrey space $M_q^p(\mathbb{R}^n)$ is another important function space with definition as follows:

Definition 1.1 *([29]) For $0 < q \le p < \infty$, the Morrey space $M_q^p(\mathbb{R}^n)$ is the collection of all measurable functions f whose Morrey space norm is*

$$\|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)} = \sup_{\substack{Q\subset\mathbb{R}^n \\ Q:cubes}} |Q|^{1/p-1/q} \|f\chi_Q\|_{L^q(\mathbb{R}^n)} < \infty.$$

This space was introduced in 1938 by Morrey in [29] in order to study the local behavior of solutions to second-order elliptic partial differential equations. In [12], by means of the theories of singular integrals and linear commutators, the authors established the regularity in Morrey spaces of strong solutions to the Dirichlet problem of nondivergence elliptic equations with VMO coefficients. For more applications of the Morrey spaces to the elliptic partial differential equations, one may see [13] for more details.

Here, we would like to mention that in many research papers, such as in [17, 19], the Morrey space is defined in another way.

Definition 1.2 *Let $0 \le \lambda \le n$ and $1 \le q < \infty$. Then for $f \in L^q_{loc}(\mathbb{R}^n)$ and any cube $B = B(x,r)$, the Morrey space $L^{q,\lambda}(\mathbb{R}^n)$ is defined by*

$$L^{q,\lambda}(\mathbb{R}^n) = \left\{ f \in L^{q,\lambda}(\mathbb{R}^n) : \|f\|_{L^{q,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{q}} \|f\|_{L^q(B(x,r))} < \infty \right\}.$$

Recall that $0 < q \le p < \infty$ and $0 \le \lambda \le n$. By checking the definitions of $\mathcal{M}_q^p(\mathbb{R}^n)$ and $L^{p,\lambda}(\mathbb{R}^n)$, it is easy to see that if we take $\lambda = (1-\frac{q}{p})n \in [0,n]$, then $L^{q,n(1-\frac{q}{p})}(\mathbb{R}^n) = \mathcal{M}_q^p(\mathbb{R}^n)$. Moreover, if we choose $p = \frac{qn}{n-\lambda} \le q$, then $\mathcal{M}_q^{\frac{qn}{n-\lambda}}(\mathbb{R}^n) = L^{q,\lambda}(\mathbb{R}^n)$. Thus, we conclude that $\mathcal{M}_q^p(\mathbb{R}^n)$ is equivalent to $L^{q,\lambda}(\mathbb{R}^n)$.

Many authors studied the boundedness of fractional-type integral operators on Morrey-type spaces. One may see [1, 17, 19, 32] for more details. For example, in [32], Spanne (but published by Peetre) proved the following theorem.

Theorem A *([32]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $1/p - 1/q = \alpha/n$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then we have*

$$\|I_\alpha f\|_{L^{q,\mu}} \le C\|f\|_{L^{p,\lambda}}.$$

Later, Adams proved the following theorem.

Theorem B *([1]) Let $1 < p < \frac{n-\lambda}{\alpha}$, $0 < \lambda < n - \alpha p$, $0 < \alpha < n$, and $1/p - 1/q = \frac{\alpha}{n-\lambda}$. Then we have*

$$\|I_\alpha(f)\|_{L^{q,\mu}} \le C\|f\|_{L^{p,\lambda}}.$$

From [18], we have the following remark.

Remark C *([18]) Let $1/q_1 = 1/p - \alpha/n$ and $1/q_2 = 1/p - \frac{\alpha}{n-\lambda}$. Moreover, we assume $\frac{\mu}{q_1} = \frac{\lambda}{p}$. Then, using the Hölder inequality, there is*

$$\|I_\alpha f\|_{L^{q_1,\mu}} \le \|I_\alpha f\|_{L^{q_2,\lambda}}.$$

Thus, from Theorem B, we see that Theorem B improves Theorem A with $1 < p < \frac{n-\lambda}{\alpha}$.

Recently, Gürbüz and Güzel [19] improved Theorem B to a general case. Before giving the main results of [19], the generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$ is defined:

Definition 1.3 *([19]) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. If $0<p<\infty$, then the generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$ is defined by*

$$L^{p,\varphi}(\mathbb{R}^n) = \left\{ f \in L^{p,\varphi}(\mathbb{R}^n) : \|f\|_{L^{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}|B(x,r)|^{-1/p}\|f\|_{L^p(B(x,r))} < \infty \right\},$$

where $|B(x,r)|$ is the Lebesgue measure of the $B(x,r)$ and $|B(x,r)| = \nu_n r^n$ with $\nu_n = |B(0,1)|$.

Obviously, if we take $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$, then $L^{p,\varphi}(\mathbb{R}^n)$ becomes $L^{p,\lambda}(\mathbb{R}^n)$.

Gürbüz and Güzel [19] proved the following theorem.

Theorem D *([19]) Suppose that $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s \le \infty$, is homogeneous of degree zero. Let $1 < s' < p < q < \infty$, $0 < \alpha < \frac{n}{p}$ and let $\varphi(x,t)$ satisfy the conditions*

$$\sup_{r<t<\infty} ess \inf_{t<\tau<\infty} \varphi(x,\tau)t^n \le C\varphi(x,r)$$

and

$$\int_r^\infty t^\alpha \varphi(x,t)^{1/p}\frac{dt}{t} \le Cr^{-\frac{\alpha p}{q-p}},$$

where C does not depend on $x \in \mathbb{R}^n$ and $r>0$. Let also $I_{\Omega,\alpha}$ be a sublinear operator satisfying

$$\left| I_{\Omega,\alpha}f(x) \le c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy \right|$$

and

$$\left| I_{\Omega,\alpha}\left(f\chi_{B(x_0,r)}\right)(x)\right| \le r^\alpha M_\Omega f(x)$$

holds for any ball $B(x_0,r)$ and the definition of $M_\Omega f(x)$ is

$$M_{\Omega,\alpha}f(x) = \sup_{t>0}|B(x,t)|^{\frac{\alpha}{n}-1}\int_{B(x,t)}|\Omega(x-y)||f(y)|dy.$$

Then we have

$$\|I_{\Omega,\alpha}f\|_{L^{q,\varphi^{1/q}}} \le C\|f\|_{L^{p,\varphi^{1/p}}}.$$

Remark E *Obviously, Theorem E is the essential improvement of Theorem B as the authors only assume $1<p<q<\infty$.*

Nowadays, Theorem B and its variants are called Adams-type inequalities since the inequality plays significant roles in studying the boundedness for the fractional integral and its commutators on the Morrey spaces (see [6, 10, 17, 19, 46] and others). Particularly, recently Sawano et al. obtained the following result.

Theorem F *([36]) Suppose that the indices* $\alpha, p_0, q_0, r_0, p, q, r$ *satisfy*

$$1 < p \le p_0 < \infty,\ 1 < q \le q_0 < \infty, 1 < r \le r_0 < \infty$$

and

$$q > r,\ 1/p_0 > \alpha/n \ge 1/q_0.$$

Also assume

$$r/r_0 = p/p_0,\ 1/p_0 + 1/q_0 - \alpha/n = 1/r_0.$$

Then, for all $f \in M_p^{p_0}(\mathbb{R}^n)$ *and* $g \in M_q^{q_0}(\mathbb{R}^n)$,

$$\|g \cdot I_\alpha(f)\|_{M_r^{r_0}(\mathbb{R}^n)} \le C \|f\|_{M_p^{p_0}(\mathbb{R}^n)} \|g\|_{M_q^{q_0}(\mathbb{R}^n)},$$

where C *is a positive constant independent of* f *and* g. *The above inequality is called an inequality of Olsen type, since it was initially proposed by Olsen in [31], and Olsen found that this inequality plays important roles in the study of the Schrödinger equation. Moreover, the Olsen-type inequality was proved in the case* $n = 3$ *by Conlon and Redondo in [9] essentially. In fact, an analogous inequality on a generalized case was obtained in [36]. Moreover, in [38] the authors obtained an Olsen-type inequality for the commutator* I_α^b *with a quite elegant method of dyadic decomposition. The reader also can see [23] on an Olsen-type inequality on the multilinear fractional integral* $I_{\alpha,m}$. *For more applications of Olsen-type inequalities to PDEs, one may see [14, 15, 37] for details.*

Motivated by the above background, we will give the Olsen-type inequalities of $I_\alpha^{\vec{A}}$ on the Morrey space where A_i has derivatives of order $N_i - 1$ in BMO$(\mathbb{R}^n)$.

Our results can be stated as follows.

Theorem 1.4 *Suppose that there exist real numbers* α, q, p, q_i, p_i $(i = 1, \cdots, m), s$, *and* t *satisfying* $0 < \alpha < mn, 1 < q_i \le p_i < \infty, 1 < q \le p < \infty, 1 < t \le s < \infty$, *and*

$$q > t, 1/p \le \alpha/n < 1/p_1 + \cdots + 1/p_m < 1.$$

Furthermore, we assume that

$$1/s = 1/p + 1/p_1 + \cdots + 1/p_m - \alpha/n, \frac{t}{s} = \frac{q_1}{p_1} = \cdots = \frac{q_m}{p_m}.$$

If A_i *has derivatives of order* $N_i - 1$ *in* $BMO(\mathbb{R}^n)$ *with* $N_i \ge 2$ *and* $N_i \in Z^+$, *then there exists a positive constant* C *independent of* f_i $(i = 1, \cdots, m)$ *and* g, *such that*

$$\|g \cdot I_\alpha^{\vec{A}} \vec{f}\|_{\mathcal{M}_t^s} \le C \prod_{i=1}^m \sum_{|\gamma_i| = N_i - 1} \|D^{\gamma_i} A_i\|_{BMO} \|g\|_{\mathcal{M}_q^p} \prod_{i=1}^m \|f_i\|_{\mathcal{M}_{q_i}^{p_i}}.$$

The method of the proof of Theorem 1.1 is also adapted to the case $q = \infty$ and $g \equiv 1$. We have the following corollary, which is also a new result and has its independent interest.

Corollary 1.5 ***(The Spanne type estimate for $I_\alpha^{\vec{A}}$)*** *Suppose that there exist real numbers α, q_i, p_i $(i = 1, \cdots, m), s$, and t satisfying $0 < \alpha < mn, 1 < q_i \le p_i < \infty, 1 < t \le s < \infty$, and*

$$\alpha/n < 1/p_1 + \cdots + 1/p_m < 1.$$

Furthermore, we assume that

$$1/s = 1/p_1 + \cdots + 1/p_m - \alpha/n, \frac{t}{s} = \frac{q_1}{p_1} = \cdots = \frac{q_m}{p_m}.$$

Then there exists a positive constant C independent of f_i $(i = 1, \cdots, m)$, such that

$$\|I_\alpha^{\vec{A}} \vec{f}\|_{\mathcal{M}_t^s} \le C \prod_{i=1}^m \sum_{|\gamma_i| = N_i - 1} \|D^{\gamma_i} A_i\|_{BMO} \|f_i\|_{\mathcal{M}_{q_i}^{p_i}}.$$

Remark 1.6 *As mentioned in Remarks C and E and comparing Theorems B and D with Corollary 1.2, it is natural to ask whether we can prove the Adams-type estimate for $I_\alpha^{\vec{A}}$ on Morrey space and generalized Morrey space. We will try to solve this problem in our future works.*

Remark 1.7 *If we take $s = t$ and $p_i = q_i$ in Corollary 1.2, we can obtain the boundedness of $I_\alpha^{\vec{A}}$ on the product L^p spaces with A_i having derivatives of order $N_i - 1$ in $BMO(\mathbb{R}^n)$ on product L^p spaces.*

Remark 1.8 *Here we would like to mention that Theorem 1.1 is not an easy consequence of Corollary 1.2 and the Hölder inequality for functions on the Morrey spaces (see (2.1) in [23, p. 1377]). Readers may see [35, 36] for details. In fact, if we want to use Corollary 1.2 and the Hölder inequality for functions on the Morrey spaces to get the Olsen-type inequality for $I_\alpha^{\vec{A}}$, we will have*

$$\|g \cdot I_\alpha^{\vec{A}} \vec{f}\|_{\mathcal{M}_t^s} \le C \prod_{i=1}^m \sum_{|\gamma_i| = N_i - 1} \|D^{\gamma_i} A_i\|_{BMO} \|g\|_{\mathcal{M}_q^p} \prod_{i=1}^m \|f_i\|_{\mathcal{M}_{q_i}^{p_i}},$$

where $\frac{q}{p} = \frac{t}{s} = \frac{q_1}{p_1} = \cdots = \frac{q_m}{p_m}$ and the other conditions are the same as in Theorem 1.1.

Remark 1.9 *Theorem 1.1 is also true if we take $N_i = 1$ $(i = 1, \cdots, m)$. Thus, we get the Olsen-type inequality for the irritated commutators of multilinear fractional integral operator $I_{\alpha, \vec{A}}$, which is defined by*

$$I_{\alpha, \vec{A}} \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{|(x - y_1, \cdots, x - y_m)|^{mn - \alpha}} \prod_{i=1}^m (A_i(x) - A_i(y_i)) d\vec{y},$$

where $A_i \in BMO(\mathbb{R}^n)$. Then we conclude that our results improve [23, Theorem 7.2] and [38, Theorem 1.1].

Remark 1.10 *As far as we know, Theorem 1.1 is also a new result even if we take $m = 1$.*

Remark 1.11 *Some basic ideas of this paper come from [33, 34, 38, 40] by using a decomposition of dyadic cubes. However, the execution of this paper becomes technically more difficult due to the fact that the structures of the multilinear operators and the Cohen–Gosselin-type operators are much more complicated than the classical commutators of fractional integrals.*

Remark 1.12 *Recently, Iida [20, 21] studied the weighted norm inequalities for the (multilinear) Hardy–Littlewood maximal operator and the (multilinear) fractional integral operator with a rough kernel on the weighted Morrey spaces by using the decomposition of dyadic cubes. As the basic ideas of [20, 21] are similar to our paper, it is natural to ask whether we can get the Olsen-type inequalities for two such operators and we will try to answer this question in our future works.*

2. Preliminaries

In this section, we will give some lemmas and definitions that will be useful throughout this paper.

Lemma 2.1 *([8]) Let b be a function on $\mathbb{R}^n$ with mth order derivatives in $L^q_{loc}(\mathbb{R}^n)$ for some $q>n$. Then*

$$|R_m(b;x,y)| \le C_{m,n}|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\gamma b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

Lemma 2.2 *([30]) Let $1\le p<\infty$. Then for any cube Q, there exists a constant $C>0$ such that*

$$\left(\frac{1}{|Q|} \int_Q |b(x)-m_Q(b)|^p dx \right)^{1/p} \le C\|b\|_{BMO}$$

for all $b \in BMO(\mathbb{R}^n)$ where $m_Q(b)$ is defined by

$$m_Q(b) = \frac{1}{|Q|}\int_Q b(x)dx.$$

Next, we introduce some maximal functions (see [26, 30]).

For a cube Q that runs over all cubes containing x, the maximal function M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|dy$$

and the definition of fractional maximal function M_α is

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}}\int_Q |f(y)|dy$$

with $0<\alpha<n$.
Furthermore, for any $p>1$, we denote

$$M^p f(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|}\int_Q |f(y)|^p dy \right)^{1/p}.$$

For $\vec{f}=(f_1,f_2,\cdots,f_m)$, the multilinear maximal function $\mathcal{M}$ is defined by

$$\mathcal{M}\vec{f}(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|}\int_Q |f_i(y_i)|dy_i.$$

Before giving the next lemma, which is the most important in this paper, we introduce the set of all dyadic grids as follows.

A dyadic grid $\mathcal{D}$ is a countable collection of cubes that satisfies the following properties:

(i) $Q \in \mathcal{D} \Rightarrow l(Q) = 2^{-k}$ for some $k \in \mathbb{Z}$.

(ii) For each $k \in \mathbb{Z}$, the set $\{Q \in \mathcal{D} : l(Q) = 2^{-k}\}$ forms a partition of $\mathbb{R}^n$.

(iii) $Q, P \in \mathcal{D} \Rightarrow Q \cap P \in \{P, Q, \emptyset\}$.

One very clear example (see [25]) for this concept is the dyadic grid that is formed by translating and then dilating the unit cube $[0,1)^n$ all over $\mathbb{R}^n$. More precisely, it is formulated as

$$\mathcal{D} = \{2^{-k}([0,1)^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

In practice, we also make extensive use of the following family of dyadic grids:

$$\mathcal{D}^t = \{2^{-k}([0,1)^n + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, t \in \{0, 1/3\}^n.$$

In [25], Lerner proved the following theorem.

Lemma 2.3 *([25]) Given any cube in $\mathbb{R}^n$, there exists a $t \in \{0, 1/3\}^n$ and a cube $Q_t \in \mathcal{D}^t$, such that $Q \subset Q_t$ and $l(Q_t) \le 6l(Q)$.*

Next, let us give a decomposition result about cubes. Suppose that Q_0 is a cube and let f be a function belonging to $L^1_{\text{loc}}(\mathbb{R}^n)$. Then we set

$$\mathcal{D}(Q_0) \equiv \{Q \in \mathcal{D} : Q \subset Q_0\}.$$

Furthermore, we denote that $3Q_0$ is the unique cube concentric to Q_0 and having the volume $3^n|Q_0|$. Denote

$$m_{3Q_0}(|\vec{f}|^p)^{1/p} = \prod_{i=1}^{m} \left(\frac{1}{|3Q_0|} \int_{3Q_0} |f_i(y_i)|^p dy_i \right)^{1/p}.$$

Next, we introduce the sparse family of Calderón–Zygmund cubes. More precisely, for each $k \in \mathbb{Z}^+$,

$$D_k \equiv \bigcup \left\{ Q : Q \in \mathcal{D}(Q_0), m_{3Q}(|\vec{f}|^p)^{1/p} > \gamma_0 A^k \right\},$$

with $\gamma_0 = m_{3Q_0}(|\vec{f}|^p)^{1/p} (p > 1)$ and $A = (2m \cdot 9^n 2^{pn})^m$.

Considering the maximal cubes with respect to inclusion, we may write

$$D_k = \bigcup_j Q_{k,j}, \tag{2.1}$$

where the cubes $\{Q_{k,j}\} \subset \mathcal{D}(Q_0)$ are nonoverlapping. That is, $\{Q_{k,j}\}$ is a family of cubes satisfying

$$\sum_j \chi_{Q_{k,j}} \le \chi_{Q_0} \tag{2.2}$$

for almost everywhere. By the maximality of $Q_{k,j}$, we get

$$\gamma_0 A^k < m_{3Q_{k,j}}(|\vec{f}|^p)^{1/p} < 2^{\frac{mn}{p}} \gamma_0 A^k < 2^{mn} \gamma_0 A^k. \tag{2.3}$$

Moreover, we have the following properties for $Q_{k,j}$:

(iv) For any fixed k, $Q_{k,j}$ are nonoverlapping for different j.

(v) If $k_1 < k_2$, then there exists i such that $Q_{k_2,j} \subset Q_{k_1,i}$ for any $j \in \mathbb{Z}$.

Next, we will use a clever idea proposed by Tanaka in [40] to decompose Q_0 as follows.

Let $E_0 = Q_0 \setminus D_1, E_{k,j} = Q_{k,j} \setminus D_{k+1}$. Then we have the following lemma.

Lemma 2.4 *The set* $\{E_0\} \bigcup \{E_{k,j}\}$ *forms a disjoint family of sets, which decomposes* Q_0, *and satisfies*

$$|Q_0| \leq 2|E_0|, \quad |Q_{k,j}| \leq 2|E_{k,j}|. \tag{2.4}$$

Proof After our paper was finished, we found that Lemma 2.4 for p in a narrow range was essentially proved by Iida [20, p. 175–176]. Here, we still give the main steps to prove Lemma 2.3 for the sake of completeness.

For the case when $p = 1$, this lemma was proved by Iida et al. [22, p. 161]. Here we only prove the case for $p > 1$.

For a fixed $Q_{k,j}$, we denote

$$A_i = \left(\prod_{l=1}^{m} \int_{3Q_{k,j}} |f_l(y_l)|^p dy_l \right)^{-\frac{1}{pm}} \left(\gamma_0 A^{k+1}\right)^{1/m} \left(\int_{3Q_{k,j}} |f_i(y_i)|^p dy_i \right)^{1/p}.$$

Obviously, there is $\prod\limits_{i=1}^{m} A_i = \gamma_0 A^{k+1}$. Then, using (2.3), we have

$$\begin{aligned}
Q_{k,j} \cap D_{k+1} &\subset \left\{x \in Q_{k,j} : \mathcal{M}\left(\chi_{3Q_{k,j}}|f_1|^p, \cdots, \chi_{3Q_{k,j}}|f_m|^p\right) > (\gamma_0 A^{k+1})^p\right\} \\
&\subset \left\{ x \in Q_{k,j} : \prod_{i=1}^{m} M(\chi_{3Q_{k,j}}|f_i|^p)(x) > (\gamma_0 A^{k+1})^p \right\} \\
&\subset \bigcup_{i=1}^{m} \left\{x \in Q_{k,j} : M(\chi_{3Q_{k,j}}|f_i|^p)(x) > A_i^p\right\}.
\end{aligned}$$

By the $L^1 - L^{1,\infty}$ boundedness of M, we get

$$\begin{aligned}
|Q_{k,j} \cap D_{k+1}| &\leq \sum_{i=1}^{m} \left|\left\{x : M(\chi_{3Q_{k,j}}|f_i|^p)(x) > A_i^p\right\}\right| \\
&\leq \sum_{i=1}^{m} \frac{3^n}{A_i^p} \int_{3Q_{k,j}} |f_i(y_i)|^p dy_i = m3^n \left(\frac{1}{\gamma_0 A^{k+1}} \prod_{i=1}^{m} \left(\int_{3Q_{k,j}} |f_i(y_i)|^p dy_i \right)^{1/p} \right)^{\frac{p}{m}}.
\end{aligned}$$

Using (2.3) again, we obtain

$$\begin{aligned}
&|Q_{k,j} \cap D_{k+1}| \leq m3^n \left(\tfrac{1}{\gamma_0 A^{k+1}} m_{3Q_{k,j}}(|\vec{f}|^p)^{1/p} \right)^{p/m} |3Q_{k,j}| \\
&\leq m3^n \left(\tfrac{2^{mn}\gamma_0 A^k}{\gamma_0 A^{k+1}} \right)^{p/m} |3Q_{k,j}| = \tfrac{m3^n 3^n 2^{pn}}{A^{\frac{p}{m}}} |Q_{k,j}| \leq \tfrac{1}{2}|Q_{k,j}|.
\end{aligned} \tag{2.5}$$

Similarly, we have

$$|D_1| \leq \tfrac{1}{2}|Q_0|. \tag{2.6}$$

Obviously, we obtain (2.4) from (2.5) and (2.6). □

Lemma 2.5 *([1]) Let* $0<\alpha<n$*,* $1<q\le p<\infty$*, and* $1<t\le s<\infty$*. Assume* $1/s=1/p-\frac{\alpha}{n}$*,* $\frac{t}{s}=\frac{q}{p}$*. Then there exists a positive constant* C *such that*

$$\|I_\alpha f\|_{\mathcal{M}_t^s}\le C\|f\|_{\mathcal{M}_q^p}.$$

Lemma 2.6 *Suppose that there exists real numbers* t,q,p *satisfying* $1<t<q\le p<\infty$*. Then we have* $\|f^t\|_{\mathcal{M}_{q/t}^{p/t}}^{1/t}=\|f\|_{\mathcal{M}_q^p}$*.*

Proof By the definition of Morrey space, we can easily prove Lemma 2.6 and we omit the details here.

3. Dyadic grids theory for $I_\alpha^{\vec{A}}$

From Lemma 2.3 and the fact $t\le s$, we know that for any cube $Q\subset\mathbb{R}^n$, there is

$$\begin{aligned}&|Q|^{1/s-1/t}\left(\int_Q|g(x)I_\alpha^{\vec{A}}(\vec{f})(x)|^tdx\right)^{1/t}\\&\le 6^n\sum_{\delta=1}^{3^n}|Q_\delta|^{1/s-1/t}\left(\int_{Q_\delta}|g(x)I_\alpha^{\vec{A}}(\vec{f})(x)|^tdx\right)^{1/t},\end{aligned}\tag{3.1}$$

where $Q_\delta\in\mathcal{D}^\delta$, $Q\subset Q_\delta$, and $l(Q_\delta)\le 6l(Q)$. □

Thus, we only need to estimate $|Q_0|^{1/s-1/t}\left(\int_{Q_0}|g(x)I_\alpha^{\vec{A}}(\vec{f})(x)|^tdx\right)^{1/t}$ with $Q_0\in\mathcal{D}^\delta$.

4. Decomposition of the operator $\|g\cdot I_\alpha^{\vec{A}}\vec{f}\|_{\mathcal{M}_t^s}$

To prove Theorem 1.1, for simplicity, we only prove for the case $m=2$ since there is no essential difference for the general case. From (ii) in Section 2, we know that for a fixed δ and each $\nu\in\mathbb{Z}$, the set $\{Q\in\mathcal{D}^\delta: l(Q)=2^{-\nu}\}$ forms a partition of $\mathbb{R}^n$. Moreover, we denote $Q\in\mathcal{D}_\nu^\delta$ with $l(Q)=2^{-\nu}$ and let $3Q$ be made up of 3^n dyadic cubes of equal size and having the same center of Q. Thus, by notations as in Section 2, we decompose $I_\alpha^{\vec{A}}$ as follows:

$$\begin{aligned}I_\alpha^{\vec{A}}\vec{f}(x)&\le\int_{\mathbb{R}^{2n}}\frac{\prod_{i=1}^{2}|f_i(y_i)R_{N_i}(A_i;x,y_i)|}{|(x-y_1,x-y_2)|^{2n-\alpha+N_1-1+N_2-1}}dy_1dy_2\\&\le C\sum_{\nu\in Z}\int_{2^{-\nu-1}<\sum_{i=1}^{2}|x-y_i|\le 2^{-\nu}}\frac{\prod_{i=1}^{2}|f_i(y_i)R_{N_i}(A_i;x,y_i)|}{|(x-y_1,x-y_2)|^{2n-\alpha+N_1-1+N_2-1}}dy_1dy_2\\&=\sum_{\nu\in Z}\sum_{Q\in\mathcal{D}_\nu^\delta}2^{\nu(2n-\alpha)}\chi_Q(x)\int_{2^{-\nu-1}<\sum_{i=1}^{2}|x-y_i|\le 2^{-\nu}}\frac{\prod_{i=1}^{2}|f_i(y_i)R_{N_i}(A_i;x,y_i)|}{|(x-y_1,x-y_2)|^{N_1+N_2-2}}dy_1dy_2.\end{aligned}$$

Now we denote $A_Q^i(x) = A_i(x) - \sum_{|\gamma_i|=N_i-1} m_Q(D^{\gamma_i}(A_i)_Q)x^{\gamma_i}$. It is obvious that

$$R_{N_i}(A_i;x,y_i) = R_{N_i}(A_Q^i;x,y_i).$$

Thus, using Lemma 2.1 and the fact $2^{-\nu-1} < |x-y_i| \le 2^{-\nu}$ with $i=1,2$, we have

$$\begin{aligned}|R_{N_i}(A_Q^i;x,y_i)| &\le |R_{N_i-1}(A_Q^i;x,y_i)| + \sum_{|\gamma_i|=N_i-1} \frac{1}{\gamma_i!}|D^{\gamma_i}A_Q^i(y_i)||x-y_i|^{N_i-1}\\ &\le C|x-y_i|^{N_i-1}\sum_{|\gamma_i|=N_i-1}\left\{\left(\frac{1}{|Q|}\int_Q |D^{\gamma_i}A_Q^i(z)|^q dz\right)^{1/q} + \left|D^{\gamma_i}A_Q^i(x)\right|\right\}\\ &\le C|x-y_i|^{N_i-1}\sum_{|\gamma_i|=N_i-1}\left(\|D^{\gamma_i}A_i\|_{\mathrm{BMO}} + D^{\gamma_i}A_Q^i(y_i)\right).\end{aligned}$$

Then, by a geometric observation, there is $B(x,2^{-\nu}) \subset 3Q$ if $x \in Q \in \mathcal{D}_\nu^\delta$. Thus, we obtain

$$\begin{aligned}I_\alpha^{\vec{A}}\vec{f}(x) \le& \sum_{\nu\in Z}\sum_{Q\in\mathcal{D}_\nu^\delta} 2^{\nu(2n-\alpha)}\chi_Q(x)\\ &\times \int_{(B(x,2^{-\nu}))^2} \frac{\prod_{i=1}^2\left(|f_i(y_i)|||x-y_i|^{N_i-1}\sum_{|\gamma_i|=N_i-1}\left(\|D^{\gamma_i}A_i\|_{\mathrm{BMO}} + D^{\gamma_i}A_Q^i(y_i)\right)|\right)}{|(x-y_1,x-y_2)|^{N_1+N_2-2}}dy_1dy_2\\ &\le \sum_{\nu\in Z}\sum_{Q\in\mathcal{D}_\nu^\delta} 2^{\nu(2n-\alpha)}\chi_Q(x)\int_{(3Q)^2}\prod_{i=1}^2|f_i(y_i)|\prod_{i=1}^2\sum_{|\gamma_i|=N_i-1}\left(\|D^{\gamma_i}A_i\|_{\mathrm{BMO}} + D^{\gamma_i}A_Q^i(y_i)\right)dy_1dy_2.\end{aligned}$$

Next, we define

$$I = g(x)\sum_{\nu\in Z}\sum_{Q\in\mathcal{D}_\nu^\delta} 2^{\nu(2n-\alpha)}\prod_{i=1}^2\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\int_{3Q}|f_i(y_i)|dy_i,$$

$$\begin{aligned}II =& g(x)\sum_{\nu\in Z}\sum_{Q\in\mathcal{D}_\nu^\delta} 2^{\nu(2n-\alpha)}\chi_Q(x)\sum_{|\gamma_1|=N_1-1}\|D^{\gamma_1}A_1\|_{\mathrm{BMO}}\int_{3Q}|f_1(y_1)|dy_1\\ &\times \sum_{|\gamma_2|=N_2-1}\int_{3Q}|f_2(y_2)D^{\gamma_2}A_Q^2(y_2)|dy_2,\end{aligned}$$

$$\begin{aligned}III =& g(x)\sum_{\nu\in Z}\sum_{Q\in\mathcal{D}_\nu^\delta} 2^{\nu(2n-\alpha)}\chi_Q(x)\sum_{|\gamma_2|=N_2-1}\|D^{\gamma_2}A_2\|_{\mathrm{BMO}}\int_{3Q}|f_2(y_2)|dy_2\\ &\times \sum_{|\gamma_1|=N_1-1}\int_{3Q}|f_1(y_1)D^{\gamma_1}A_Q^1(y_1)|dy_1,\end{aligned}$$

and

$$IV = g(x)\sum_{\nu\in Z}\sum_{Q\in\mathcal{D}_\nu^\delta} 2^{\nu(2n-\alpha)}\chi_Q(x)\sum_{|\gamma_1|=N_1-1}\int_{3Q}|f_1(y_1)D^{\gamma_1}A_Q^1(y_1)|dy_1$$
$$\times\sum_{|\gamma_2|=N_2-1}\int_{3Q}|f_2(y_2)D^{\gamma_2}A_Q^2(y_2)|dy_2.$$

By the above estimates and notations, it is easy to get

$$\|g\cdot I_\alpha^{\vec{A}}\vec{f}\|_{\mathcal{M}_t^s} \le \|I\|_{\mathcal{M}_t^s} + \|II\|_{\mathcal{M}_t^s} + \|III\|_{\mathcal{M}_t^s} + \|IV\|_{\mathcal{M}_t^s}.$$

Next, we will give the estimates of $\|I\|_{\mathcal{M}_t^s}, \|II\|_{\mathcal{M}_t^s}, \|III\|_{\mathcal{M}_t^s}$, and $\|IV\|_{\mathcal{M}_t^s}$ respectively.

5. Estimates of $\|I\|_{\mathcal{M}_t^s}$

For I, we have

$$\begin{aligned}
I &\le \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\chi_Q(x)g(x)\sum_{\nu\in Z}\sum_{Q\in\mathcal{D}_\nu^\delta}2^{\nu(n-\alpha)}|Q||Q|\inf_{y_1\in Q}Mf_1(y_1)\inf_{y_2\in Q}Mf_2(y_2)\\
&\le \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}|g(x)|\sum_{\nu\in Z}\int_{2^{-\nu-1}<\sum_{i=1}^{2}|x-y_i|\le 2^{-\nu}}\frac{\prod_{i=1}^{2}Mf_i(y_i)}{|(x-y_1,x-y_2)|^{2n-\alpha}}dy_1dy_2\\
&\le \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}|g(x)|\int_{(\mathbb{R}^n)^2}\frac{\prod_{i=1}^{2}M(Mf_i)(y_i)}{|(x-y_1,x-y_2)|^{2n-\alpha}}dy_1dy_2\\
&= \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}|g(x)|I_{\alpha,2}(\prod_{i=1}^{2}M(Mf_i))(x).
\end{aligned}$$

Now, by [23, Theorem 7.2] and the boundedness of the Hardy–Littlewood maximal function on the Morrey space (see [7]), we obtain

$$\begin{aligned}
\|I\|_{\mathcal{M}_t^s} &\le C\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|g\|_{\mathcal{M}_q^p}\prod_{i=1}^{2}\|M(Mf_i)\|_{\mathcal{M}_{q_i}^{p_i}}\\
&\le C\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|g\|_{\mathcal{M}_q^p}\prod_{i=1}^{2}\|f_i\|_{\mathcal{M}_{q_i}^{p_i}}.
\end{aligned} \tag{3.2}$$

6. Estimates for $\|IV\|_{\mathcal{M}_t^s}$

6.1. Decomposition of IV

Next we give the estimates of $\|IV\|_{\mathcal{M}_t^s}$. For any fixed cube $Q_0 \in \mathcal{D}^\delta$, as $Q \in \mathcal{D}_\nu^\delta$, we may denote IV_1 and IV_2 as follows:

$$IV_1 = g(x) \sum_{\nu \in Z} \sum_{Q \in \mathcal{D}_\nu^\delta, Q \supset Q_0} \chi_Q(x) 2^{\nu(2n-\alpha)} \sum_{|\gamma_1|=N_1-1} \int_{3Q} |f_1(y_1) D^{\gamma_1} A_Q^1(y_1)| dy_1 \times \sum_{|\gamma_2|=N_2-1} \int_{3Q} |f_2(y_2) D^{\gamma_2} A_Q^2(y_2)| dy_2$$

and

$$IV_2 = g(x) \sum_{\nu \in Z} \sum_{Q \in \mathcal{D}_\nu^\delta, Q \subset Q_0} \chi_Q(x) 2^{\nu(2n-\alpha)} \sum_{|\gamma_1|=N_1-1} \int_{3Q} |f_1(y_1) D^{\gamma_1} A_Q^1(y_1)| dy_1 \times \sum_{|\gamma_2|=N_2-1} \int_{3Q} |f_2(y_2) D^{\gamma_2} A_Q^2(y_2)| dy_2.$$

Thus, we may decompose IV as $IV = IV_1 + IV_2$.

6.2. Estimates of $\|IV_1\|_{\mathcal{M}_t^s}$.

For IV_1, let Q_k be the unique cube containing Q_0 and satisfying $|Q_k| = 2^{kn}|Q_0|$. Set $\nu = -\log_2 |Q_k|^{\frac{1}{n}}$. Then we will give the estimates of E_k, where E_k is defined by

$$E_k = |Q_0|^{1/s-1/t} \left\{ \int_{Q_0} \left| 2^{\nu(2n-\alpha)} g(x) \prod_{i=1}^{2} \sum_{|\gamma_i|=N_i-1} \int_{3Q_k} |f_i(y_i)||D^{\gamma_i} A_{Q_k}(y_i)| dy_i \right|^t dx \right\}^{1/t}.$$

By the Hölder inequality, we obtain

$$E_k \leq |Q_0|^{1/s-1/t} \left(\int_{Q_0} |2^{\nu(2n-\alpha)} g(x)|^t dx \right)^{1/t} \prod_{i=1}^{2} \sum_{|\gamma_i|=N_i-1} \int_{3Q_k} |f_i(y_i)||D^{\gamma_i} A_{Q_k}(y_i)| dy_i$$

$$\leq |Q_0|^{1/s-1/t}\left(\int_{Q_0}|2^{\nu(2n-\alpha)}g(x)|^t dx\right)^{1/t}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\left(\int_{3Q_k}|f_i(y_i)|^{q_i}dy_i\right)^{1/q_i}$$

$$\times\left(\int_{3Q_k}|D^{\gamma_i}A_{Q_k}(y_i)|^{\frac{q_i}{q_i-1}}dy_i\right)^{1-1/q_i}$$

$$= |Q_0|^{1/s-1/t}\left(\int_{Q_0}|2^{\nu(2n-\alpha)}g(x)|^t dx\right)^{1/t}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\left(\int_{3Q_k}|f_i(y_i)|^{q_i}dy_i\right)^{1/q_i}$$

$$\times\left(\int_{3Q_k}|D^{\gamma_i}A_i(y_i)-m_{Q_k}(D^{\gamma_i}A_i)|^{\frac{q_i}{q_i-1}}dy_i\right)^{1-1/q_i}$$

$$\leq C|Q_0|^{1/s-1/t}\left(\int_{Q_0}|2^{\nu(2n-\alpha)}g(x)|^t dx\right)^{1/t}\prod_{i=1}^{2}\left(\int_{3Q_k}|f_i(y_i)|^{q_i}dy_i\right)^{1/q_i}$$

$$\times\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}|Q_k|^{1-1/q_i}$$

$$\leq \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}|3Q_k|^{1/p_i-1/q_i}\left(\int_{3Q_k}|f_i(y_i)|^{q_i}dy_i\right)^{1/q_i}|3Q_k|^{1/q_i-1/p_i}$$

$$\times|Q_k|^{1-1/q_i}2^{\nu(2n-\alpha)}|Q_0|^{1/p-1/q}\left(\int_{Q_0}|g(x)|^q dx\right)^{1/q}|Q_0|^{1/s-1/q}|Q_0|^{1/q-1/p}$$

$$\leq C\|g\|_{\mathcal{M}^p_q}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}}$$

$$\times|3Q_k|^{1-1/p_1}|3Q_k|^{1-1/p_2}2^{\nu(2n-\alpha)}|Q_0|^{1/s-1/p}.$$

As $2^{\nu(2n-\alpha)}=\left(2^{-\log_2|Q_k|^{\frac{1}{n}}}\right)^{2n-\alpha}=|Q_k|^{-\frac{1}{n}(2n-\alpha)}=|Q_k|^{\frac{\alpha}{n}-2}$, we obtain

$$|3Q_k|^{1-1/p_1}|3Q_k|^{1-1/p_2}2^{\nu(2n-\alpha)}|Q_0|^{1/s-1/p}=2^{kn(\frac{\alpha}{n}-1/p_1-1/p_2)}.$$

Recall that Q_k is the unique cube containing Q_0 and satisfying $|Q_k|=2^{kn}|Q_0|$. By the condition that $\frac{\alpha}{n}<1/p_1+1/p_2$ and the definitions of E_k and IV_1, we have

$$|Q_0|^{1/s-1/t}\left(\int_{Q_0}|IV_1|^t dx\right)^{1/t}\leq C\|g\|_{\mathcal{M}^p_q}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}},$$

which implies

$$\|IV_1\|_{\mathcal{M}^s_t}\leq C\|g\|_{\mathcal{M}^p_q}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}}. \tag{3.3}$$

6.3. Estimates of $\|IV_2\|_{\mathcal{M}^s_t}$.

Finally, we will give the estimates of $\|IV_2\|_{\mathcal{M}^s_t}$. We decompose this procession into five parts.

6.3.1. Duality theory for IV_2.

As $t>1$, we may choose a function $\omega \in L^{t'}(Q_0)$, such that

$$\left(\int_{Q_0} |IV_2|^t dt\right)^{1/t} \le 2\int_{Q_0} |IV_2|\omega(x)dx. \tag{3.4}$$

6.3.2. Decomposition for $\int_{Q_0} |IV_2|\omega(x)dx$.

Now we recall some notations in Section 2. For any $p>1$, we set

$$\mathcal{D}_0^\delta(Q_0) \equiv \{Q \in \mathcal{D}^\delta(Q_0) : m_{3Q}(|\vec{f}|^p)^{1/p} \le \gamma_0 A\}$$

and

$$\mathcal{D}_{k,j}^\delta(Q_0) \equiv \{Q \in \mathcal{D}^\delta(Q_0) : Q \subset Q_{k,j}, \gamma_0 A^k < m_{3Q}(|\vec{f}|^p)^{1/p} \le \gamma_0 A^{k+1}\},$$

where γ_0 and A are the same as in Section 2 and $\mathcal{D}^\delta(Q_0) \equiv \{Q \in \mathcal{D}^\delta : Q \subset Q_0\}$. Thus, we have

$$\mathcal{D}^\delta(Q_0) = \mathcal{D}_0^\delta(Q_0) \cup \bigcup_{k,j} \mathcal{D}_{k,j}^\delta(Q_0).$$

We obtain

$$\begin{aligned}
\int_{Q_0} |IV_2|\omega(x)dx &= \sum_{Q\in\mathcal{D}_0^\delta(Q_0)} 2^{\nu(2n-\alpha)} \int_Q g(x)\omega(x)dx \\
&\quad \times \prod_{i=1}^{2} \sum_{|\gamma_i|=N_i-1} \int_{3Q} |f_i(y_i)||D^{\gamma_i}A_Q^i(y_i)|dy_i \\
&\quad + \sum_{k,j} \sum_{Q\in\mathcal{D}_{k,j}^\delta(Q_0)} 2^{\nu(2n-\alpha)} \int_Q g(x)\omega(x)dx \\
&\quad \times \prod_{i=1}^{2} \sum_{|\gamma_i|=N_i-1} \int_{3Q} |f_i(y_i)||D^{\gamma_i}A_Q^i(y_i)|dy_i \\
&= A + B.
\end{aligned} \tag{3.5}$$

6.3.3. Estimates of B.

To estimate B, first we assume that ℓ is a positive number slightly larger than 1. Then, using the Hölder inequality and Lemma 2.2, we have

$$\begin{aligned}
&\prod_{i=1}^{2} \int_{3Q} |f_i(y_i)||D^{\gamma_i}A_Q^i(y_i)|dy_i \\
&\le \prod_{i=1}^{2} \int_{3Q} |f_i(y_i)||D^{\gamma_i}A_i(y_i) - m_Q(D^{\gamma_i}A_i)|dy_i \\
&\le \prod_{i=1}^{2} \left(\int_{3Q} |f_i(y_i)|^\ell dy_i\right)^{1/\ell} \left(\int_{3Q} |D^{\gamma_i}A_i(y_i) - m_Q(D^{\gamma_i}A_i)|^{\frac{\ell}{\ell-1}} dy_i\right)^{1-1/\ell} \\
&\le C\prod_{i=1}^{2} \left[\left(\frac{1}{|3Q|}\int_{3Q} |f_i(y_i)|^\ell dy_i\right)^{1/\ell} |3Q|^{1/\ell+1-1/\ell} \|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\right] \\
&= C\prod_{i=1}^{2} \|D^{\gamma_i}A_i\|_{\mathrm{BMO}} m_{3Q}(|\vec{f}|^\ell)^{1/\ell}|Q|^2.
\end{aligned}$$

Moreover, by a simple computation, we have the following facts:

$$\int_Q g(x)\omega(x)dx = \frac{|Q|}{|Q|}\int_Q g(x)\omega(x)dx \le \int_Q M(g\omega)(x)dx.$$

Thus, by the above estimates, Lemma 2.4, inequality (2.3), the definition of $\mathcal{D}_{k,j}$, and the fact that $\alpha/n < 1$, we obtain

$$\begin{aligned}
B &= \sum_{Q\in\mathcal{D}^{\delta}_{k,j}(Q_0)} 2^{\nu(2n-\alpha)}\int_Q g(x)\omega(x)dx \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\int_{3Q}|f_i(y_i)||D^{\gamma_i}A^i_Q(y_i)|dy_i\\
&\le \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}|Q|^{\frac{\alpha}{n}}\int_Q M(g\omega)(x)dx\, m_{3Q}(|\vec{f}|^{\ell})^{1/\ell}\\
&\le C\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}|Q_{k,j}|^{\frac{\alpha}{n}}\int_{Q_{k,j}} M(g\omega)(x)dx\, m_{3Q_{k,j}}(|\vec{f}|^{\ell})^{1/\ell}\\
&\le C\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}|Q_{k,j}|^{\frac{\alpha}{n}} m_{Q_{k,j}}[M(g\omega)] m_{3Q_{k,j}}(|\vec{f}|^{\ell})^{1/\ell}|Q_{k,j}|\\
&\le C\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}|Q_{k,j}|^{\frac{\alpha}{n}} m_{Q_{k,j}}[M(g\omega)] m_{3Q_{k,j}}(|\vec{f}|^{\ell})^{1/\ell}|E_{k,j}|.
\end{aligned}$$

By the Hölder inequality, there exists a real number θ satisfying $1 < t < \theta < q$, such that

$$M(g\omega) \le M^{\theta'}\omega\cdot M^{\theta}g.$$

Thus, we can conclude that

$$m_{Q_{k,j}}[M(g\omega)] \le \left(m_{Q_{k,j}}((M^{\theta}\omega)^{q'})\right)^{1/q'}\left(m_{Q_{k,j}}((M^{\theta}\omega)^{q})\right)^{1/q}.$$

Now we give the following estimates, which will be used later:

$$\begin{aligned}
&|Q_{k,j}|^{1/p}\left(m_{Q_{k,j}}((M^{\theta}\omega)^{q})\right)^{1/q}\\
&= |Q_{k,j}|^{1/p}\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}\left|M^{\theta}g(x)\right|^{q}dx\right)^{1/q}\\
&= \left(|Q_{k,j}|^{\frac{\theta}{p}}\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}M(|g|^{\theta})(x)^{q/\theta}dx\right)^{\theta/q}\right)^{1/\theta}\\
&\le C\left(|Q_{k,j}|^{\frac{\theta}{pq}}\int_{Q_{k,j}}|g(x)|^{\theta\frac{q}{\theta}}dx\right)^{\frac{1}{\theta}\frac{\theta}{q}} = |Q_{k,j}|^{1/p-1/q}\left(\int_{Q_{k,j}}|g(x)|^{q}dx\right)^{1/q} \le C\|g\|_{\mathcal{M}^p_q},
\end{aligned}$$

where we have used the boundedness of the Hardy–Littlewood maximal function on the Morrey space in the third inequality (see [7]).

Thus, we get

$$\begin{aligned} B &= \sum_{Q\in\mathcal{D}^{\delta}_{k,j}(Q_0)} 2^{\nu(2n-\alpha)}\int_Q g(x)\omega(x)dx \\ &\times \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\int_{3Q}|f_i(y_i)||D^{\gamma_i}A^i_Q(y_i)|dy_i \\ &\le C\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|g\|_{\mathcal{M}^p_q}|Q_{k,j}|^{\frac{\alpha}{n}-\frac{1}{p}}(m_{3Q_{k,j}}(|\vec{f}|^{\ell}))^{1/\ell} \\ &\times\left(m_{Q_{k,j}}((M^{\theta'}\omega)^q)\right)^{1/q'}|E_{k,j}|. \end{aligned} \tag{3.6}$$

6.3.4. Estimates of A.

Similarly, we have

$$\begin{aligned} A &= \sum_{Q\in\mathcal{D}^{\delta}_{0}(Q_0)} 2^{\nu(2n-\alpha)}\int_Q g(x)\omega(x)dx \\ &\times \prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\int_{3Q}|f_i(y_i)||D^{\gamma_i}A^i_Q(y_i)|dy_i \\ &\le C\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|g\|_{\mathcal{M}^p_q}|Q_0|^{\frac{\alpha}{n}-\frac{1}{p}}(m_{3Q_0}(|\vec{f}|^{\ell}))^{1/\ell} \\ &\times\left(m_{Q_0}((M^{\theta'}\omega)^q)\right)^{1/q'}|E_0|. \end{aligned} \tag{3.7}$$

6.3.5. Estimates of $|Q_0|^{1/s-1/t}\int_{Q_0}|IV_2|\omega(x)dx$.

Combining the estimates of (3.5)–(3.7) and using the fact that $\{E_0\}\bigcup\{E_{k,j}\}$ forms a disjoint family of decompositions for Q_0, we have

$$\begin{aligned} &|Q_0|^{1/s-1/t}\int_{Q_0}|IV_2|\omega(x)dx \\ &\le C|Q_0|^{1/s-1/t}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|g\|_{\mathcal{M}^p_q}\int_{Q_0}M^{q'}(M^{\theta'}\omega)(x)\prod_{i=1}^{2}(M_{\beta_i}(|f_i|^{\ell})(x)^{1/\ell}dx \\ &\le C|Q_0|^{1/s-1/t}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|g\|_{\mathcal{M}^p_q}\left(\int_{Q_0}M^{q'}(M^{\theta'}\omega)(x)^{t'}dx\right)^{1/t'} \\ &\times\left(\int_{Q_0}\prod_{i=1}^{2}(M_{\beta_i}(|f_i|^{\ell})(x)^{t/\ell}dx\right)^{1/t}, \end{aligned}$$

where M_{β_i} denotes the fractional maximal function and $\sum_{i=1}^{2}\beta_i=\ell\alpha-\frac{n\ell}{p}>0$.

By the fact that $\frac{t'}{\theta'}>1$ and $\frac{t'}{q'}>1$, we can easily get $\left(\int_{Q_0}M^{q'}(M^{\theta'}\omega)(x)^{t'}dx\right)^{1/t'}\le C$ and it remains to give the estimate of $|Q_0|^{1/s-1/t}\left(\int_{Q_0}\prod_{i=1}^{2}(M_{\beta_i}(|f_i|^{\ell})(x)^{t/\ell}dx\right)^{\frac{1}{t}}$.

Recall that ℓ is slightly larger than 1. Then by Lemmas 2.5–2.6, the fact that $M_{\beta_i}(|f_i|^{\ell})(x)\le$

$CI_{\beta_i}(|f_i|^\ell)(x)$, the conditions of Theorem 1.1, and the Hölder inequality for functions on the Morrey space (see (2.1) in [23, p. 1377]), we obtain

$$\begin{aligned}
&|Q_0|^{1/s-1/t}\left(\int_{Q_0}\prod_{i=1}^{2}(M_{\beta_i}(|f_i|^\ell)(x)^{t/\ell}dx\right)^{\frac{1}{t}}\\
&\leq \sup_{Q_0}\left(|Q_0|^{\frac{1}{s/\ell}-\frac{1}{t/\ell}}\left(\int_{Q_0}\prod_{i=1}^{2}I_{\beta_i}(|f_i|^\ell)(x)^{t/\ell}dx\right)^{\frac{\ell}{t}}\right)^{1/\ell}\\
&=\left\|\prod_{i=1}^{2}I_{\beta_i}(|f_i|^\ell)\right\|^{1/\ell}_{\mathcal{M}^{\frac{s}{\ell}}_{\frac{t}{\ell}}} \leq \|I_{\beta_1}(|f_1|^\ell)\|_{\mathcal{M}^{\mu_1/\ell}_{\nu_1/\ell}}\|I_{\beta_2}(|f_2|^\ell)\|_{\mathcal{M}^{\mu_2/\ell}_{\nu_2/\ell}}\\
&\leq C\|f_1^\ell\|^{1/\ell}_{\mathcal{M}^{p_1/\ell}_{q_1/\ell}}\|f_2^\ell\|^{1/\ell}_{\mathcal{M}^{p_2/\ell}_{q_2/\ell}} = C\prod_{i=1}^{2}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}},
\end{aligned}$$

where $\frac{1}{\mu_i/\ell}=\frac{1}{p_i/\ell}-\beta_i/n$ and $\frac{s}{t}=\frac{p_i}{q_i}=\frac{\mu_i}{\nu_i}\geq 1$ with $i=1,2$.
Combining the above estimates, we get

$$|Q_0|^{1/s-1/t}\int_{Q_0}|IV_2|\omega(x)dx \leq C\|g\|_{\mathcal{M}^p_q}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}},$$

which implies

$$\|IV_2\|_{\mathcal{M}^s_t} \leq C\|g\|_{\mathcal{M}^p_q}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}}. \tag{3.8}$$

Thus, by (3.3) and (3.8), we conclude that

$$\|IV\|_{\mathcal{M}^s_t} \leq C\|g\|_{\mathcal{M}^p_q}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}}. \tag{3.9}$$

7. Estimates of $\|II\|_{\mathcal{M}^s_t}$ and $\|III\|_{\mathcal{M}^s_t}$

Now it remains to give the estimates of $\|II\|_{\mathcal{M}^s_t}$ and $\|III\|_{\mathcal{M}^s_t}$, respectively. Using similar arguments as in the estimates of $\|I\|_{\mathcal{M}^s_t}$ and $\|III\|_{\mathcal{M}^s_t}$, we can easily get

$$\|II\|_{\mathcal{M}^s_t} \leq C\|g\|_{\mathcal{M}^p_q}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}} \tag{3.10}$$

and

$$\|III\|_{\mathcal{M}^s_t} \leq C\|g\|_{\mathcal{M}^p_q}\prod_{i=1}^{2}\sum_{|\gamma_i|=N_i-1}\|D^{\gamma_i}A_i\|_{\mathrm{BMO}}\|f_i\|_{\mathcal{M}^{p_i}_{q_i}}. \tag{3.11}$$

8. Proof of Theorem 1.1

Recalling (3.1) and combining the estimates of (3.2) and (3.9)–(3.11), we finish the proof of Theorem 1.1.

9. A final remark

In [14], Gala et al. proved the following Olsen-type inequality.

Theorem 9.1 *([14]) Let $0<\alpha<\frac{3}{2}$. Then for $n=3$ and $p>1$, there is*

$$\|g\cdot I_\alpha(f)\|_{L^2(\mathbb{R}^3)}\le C\|f\|_{L^2(\mathbb{R}^3)}\|g\|_{\mathcal{M}^{3/\alpha}_{L^2 log^p L(\mathbb{R}^3)}}.$$

Here $\mathcal{M}^{3/\alpha}_{L^2 log^p L(\mathbb{R}^3)}$ denotes the Orlicz–Morrey space and one may find its definition from [14, p. 1322].

Using Theorem 9.1, Gala et al. [14] improved the known regularity criterion of the weak solution for the magneto-micropolar fluid equations in the Orlicz–Morrey space $\mathcal{M}^{3/\alpha}_{L^2\log^p L(\mathbb{R}^3)}$ ([14, Theorem 1.3]).

Moreover, Gala et al. [15] also showed that Theorem 9.1 can also be applied to the 3D incompressible magneto-hydrodynamic (MHD) equations and they established Serrin's uniqueness result of the Leray weak solution for the 3D incompressible MHD equations in Orlicz–Morrey spaces ([15, Theorem 3.1]).

As the variation problem plays important roles in the study of Schrödinger equations, Sawano et al. [37] used Theorem F to solve a problem related to the variation problem (see [37, Theorem 6.1]). Moreover, Sawano et al. [37] also used Theorem F to improve the famous Sobolev–Hardy inequalities (see [37, Theorem 6.3]).

Thus, it is natural to ask the whether the multilinear version of the Olsen-type inequalities can be applied to PDEs and we will consider this problem in our future works.

10. Appendix

In this appendix, we recall some definitions of Orlicz–Morrey spaces, nondivergence elliptic equations, magneto-micropolar fluid equations, 3D incompressible MHD equations, and the Sobolev–Hardy inequality from [12, 14, 15, 37].

Definition G *([12], Dirichlet problem on the second-order elliptic equation in nondivergence form)*

$$\begin{cases} Lu\equiv \sum_{i,j=1}^n a_{i,j}(x)u_{x_i,x_j}=f & a.e. \;\; in \quad \Omega \\ u=0 & on \quad \partial\Omega \end{cases}$$

where $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$; Ω is a bounded domain $\mathcal{C}^{1,1}$ of $\mathbb{R}^n$; the coefficients $\{a_{i,j}\}$ of L are symmetric and uniformly elliptic, i.e. for some $\nu\ge 1$ and every $\xi\in\mathbb{R}^n$,

$$a_{i,j}(x)=a_{j,i}(x) \qquad and \qquad \nu^{-1}|\xi|^2\le\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j\le\nu|\xi|^2$$

with a.e. $x\in\Omega$.

Definition H *([14], Orlicz–Morrey space) For $p\in\mathbb{R}$ and $1<\mu<\nu<\infty$, the Orlicz–Morrey space $\mathcal{M}^\nu_{L^\mu log^p L}(\mathbb{R}^n)$ is defined by*

$$\mathcal{M}^\nu_{L^\mu log^p L}(\mathbb{R}^n)=\left\{f\in\mathcal{M}^\nu_{L^\mu log^p L(\mathbb{R}^n)}:\|f\|_{\mathcal{M}^\nu_{L^\mu log^p L}(\mathbb{R}^n)}:=\sup_{r>0,x\in\mathbb{R}^n} r^{n/\nu}\|f\|_{B(x,r)L^\mu log^p L}<\infty\right\},$$

where $\|f\|_{B(x,r)L^{\mu}log^{p}L}$ *denotes the* $t^{\mu}log^{p}(3+t)$ *average given by*

$$\|f\|_{B(x,r)L^{\mu}log^{p}L} := \inf\left\{\lambda>0 : \frac{1}{|B(x,r)|}\int_{B(x,r)}\left(\frac{|f(x)|}{\lambda}\right)^{\mu} log\left(3+\frac{|f(x)|}{\lambda}\right)^{p} dx \le 1\right\}.$$

Definition I *([14], Three-dimensional magneto-micropolar fluid equations) Let* $u=(u_1(x,t), u_2(x,t), u_3(x,t))$ *the velocity of the fluids at a point* $x\in\mathbb{R}^3$, $t\in[0,T)$. *The functions* $\omega=(\omega_1(x,t),\omega_2(x,t),\omega_3(x,t))$, $b=(b_1(x,t),b_2(x,t)b_3(x,t))$, *and* $p=p(x,t)$ *denote respectively the microrotational velocity, the magnetic field, and the hydrostatic pressure. Then the three-dimensional magneto-micropolar fluid equations can be stated as*

$$\begin{cases} \partial_t u-(u\cdot\nabla)u-(\mu+\chi)\Delta u-(b\cdot\nabla)b+\nabla(p+b^2)-\chi\nabla\times\omega=0,\\ \partial_t\omega-\gamma\Delta\omega-k\nabla div\omega+2k\omega+(u\cdot\nabla)\omega-\chi\nabla\times u=0,\\ \partial_t b-\nu\Delta b+(u\cdot\nabla)b-(b\cdot\nabla)u=0,\\ divu=divb=0,\\ u(x,0)=u_0(x), b(x,0)=b_0(x), \omega(x,0)=\omega_0(x), \end{cases}$$

where u_0, ω_0, *and* b_0 *are the prescribed initial data for the velocity and angular velocity and magnetic field such that* u_0 *and* b_0 *are divergence-free;* $divu_0=0$ *and* $divb_0=0$. *The constant* μ *is the kinematic viscosity,* χ *denotes the vortex viscosity, and* k *and* γ *are spin viscosities.*

Definition J *([15], 3D incompressible MHD equations)*

$$\begin{cases} \partial_t u-\Delta u+(u\cdot\nabla)u+\nabla p+\frac{1}{2}\nabla|b|^2-(b\cdot\nabla)b=0,\\ \partial_t b-\Delta b+(u\cdot\nabla)b-(b\cdot\nabla)u=0,\\ div(u)=div(b)=0,\\ u(\cdot,0)=u_0, \quad b(\cdot,0)=b_0, \end{cases}$$

where $u=u(x,t)$ *is the velocity field,* $b\in\mathbb{R}^3$ *is the magnetic field, and* $p=p(x,t)$ *is the scalar pressure while* u_0 *and* b_0 *are given initial velocity and initial magnetic field with* $divu_0=divb_0=0$ *in the sense of distribution.*

Definition K *([37], Sobolev–Hardy inequality) Let* $0\le s\le 2$. *Then we have*

$$\|u\|_{L^{\frac{2n-2s}{n-2}}(|x|^{-s}dx)} \le C\|\nabla u\|_{L^2}.$$

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China under Grant Nos. 11561057, 11471309, 11771195, and 11761057 and the Natural Science Foundation of Jiangxi Province under Grant No. 20151BAB211002. The first author would like to thank Professor Y Sawano for giving his publication [38] to him. Furthermore, the first author would like to express his gratitude to Professor D Fan for his kind remarks, which made this paper more readable. Finally, both authors would like to express their gratitude to the anonymous referees for valuable suggestions that improved the quality of this paper.

References

[1] Adams R. A note on Riesz potential. Duke Math J 1975; 42: 765-778.

[2] Bajsanski B, Coifman R. On singular integrals. In: Proceedings of Symposia in Pure Mathematics. Providence, RI, USA: American Mathematical Society, 1966, pp. 1-17.

[3] Calderón AP. Commutators of singular integral operators. P Natl Acad Sci USA 1965; 53: 1092-1099.

[4] Chanillo S. A note on commutators. Indiana U Math J 1982; 31: 7-16.

[5] Chen S, Wu H. Multiple weighted estimates for commutators of multilinear fractional integral operators. Sci China Math 2013; 56: 1-18.

[6] Chen Y, Ding Y, Wang X. A characterization of the boundedness for the commutator of fractional integral on Morrey space. Sci Sin Math 2012; 42: 879-886 (in Chinese).

[7] Chiarenza F, Frasca F. Morrey spaces and Hardy-Littlewood maximal function. Rend Math Appl 1987; 7: 273-279.

[8] Cohen J, Gosselin J. A BMO estimate for multilinear singular integrals. Illinois J Math 1986; 30: 445-464.

[9] Conlon J, Redondo J. Estimates on the solution of an elliptic equation related to Brownian motion with drift. Rev Mat Iberoam 1995; 11: 1-65.

[10] Ding Y. A characterization of BMO via commutators for some operators. Northeast Math J 1997; 13: 422-432.

[11] Ding Y. A note on multilinear fractional integrals with rough kernel. Adv Math (China) 2001; 3: 238-246.

[12] Fan D, Lu S, Yang D. Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients. Georgian Math J 1998; 5: 425-440.

[13] Fazio G, Hakim D, Sawano Y. Elliptic equations with discontinuous coefficients in generalized Morrey spaces. Eur J Math 2017; 3: 728-762.

[14] Gala S, Sawano Y, Tanaka H. A new Beale-Kato-Majda criteria for the 3D magneto-micropolar fluid equations in the Orlicz-Morrey space. Math Methods Appl Sci 2012; 35: 1321-1334.

[15] Gala S, Sawano Y, Tanaka H. On the uniqueness of weak solutions of the 3D MHD equations in the Orlicz-Morrey space. Appl Anal 2013; 92: 776-783.

[16] Grafakos L. On multilinear fractional integrals. Stud Math 1992; 102: 49-56.

[17] Gürbüz F. Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces. Canad Math Bull 2017; 60: 131-145.

[18] Gürbüz F. Multi-sublinear operators generated by multilinear fractional integral operators and commutators on the product generalized local Morrey spaces. Adv Math (China) (in press).

[19] Gürbüz F, Güzel G. A characterization for the Adams type boundedness of sublinear operators with rough kernels generated by fractional integrals and their commutators on generalized Morrey spaces. International Journal of Applied Mathematics and Statistics 2018; 57: 72-82.

[20] Iida T. Weighted norm inequalities on Morrey spaces for linear and multilinear fractional integrals with homogeneous kernels. Taiwan J Math 2014; 18: 147-185.

[21] Iida T. The boundedness of the Hardy-Littlewood maximal operator and multilinear maximal operator in weighted Morrey type spaces. J Funct Spaces 2014; 2014: 648251.

[22] Iida T, Sato E, Sawano Y, Tanaka H. Weighted norm inequalities for multilinear fractional operators on Morrey spaces. Stud Math 2011; 205: 139-170.

[23] Iida T, Sato E, Sawano Y, Tanaka H. Multilinear fractional integrals on Morrey spaces. Acta Math Sin (Engl Ser) 2012; 28: 1375-1384.

[24] Kenig CE, Stein EM. Multilinear estimates and functional integration. Math Res Lett 199; 6: 1-15.

[25] Lerner AK. On an estimate of Calderón-Zygmund operators by dyadic positive operators. J Anal Math 2013; 121: 141-161.

[26] Lerner AK, Ombrosi S, Pérez C, Torres R, Trujillo-Gonzales R. New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory. Adv Math 2009; 220: 1222-1264.

[27] Lu S, Yan D. L^p-boundedness of multilinear oscillatory singular integrals with Calderön-Zygmund kernel. Sci China (Ser A) 2002; 45: 196-213.

[28] Mo H, Yu D, Zhou H. Generalized higher commutators generated by the multilinear fractional integrals and Lipschitz functions. Turk J Math 2014; 38: 851-861.

[29] Morrey CB. On the solutions of quasi-linear elliptic partial differential equations. T Am Math Soc 1938; 43: 126-166.

[30] Muckenhoupt B, Wheeden, R. Weighted bounded mean oscillation and the Hilbert transform. Stud Math 1976; 54: 221-237.

[31] Olsen P. Fractional integration, Morrey spaces and Schrödinger equation. Commun Part Diff Eq 1995; 20: 2005-2055.

[32] Peetre J. On the theory of $\mathcal{L}_{p,\lambda}$. J Funct Anal 1969; 4: 71-87.

[33] Pérez C. Two weighted inequalities for potential and fractional type maximal operators. Indiana U Math J 1994; 43: 1-28.

[34] Pérez C. Sharp L^p-weighted Sobolev inequalities. Ann I Fourier 1995; 45: 809-824.

[35] Sawano Y, Sugano S, Tanaka H. A note on generalized fractional integral on generalized Morrey spaces. Bound Value Probl 2009; 2009: 835865.

[36] Sawano Y, Sugano S, Tanaka H. Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces. T Am Math Soc 2011; 363: 6481-6503.

[37] Sawano Y, Sugano S, Tanaka H. Olsen's inequality and its applications to Schrödinger equations. In: Harmonic Analysis and Nonlinear Partial Differential Equations. Kyoto University, Japan: RIMS Kôkyûroku Bessatsu, 2011, pp. 51-80.

[38] Sawano Y, Sugano S, Tanaka H. A bilinear estimate for commutators of fractional integral operators. In: Harmonic Analysis and Nonlinear Partial Differential Equations. Kyoto University, Japan: RIMS Kôkyûroku Bessatsu, 2013, pp. 155-170.

[39] Stein EM. Singular Integrals and Differentiability Properties of Functions. Princeton, NJ, USA: Princeton University Press, 1970.

[40] Tanaka H. Morrey spaces and fractional integrals. J Aust Math Soc 2010; 88: 247-259.

[41] Wang C, Zhang Z. A new proof of Wu's theorem on vortex sheets. Sci China Math 2012; 55: 1449-1462.

[42] Wu Q, Yang D. On fractional multilinear singular integrals. Math Nachr 2002; 239-240: 215-235.

[43] Wu S. Mathematical analysis of vortex sheets. Comm Pure Appl Math 2006; 59: 1065-1206.

[44] Xue Q. Weighted estimates for the iterated commutators of multilinear maximal and fractional type operators. Stud Math 2013; 217: 97-122.

[45] Yan D. Some problems on multilinear singular integral operators and multilinear oscillatory singular integral operators. PhD, Beijing Normal University, Beijing, China, 2001. 2005.

[46] Yu X, Tao X. Boundedness for a class of generalized commutators on λ-central Morrey space. Acta Math Sin (Engl Ser) 2013; 29: 1917-1926.