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Positive solutions of Neumann problems for a discrete system coming from models of house burglary

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Abstract: We show existence results of positive solutions of Neumann problems for a discrete system:

$$
\begin{aligned}
&\eta\Delta^2(A_{k-1}-A^0_{k-1})-A_k+A^0_k+N_kA_k=0,\quad k\in[2,n-1]_{\mathbb{Z}},\\
&\Delta\big(\Delta N_{k-1}-2N_k\frac{\Delta A_{k-1}}{A_k}\big)-N_kA_k+A^1_k-A^0_k=0,\quad k\in[2,n-1]_{\mathbb{Z}},\\
&\Delta A_1=0=\Delta A_{n-1},\quad \Delta N_1=0=\Delta N_{n-1},
\end{aligned}
$$

where the assumptions on η, A^0_k, and A^1_k are motivated by some mathematics models for house burglary. Our results are based on the topological degree theory.

Key words: Neumann boundary value problems, nonconstant positive solutions, topological degree theory

1. Introduction

We will denote the integer set by $\mathbb{Z}$. For any $a,b\in\mathbb{Z}$ with $a<b$, setting $[a,b]_{\mathbb{Z}}:=\{a,a+1,\cdots,b\}$.

In this paper, we study the existence of positive solutions of the difference systems:

$$
\begin{aligned}
&\eta\Delta^2(A_{k-1}-A^0_{k-1})-A_k+A^0_k+N_kA_k=0,\quad k\in[2,n-1]_{\mathbb{Z}},\\
&\Delta\big(\Delta N_{k-1}-2N_k\frac{\Delta A_{k-1}}{A_k}\big)-N_kA_k+A^1_k-A^0_k=0,\quad k\in[2,n-1]_{\mathbb{Z}},\\
&\Delta A_1=0=\Delta A_{n-1},\quad \Delta N_1=0=\Delta N_{n-1},
\end{aligned}
\tag{1}
$$

where $\eta>0$ is a constant, $\mathbf{A}^0=(A^0_1,\cdots,A^0_n)\in\mathbb{R}^n$, $\mathbf{A}^1=(A^1_1,\cdots,A^1_n)\in\mathbb{R}^n$, and $A^1_k>A^0_k>0$ for each $k\in[1,n]_{\mathbb{Z}}$. The homogeneous Neumann boundary condition implies that the system is self-contained with zero flux across the boundary.

A solution of (1) is a couple of real vector functions $(\mathbf{A},\mathbf{N})\in\mathbb{R}^n\times\mathbb{R}^n$ satisfying the system. We are interested in positive solutions, that is, $A_k>0$ and $N_k\geq 0$ with $N_k\not\equiv 0$ for all $k\in[1,n]_{\mathbb{Z}}$.

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This problem is motivated by Neumann boundary value problems for differential systems:

$$\begin{aligned}&\eta(A - A^0(x))'' - A + A^0(x) + NA = 0, \quad x \in (0, L),\\&(N' - 2N\frac{A'}{A})' - NA + +A^1(x) - +A^0(x) = 0, \quad x \in (0, L),\\&A'(0) = 0 = A'(L), \quad N'(0) = 0 = N'(L),\end{aligned} \tag{2}$$

which is a one-dimensional problem associated with a very successful model for house burglaries [17], see also the related papers [2, 4, 11, 12, 16]. In most of these models, $\eta > 0$ is the diffusion rate of attractiveness, A^0 is the intrinsic (static) component of attractiveness, A^1 is the average attractiveness, A is the attractiveness of a house for burglary, and N is the density of burglars. Thus, in the discrete case, the restrictions $A_k > 0$ and $N_k \geq 0$ with $N_k \not\equiv 0$ for all $k \in [1, n]_{\mathbb{Z}}$ appear as natural. Note that [5] is the consequence of the discretization of differential problems [11]. For some results on nonlinear difference problems, see [1, 3, 5–10, 15] and the references therein.

However, the discrete analogues of (2) have received almost no attention. In this article, we will discuss them in detail. We assume that the following conditions are satisfied:

(H1) $\mathbf{A}^0 = (A_1^0, \cdots, A_n^0) \in \mathbb{R}^n$, $\Delta A_1^0 = 0 = \Delta A_{n-1}^0$ and $A_k^0 > 0$, $k \in [1, n]_{\mathbb{Z}}$.

(H2) $\mathbf{A}^1 = (A_1^1, \cdots, A_n^1) \in \mathbb{R}^n$ and $A_k^1 > 0$, $k \in [1, n]_{\mathbb{Z}}$.

(H3) $A_k^1 > A_k^0$, $k \in [1, n]_{\mathbb{Z}}$.

If $\mathbf{A}^0, \mathbf{A}^1$ are positive constants, i.e. $\mathbf{A}^0 = (A^0, \cdots, A^0), \mathbf{A}^1 = (A^1, \cdots, A^1)$, system (1) admits the unique positive constant solution $\mathbf{A} = (A, \cdots, A), \mathbf{N} = (N, \cdots, N)$, where

$$A = A^1, \quad N = \frac{A^1 - A^0}{A^1}$$

under the condition that $A^1 > A^0$.

A natural question is whether or not nonconstant positive solutions still exist when $\mathbf{A}^0, \mathbf{A}^1$ are no longer constants? To this end, under the assumptions (H1)–(H3), we obtain the following result.

Theorem 1 *Let $\eta > 0$ be a constant. Under the assumptions (H1)–(H3), system (1) admits at least one nonconstant positive solution.*

The purpose of this paper was to show that analogues of existence results of solutions for differential problems proved in [12] hold for the corresponding difference systems. However, some basic ideas from differential calculus are not necessarily available in the field of difference equations such as the intermediate value theorem, the mean value theorem, and Rolle's theorem. Thus, new challenges are faced and innovation is required. The proof is elementary and relies on Brouwer degree theory [13, 14].

Throughout this paper, we use the following notations and conventions. Given $n \in \mathbb{N} (n \geq 4)$ and $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Define $(\Delta x_1, \cdots, \Delta x_{n-1}) \in \mathbb{R}^{n-1}$ by $\Delta x_m = x_{m+1} - x_m$, $m \in [1, n-1]_{\mathbb{Z}}$. For every $l, m \in \mathbb{N}$ with $m > l$, we set $\sum_{k=m}^{l} x_k = 0$.

Let us introduce the vector space

$$V^{n-2} = \{\mathbf{x} \in \mathbb{R}^n : \Delta x_1 = 0 = \Delta x_{n-1}\}, \tag{3}$$

endowed with the orientation of $\mathbb{R}^n$. Its elements can be associated with the coordinates $(x_2, \cdots, x_{n-1})$ and correspond to the elements of $\mathbb{R}^n$ of the form:

$$(x_2, x_2, x_3, \cdots, x_{n-2}, x_{n-1}, x_{n-1}).$$

We use the norm $\|\mathbf{x}\| := \max_{k\in[2,n-1]_{\mathbb{Z}}} |x_k|$ in V^{n-2}, and $\max_{k\in[1,n-2]_{\mathbb{Z}}} |x_k|$ in $\mathbb{R}^{n-2}$.

2. A priori estimates

Let us consider the homotopy corresponding to system (1) for $\lambda \in (0,1]$,

$$\eta\Delta^2(A_{k-1} - A^0_{k-1}) = \lambda(A_k - A^0_k - N_k A_k), \quad k \in [2, n-1]_{\mathbb{Z}}, \quad \Delta A_1 = 0 = \Delta A_{n-1}, \tag{4}$$

$$\Delta\big(\Delta N_{k-1} - 2\lambda N_k \frac{\Delta A_{k-1}}{A_k}\big) = \lambda(N_k A_k - A^1_k + A^0_k), \quad k \in [2, n-1]_{\mathbb{Z}}, \quad \Delta N_1 = 0 = \Delta N_{n-1}. \tag{5}$$

Obviously, for $\lambda = 1$, (4) and (5) reduce to (1).

Setting

$$\min \mathbf{B} := \min_{k\in[1,p]_{\mathbb{Z}}} B_k, \ \max \mathbf{B} := \max_{k\in[1,p]_{\mathbb{Z}}} B_k \text{ and } \overline{\mathbf{B}} = \frac{1}{n-2}\sum_{k=2}^{n-1} B_k, \quad \mathbf{B} \in \mathbb{R}^p.$$

Lemma 1 *Let $(\mathbf{A}, \mathbf{N})$ be a solution of (4) and (5) for some $\lambda \in (0,1]$. Then,*

$$\overline{\mathbf{A}} = \overline{\mathbf{A}^1}. \tag{6}$$

Proof Adding the two equations in (4) and (5), and then summing from $k = 2$ to $n-1$, we are led to $\overline{\mathbf{A}} = \overline{\mathbf{A}^1}$ by the Neumann boundary conditions. □

Lemma 2 *Let $(\mathbf{A}, \mathbf{N})$ be a positive solution of (4) and (5) for some $\lambda \in (0,1]$. Then,*

$$\overline{\mathbf{NA}} = \overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}. \tag{7}$$

Proof Summing the equation of (4) from $k = 2$ to $n-1$, we have that

$$\sum_{k=2}^{n-1}(A_k - A^0_k) = \sum_{k=2}^{n-1} N_k A_k,$$

which combines with Lemma 1 yields (7). □

Lemma 3 *Let (H3) hold. Let $(\mathbf{A}, \mathbf{N})$ be a positive solution of (4) and (5) for some $\lambda \in (0,1]$. Then, for any $k \in [2, n-1]_{\mathbb{Z}}$,*

$$|\Delta A_k| \le (n-2)\max|\Delta^2\mathbf{A}^0| + \frac{2(n-2)}{\eta}\overline{\mathbf{A}^1}, \tag{8}$$

$$A_k \le \overline{\mathbf{A}^1} + (n-2)^2 \max|\Delta^2\mathbf{A}^0| + \frac{2(n-2)^2}{\eta}\overline{\mathbf{A}^1} := C_1. \tag{9}$$

Proof According to (4), it is apparent that

$$|\Delta^2 A_{k-1}| \le |\Delta^2 A^0_{k-1}| + \frac{\lambda}{\eta}|A_k - A^0_k - N_k A_k| \le |\Delta^2 A^0_{k-1}| + \frac{1}{\eta}(A_k + A^0_k + N_k A_k), \quad k \in [2, n-1]_{\mathbb{Z}}.$$

Combining this with (6) and (7) implies that

$$\begin{aligned} |\Delta A_k| =& |\sum_{i=2}^{k} \Delta^2 A_{i-1}| \le \sum_{i=2}^{n-1} |\Delta^2 A_{i-1}| \\ \le& (n-2)\max|\Delta^2 \mathbf{A}^0| + \frac{n-2}{\eta}(\overline{\mathbf{A}^1} + \overline{\mathbf{A}^0} + \overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}) \\ =& (n-2)\max|\Delta^2 \mathbf{A}^0| + \frac{2(n-2)}{\eta}\overline{\mathbf{A}^1}. \end{aligned} \tag{10}$$

On the other hand, there exists $k_1 \in [2, n-1]_{\mathbb{Z}}$ such that $\min \mathbf{A} = A_{k_1}$, then $A_{k_1} \le \overline{\mathbf{A}}$. Thus, by virtue of (6) and (10), it follows that for every $k \in [2, n-1]_{\mathbb{Z}}$,

$$\begin{aligned} A_k =& A_{k_1} + \sum_{i=k_1}^{k-1} \Delta A_i \le \overline{\mathbf{A}} + \sum_{i=2}^{n-1} \Delta A_i \\ \le& \overline{\mathbf{A}^1} + (n-2)^2 \max|\Delta^2 \mathbf{A}^0| + \frac{2(n-2)^2}{\eta}\overline{\mathbf{A}^1}. \end{aligned}$$

□

Lemma 4 *Assume (H3). Let* $(\mathbf{A}, \mathbf{N})$ *be a positive solution of (4) and (5) for some* $\lambda \in (0, 1]$. *Then, for any* $k \in [2, n-1]_{\mathbb{Z}}$,

$$A_k \ge \min \mathbf{A}^0 := C_2. \tag{11}$$

Proof Since there exists $k_1 \in [2, n-1]_{\mathbb{Z}}$ such that $\min(\mathbf{A} - \mathbf{A}^0) = A_{k_1} - A^0_{k_1}$, then

$$0 \le \eta\Delta^2(A_{k_1} - A^0_{k_1}) = \lambda(A_{k_1} - A^0_{k_1} - N_{k_1}A_{k_1}) \le A_{k_1} - A^0_{k_1}.$$

This implies that $A_k \ge A^0_k$ for $k \in [2, n-1]_{\mathbb{Z}}$. Now, we may deduce that

$$A_k \ge \min \mathbf{A} \ge \min \mathbf{A}^0.$$

□

Corollary 1 *Assume (H3). Let* $(\mathbf{A}, \mathbf{N})$ *be a positive solution of (4) and (5) for some* $\lambda \in (0, 1]$, *then, for any* $k \in [2, n-1]_{\mathbb{Z}}$,

$$\Big|\frac{\Delta A_k}{A_k}\Big| \le \frac{(n-2)\max|\Delta^2 \mathbf{A}^0| + 2(n-2)\overline{\mathbf{A}^1}}{\eta C_2} := C_3. \tag{12}$$

Lemma 5 *Assume that (H3) holds. Let* $(\mathbf{A}, \mathbf{N})$ *be a positive solution of (4) and (5) for some* $\lambda \in (0, 1]$. *Then,*

$$\frac{\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}}{C_1} \le \overline{\mathbf{N}} \le \frac{\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}}{C_2}. \tag{13}$$

Proof From (7), (9), (11), and the inequalities

$$(\min \mathbf{A}) \sum_{k=2}^{n-1} N_k \leq \sum_{k=2}^{n-1} N_k A_k \leq (\max \mathbf{A}) \sum_{k=2}^{n-1} N_k,$$

we may get the desired result. □

Lemma 6 *Assume (H3). Let* $(\mathbf{A}, \mathbf{N})$ *be a positive solution of (4) and (5) for some* $\lambda \in (0,1]$. *Then,*

$$\Big|\Delta N_k - 2\lambda N_{k+1} \frac{\Delta A_k}{A_{k+1}}\Big| \leq 2(n-2)(\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}), \quad k \in [2, n-1]_{\mathbb{Z}}. \tag{14}$$

Proof Owing to (5), it can be easily seen that

$$|\Delta(\Delta N_{k-1} - 2\lambda N_k \frac{\Delta A_{k-1}}{A_k})| \leq |N_k A_k + A_k^1 - A_k^0| = N_k A_k + A_k^1 - A_k^0, \quad k \in [2, n-1]_{\mathbb{Z}}.$$

Hence, using the boundary conditions and (7), it is not difficult to prove that for every $k \in [2, n-1]_{\mathbb{Z}}$,

$$\begin{aligned} &\Big|\Delta N_k - 2\lambda N_{k+1} \frac{\Delta A_k}{A_{k+1}}\Big| = \Big|\sum_{i=2}^{k} \Delta(\Delta N_{i-1} - 2\lambda N_i \frac{\Delta A_{i-1}}{A_i})\Big| \\ &\leq \sum_{k=2}^{n-1} (N_k A_k + A_k^1 - A_k^0) \leq (n-2)(\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0} + \overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}) \\ &= 2(n-2)(\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}). \end{aligned}$$

□

Lemma 7 *Assume (H3). Let* $(\mathbf{A}, \mathbf{N})$ *be a positive solution of (4) and (5) for some* $\lambda \in (0,1]$. *Then,*

$$|N_k - \overline{\mathbf{N}}| \leq 2(n-2)(\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0})\Big[\frac{C_3}{C_2} + (n-2)\Big] := C_4, \quad k \in [2, n-1]_{\mathbb{Z}}. \tag{15}$$

Proof From (14), we have

$$|\Delta N_k| \leq 2N_{k+1}\Big|\frac{\Delta A_k}{A_{k+1}}\Big| + 2(n-2)(\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}), \quad k \in [2, n-1]_{\mathbb{Z}}.$$

Let now $k_1 \in [2, n-1]_{\mathbb{Z}}$ be a minimum point of $\mathbf{N}$, then $\overline{\mathbf{N}} \geq N_{k_1}$. Then it follows from (12) and (13) that for any $k \in [2, n-1]_{\mathbb{Z}}$,

$$\begin{aligned} |N_k - \overline{\mathbf{N}}| \leq & |N_k - N_{k_1}| = |\sum_{i=k_1}^{k-1} \Delta N_i| \\ & \leq \sum_{k=2}^{n-1} |\Delta N_k| \leq 2C_3 \sum_{k=2}^{n-1} N_k + 2(n-2)^2(\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}) \\ & \leq 2(n-2)(\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0})\Big[\frac{C_3}{C_2} + (n-2)\Big]. \end{aligned}$$

□

Corollary 2 *Assume (H3). Let* $(\mathbf{A},\mathbf{N})$ *be a positive solution of (4) and (5) for some* $\lambda\in(0,1]$*. Then, for any* $k\in[2,n-1]_{\mathbb{Z}}$*,*

$$N_k \le C_4 + \frac{\overline{\mathbf{A}^1}-\overline{\mathbf{A}^0}}{C_2} := C_5. \tag{16}$$

3. Proof of the main result

This section is devoted to the existence of nonconstant positive solutions to system (1). For our purpose, we need some preliminary results.

Lemma 8 *For any* $\lambda\in(0,1]$*,* $(\mathbf{A},\mathbf{N})$ *is a solution of (4) and (5) if and only if* $(\mathbf{A},\mathbf{N})$ *is a solution of the following system:*

$$A_k = A_2 + A_k^0 - A_2^0 - \overline{(\mathbf{A}-\mathbf{A}^0-\mathbf{N}\mathbf{A})} + \frac{\lambda}{\eta}\sum_{j=2}^{k-1}\sum_{i=2}^{j}(A_i - A_i^0 - N_iA_i), \quad k\in[1,n]_{\mathbb{Z}}, \tag{17}$$

$$N_k = N_2 - \overline{(\mathbf{N}\mathbf{A}-\mathbf{A}^1+\mathbf{A}^0)} + \lambda\sum_{i=2}^{k}2N_i\frac{\Delta A_{i-1}}{A_i} + \lambda\sum_{j=2}^{k-1}\sum_{i=2}^{j}(N_iA_i - A_i^1 + A_i^0), \quad k\in[1,n]_{\mathbb{Z}}. \tag{18}$$

Proof Suppose $(\mathbf{A},\mathbf{N})$ is a solution of the system (17) and (18). By taking $k=2$ in both equations, we find that

$$\overline{(\mathbf{A}-\mathbf{A}^0-\mathbf{N}\mathbf{A})}=0, \quad \overline{(\mathbf{N}\mathbf{A}-\mathbf{A}^1+\mathbf{A}^0)}=0. \tag{19}$$

On the other hand, according to (18) and (19), it becomes apparent that for any $k\in[1,n-1]_{\mathbb{Z}}$,

$$\Delta A_k = \Delta A_k^0 + \frac{\lambda}{\eta}\sum_{i=2}^{k}(A_i - A_i^0 - N_iA_i), \tag{20}$$

$$\Delta N_k = \lambda[2N_{k+1}\frac{\Delta A_k}{A_{k+1}} + \sum_{i=2}^{k}(N_iA_i - A_i^1 + A_i^0)]. \tag{21}$$

It should be noted that we obtain $\Delta A_1 = 0 = \Delta N_1$ if $k=1$ and $\Delta A_{n-1}=0=\Delta N_{n-1}$ if $k=n-1$ by using (19). Finally, it is not difficult to verify that

$$\Delta^2(A_{k-1}-A_{k-1}^0) = \frac{\lambda}{\eta}(A_k - A_k^0 - N_kA_k), \quad k\in[2,n-1]_{\mathbb{Z}},$$

and

$$\Delta^2 N_{k-1} = \lambda\big[\Delta\big(2N_k\frac{\Delta A_{k-1}}{A_k}\big) + (N_kA_k - A_k^1 + A_k^0)\big], \quad k\in[2,n-1]_{\mathbb{Z}},$$

which is equivalent to systems (4) and (5). Similarly, the proof of the converse is valid. □

Let us now take $0<R_2<C_2\le C_1<R_1$, $C_3C_1<R_3$ and $C_5<R_5$, where C_1,C_2,C_3 and C_5 are respectively given in (9),(11),(12), and (16).

Define the vector space $E = V^{n-2} \times V^{n-2}$ by the usual norm $\|(\mathbf{A}, \mathbf{N})\|_E = \|\mathbf{A}\| + \|\Delta\mathbf{A}\| + \|\mathbf{N}\|$. Set the bounded set $\Omega \subset E$,

$$\Omega = \{(\mathbf{A}, \mathbf{N}) \in E : \ R_2 < A_k < R_1, \ |\Delta A_k| < R_3, 0 \le N_k < R_5, \ k \in [2, n-1]_{\mathbb{Z}}\}. \tag{22}$$

Define the operator $\mathcal{F} : \Omega \times [0,1] \to E$,

$$\mathcal{F}(\mathbf{A}, \mathbf{N}, \lambda) = \begin{pmatrix} A_2 + A_k^0 - A_2^0 - \overline{(\mathbf{A} - \mathbf{A}^0 - \mathbf{N}\mathbf{A})} + \frac{\lambda}{\eta} \sum_{j=2}^{k-1} \sum_{i=2}^{j} (A_i - A_i^0 - N_i A_i), \\ N_2 - \overline{(\mathbf{N}\mathbf{A} - \mathbf{A}^1 + \mathbf{A}^0)} + \lambda \sum_{i=2}^{k} 2N_i \frac{\Delta A_{i-1}}{A_i} + \lambda \sum_{j=2}^{k-1} \sum_{i=2}^{j} (N_i A_i - A_i^1 + A_i^0) \end{pmatrix}. \tag{23}$$

By Lemma 8, we deduce that $(\mathbf{A}, \mathbf{N})$ is a positive solution of systems (4) and (5) if and only if $(\mathbf{A}, \mathbf{N})$ is a fixed point of $\mathcal{F}$.

Lemma 9 *$(\mathbf{A}, \mathbf{N}) \in \bar{\Omega}$ is a fixed point of $\mathcal{F}(\cdot, 0)$ if and only if $\mathbf{A} - \mathbf{A}^0$ and $\mathbf{N}$ are constants, set $\mathbf{A} = \mathbf{A}^0 + \mathbf{B}$ and $\mathbf{B} = (B, \cdots, B)$, $\mathbf{N} = (N, \cdots, N)$, where $(\mathbf{B}, \mathbf{N})$ satisfies the algebraic system*

$$B - N\overline{\mathbf{A}^0} - NB = 0, \quad N\overline{\mathbf{A}^0} + NB - \overline{\mathbf{A}^1} + \overline{\mathbf{A}^0} = 0, \tag{24}$$

whose unique solution is given by:

$$B = \overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}, \quad N = \frac{\overline{\mathbf{A}^1} - \overline{\mathbf{A}^0}}{\overline{\mathbf{A}^1}}. \tag{25}$$

Proof Since $(\mathbf{A}, \mathbf{N}) \in \bar{\Omega}$ is a fixed point of $\mathcal{F}(\cdot, 0)$ if and only if

$$A_k = A_2 + A_k^0 - A_2^0 - \overline{(\mathbf{A} - \mathbf{A}^0 - \mathbf{N}\mathbf{A})}, \quad N_k = N_2 - \overline{(\mathbf{N}\mathbf{A} - \mathbf{A}^1 + \mathbf{A}^0)},$$

i.e. if and only if

$$\overline{(\mathbf{A} - \mathbf{A}^0 - \mathbf{N}\mathbf{A})} = 0, \quad \overline{(\mathbf{N}\mathbf{A} - \mathbf{A}^1 + \mathbf{A}^0)} = 0, \tag{26}$$

and $B_k = A_k - A_k^0 \equiv A_2 - A_2^0$ and $N_k \equiv N_2$ for any $k \in [2, n-1]_{\mathbb{Z}}$, that is, $\mathbf{B}$ and $\mathbf{N}$ are constants, combining this with (26), we have that (24) and (25) hold. □

Proof of Theorem 1 It is easy to see that solving (1) is equivalent to finding a fixed point of $\mathcal{F}(\cdot, 1)$ in $\bar{\Omega}$. Furthermore, from the definition of Ω and Lemmas 3 and 4, Corollaries 1 and 2, we have that $\mathcal{F}(\cdot, \lambda)$ has no fixed point in $\partial\Omega$ for all $\lambda \in (0, 1]$. By using the homotopy invariance of degree, we can conclude that

$$\deg[I - \mathcal{F}(\cdot, 1), \Omega, 0] = \deg[I - \mathcal{F}(\cdot, 0), \Omega, 0]. \tag{27}$$

On the other hand, by Lemma 9, any fixed point of $\mathcal{F}(\cdot, 0)$ has the form $(\mathbf{A}^0 + \mathbf{B}, \mathbf{N})$ with $\mathbf{B}$ and $\mathbf{N}$ are constants satisfying (25), so that $R_2 - A_k^0 < A_k - A_k^0 < R_1 - A_k^0$ and $0 \le N_k < R_5$. Set $A_k = A_k^0 + B$. Now the invariance of the topological degree by translation yields that

$$\deg[I - \mathcal{F}(\cdot, 0), \Omega, 0] = \deg[(I - \mathcal{F})(\mathbf{A}^0 + \cdot, \cdot, 0), \Omega - (\mathbf{A}^0, 0), 0],$$

where

$$(I-\mathcal{F})(\mathbf{A}^0+\mathbf{B},\mathbf{N},0) = (B_k - B_2 + \overline{\mathbf{B}-\mathbf{N}\mathbf{A}^0-\mathbf{N}\mathbf{B}},\ N_k - N_2 + \overline{\mathbf{N}\mathbf{A}^0+\mathbf{N}\mathbf{B}-\mathbf{A}^1+\mathbf{A}^0}) =: (I-\tilde{\mathcal{F}})(\mathbf{B},\mathbf{N}).$$

Then it follows from the reduction theorem of the topological degree that

$$\deg[I-\tilde{\mathcal{F}}, \Omega-(\mathbf{A}^0,0),0] = \deg[(I-\tilde{\mathcal{F}})|_{\mathbb{R}^2}, (\Omega-(\mathbf{A}^0,0))\cap\mathbb{R}^2, 0] = \deg[\mathcal{G}, (R_2-\max\mathbf{A}^0, R_1-\min\mathbf{A}^0)\times[0,R_5),0],$$

where $\mathcal{G}:[R_2-\max\mathbf{A}^0, R_1-\min\mathbf{A}^0]\times[0,R_5]\to\mathbb{R}$ is defined by

$$\mathcal{G}(B,N) = (B - N\overline{\mathbf{A}^0} - NB, N\overline{\mathbf{A}^0} + NB - \overline{\mathbf{A}^1} + \overline{\mathbf{A}^0}).$$

Since the unique zero of $\mathcal{G}$ is given by (25). Therefore,

$$\deg[\mathcal{G}, (R_2-\max\mathbf{A}^0, R_1-\min\mathbf{A}^0)\times(0,R_5),0] = 1.$$

Consequently,

$$\deg[I-\mathcal{F}(\cdot,1),\Omega,0]=1,$$

and hence, the existence property of degree theory implies that $\mathcal{F}(\cdot,1)$ has at least one fixed point in Ω, i.e. system (1) admits at least one nonconstant positive solution. □

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