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The density theorem for hermitian K-theory

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Abstract: Karoubi's density theorem was first proved in Benayat's thesis and then cited and used in several books and articles. As K-theory is a special case of hermitian ${}_{\varepsilon}L$-theory, a natural question is whether such a theorem is still true in the latter theory. The purpose of this article is to show that it is indeed the case.

Key words: Hermitian algebra, sesquilinear form, hyperbolic form, spectrum, polar form, K-theory

1. Introduction

Functors of algebraic topology are numerous and often easy to define. However, although their algebraic properties are well known, the major task and the most difficult problem is to compute their values for interesting objects. Karoubi's density theorem [2,5,10] says that if A is densely and continuously included in the Banach algebra B and units of A are those of B belonging to A, then $K(A)$ and $K(B)$ are isomorphic; it was first proved in Benayat's thesis [2] to compute the K-theory of the Banach algebra of absolutely summable Laurent series in n variables. The theorem raises the question (known as Swan's problem) whether, under the same hypotheses, there is equality of stable ranks [8]. It also allows the extension of topological K-theory to a whole class of dense subalgebras of the algebras involved and to Frechet algebras and dense subalgebras of these [1,6–8]. Since K-theory is a special case of hermitian ${}_{\varepsilon}L$-theory, it is natural to ask whether the density theorem is still valid for the latter, allowing us to extend the mentioned problems to the hermitian situation; in this article, we answer the question positively. We refer to [4] for the basics concerning Banach categories.

2. Notations, definitions, review

Let $\mathcal{C}$ be a Banach group category, i.e. an additive category $\mathcal{C}$ such that the group of morphisms $\mathcal{C}(M,N)$ is a Banach group and the composition of morphisms $\mathcal{C}(M,N)\times\mathcal{C}(N,P)\longrightarrow\mathcal{C}(M,P)$ is continuous; a transposition functor is a contravariant Banach functor $t:\mathcal{C}\longrightarrow\mathcal{C}$ such that $t\circ t$ is naturally isomorphic to the identity functor $1_{\mathcal{C}}$. In the sequel, we will always identify $t^2(M)$ and $t^2(f)$ with M and f respectively (M object and f morphism in $\mathcal{C}$). The pair $(\mathcal{C},t)$ is called a hermitian category.

Example 1 *A hermitian ring is a Banach ring endowed with a bounded involution* $(a\longrightarrow\overline{a})$*. Let* $\mathcal{P}(A)$ *be the category of projective right* A*-modules of finite type. We can endow it with the following transposition: if* P

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is an object in $\mathcal{P}(A)$, *we put*

$$ {}^{t}P=\left\{\begin{array}{c} f:P\longrightarrow A:\ f \text{ is a homomorphism of abelian groups and } f(xa)=\overline{a}f(x) \\ \text{for } a\in A \text{ and } x\in P \end{array}\right\} $$

The formula $(fa)(x)=f(x)a$ *endows* ${}^{t}P$ *with a right* A*-module structure. Since* P *is projective and of finite type,* ${}^{t}P$ *is also of finite type and projective;* ${}^{t}({}^{t}P)$ *is canonically isomorphic to* P. *If* $g:P\longrightarrow Q$ *is a morphism, then composition with* g *defines a morphism* ${}^{t}g:\ {}^{t}Q\longrightarrow {}^{t}P$.

Example 2 *Let* X *be an involutive compact space (i.e.* X *is endowed with a continuous application* $\sigma:X\longrightarrow X$ *such that* $\sigma^{2}=1_{X}$*) and* A *a hermitian ring. We consider the category* $\mathcal{E}(X,A)$ *of locally trivial fiber spaces over* X *with fibers in* $\mathcal{P}(A)$. *The previously defined functor, being continuous, induces a transposition functor still written* $t:\mathcal{E}(X,A)\longrightarrow\mathcal{E}(X,A)$.

Let M be an object in $(\mathcal{C},t)$. A sesquilinear form on M is a morphism $\varphi:M\longrightarrow {}^{t}M$; it is nondegenerate if it is an isomorphism in the category. We write $Sesq(M)$ for the abelian group of sesquilinear forms on M. We have an involution $T:Sesq(M)\longrightarrow Sesq(M)$ defined by $T(\varphi)$ being the composition $M\approx {}^{t}({}^{t}M)\xrightarrow{{}^{t}\varphi} {}^{t}M$. Let $\varepsilon=(+1)$ or (-1) and consider the following complex of abelian groups: $\cdots\longrightarrow Sesq(M)\xrightarrow{1+T\varepsilon} Sesq(M)\xrightarrow{1-T\varepsilon} Sesq(M)\longrightarrow\cdots$

We assume, once and for all, that we can divide by 2 in any $Sesq(M)$. Then the above complex is acyclic for, if $\varphi\in Ker(1-T\varepsilon)$, then $\psi=\frac{1}{2}\varphi$ has image φ by $(1+T\varepsilon)$. We have the obvious isomorphisms:

$$Coker(1-T\varepsilon)=Sesq(M)\diagup\operatorname{Im}(1-T\varepsilon)=Sesq(M)\diagup Ker(1+T\varepsilon)\approx\operatorname{Im}(1+T\varepsilon)\approx Ker(1-T\varepsilon).$$

Definition 3 *An* ε*-hermitian form on* M *is an element of* $Coker(1-T\varepsilon)$.

From what precedes, we can identify an ε-hermitian form on M to a $\varphi:M\longrightarrow {}^{t}M$ such that ${}^{t}\varphi=\varepsilon\varphi$.

Definition 4
1. *A hermitian object is a pair* (M,φ) *where* $M\in Ob(\mathcal{C})$ *and* $\varphi:M\longrightarrow {}^{t}M$ *is an isomorphism in* $\mathcal{C}$ *such that* ${}^{t}\varphi=\varepsilon\varphi$.
2. *Let* (M,φ) *and* (N,ψ) *be two hermitian objects; a* $\mathcal{C}$*-morphism* $f:M\longrightarrow N$ *is unitary if* $\varphi={}^{t}f\circ\psi\circ f$ *and it is an isometry if, moreover, it is an isomorphism. We put* $U_{\varphi}(M)$ *for the group of unitary automorphisms (i.e. isometries) of* (M,φ).

The adjoint of $f:M\longrightarrow N$ is $f^{*}=\varphi^{-1}\circ\ {}^{t}f\circ\psi:N\longrightarrow M$; it is clear that if $f\in U_{\varphi}(M)$ then $f^{*}\circ f=\varphi^{-1}\circ^{t}f\circ\psi\circ f=1_{M}$, that is $f^{*}=f^{-1}$.

2.1. Hyperbolic forms

The following $\begin{pmatrix}0 & 1_{{}^{t}M}\\ \varepsilon 1_{M} & 0\end{pmatrix}:M\oplus\ {}^{t}M\longrightarrow {}^{t}M\oplus M$ is an ε-hermitian form on $M\oplus\ {}^{t}M$ and will be written $\begin{pmatrix}0 & 1\\ \varepsilon & 0\end{pmatrix}$. This hermitian object is written $H(M)$ and called a hyperbolic form on M. In the case of the two fundamental examples, we know from [4] that any hermitian object (P,θ) is a direct factor of a hyperbolic form, i.e. there is a hermitian object (P', θ') such that $(P, \theta) \oplus (P', \theta')$ is isometric to a hyperbolic form $H(M)$. An isometry on $H(M)$ is a morphism $u : M \oplus {}^tM \longrightarrow M \oplus {}^tM$ such that its adjoint u^* for $\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$ satisfies $u^* = u^{-1}$; we write $U(H(M))$ for the group of these isometries.

2.2. Definition of ${}_\varepsilon L(\mathcal{C})$

Let ${}_\varepsilon Q(\mathcal{C})$ be the category whose objects are the hermitian objects and the morphisms are those which are unitary. We define ${}_\varepsilon L(\mathcal{C})$ as the Grothendieck group $K({}_\varepsilon Q(\mathcal{C}))$. Explicitly, let M be the monoid of isometry classes of hermitian objects in $\mathcal{C}$; then ${}_\varepsilon L(\mathcal{C})$ is the symmetrized group of M. An element of ${}_\varepsilon L(\mathcal{C})$ is a formal difference $[M, \varphi] - [N, \psi]$, where $[M, \varphi]$ is the isometry class of (M, φ). In particular, ${}_\varepsilon L(P(A))$ and ${}_\varepsilon L(\mathcal{E}(X, A))$ are written respectively as ${}_\varepsilon L(A)$ and ${}_\varepsilon L(X, A)$. Recall that the addition law in ${}_\varepsilon L(A)$ is the obvious direct sum $[M, \varphi] \oplus [N, \psi] \equiv [M \oplus N, \varphi \oplus \psi]$. Any element of ${}_\varepsilon L(A)$ can be written as $[H(A^n)] - [M, \varphi]$ and a generator can be given in two ways:

1. *As the image of a projector*

   Let (M, φ) be an ε-hermitian A-module; then there is a self-adjoint projector $p : H(A^n) \longrightarrow H(A^n)$ such that (M, φ) is isometric to $\left(\mathrm{Im} p, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}_{|\mathrm{Im}\ p} \right)$.

2. *As an involution*

   There is a self-adjoint involution $\nu : H(A^n) \longrightarrow H(A^n)$, i.e. $\nu^2 = 1$ and $\nu^* = \nu$, such that (M, φ) is isometric to $\left(Ker(\nu - 1), \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}_{|Ker(\nu-1)} \right)$.

3. The density theorem for ${}_\varepsilon L$

Let $k = \mathbb{R}$ or $\mathbb{C}$; $\mathbb{R}$ is endowed with the identity involution $\left(\lambda \longrightarrow \overline{\lambda}, \forall \lambda \in \mathbb{R}\right)$ while $\mathbb{C}$ can be provided with either the identity involution or the complex conjugation. A k-hermitian algebra is a k-algebra A that is a hermitian ring $(a \longrightarrow \overline{a})$ complying with the following relation: $\overline{\lambda \cdot a} = \overline{\lambda} \cdot \overline{a}, \forall \lambda \in k, \forall a \in A$. We write $U(A)$ for the group of invertible elements of A.

Theorem 5 *Let A and B be two k-hermitian algebras such that:*

1. *A is continuously and densely included in B*
2. *The involution of A is the restriction of that of B*
3. *$U(A) = A \cap U(B)$*

   Then the natural homomorphism ${}_\varepsilon L(A) \longrightarrow {}_\varepsilon L(B)$ is an isomorphism.

Proof Let M_A and M_B be the abelian monoids of isometry classes of ε-hermitian A- and B-modules respectively. Extension of scalars gives the morphism of monoids $T : M_A \longrightarrow M_B, [M, \varphi] \longrightarrow \left[M_B = M \underset{A}{\otimes} B, \varphi_B = \varphi \otimes 1 \right]$ inducing the ring homomorphism $\widetilde{T} : [M, \varphi] - [N, \psi] \longrightarrow [M_B, \varphi_B] - [N_B, \psi_B]$ from ${}_\varepsilon L(A)$ to ${}_\varepsilon L(B)$. □

1. $\widetilde{T}$ is injective

Let $p : H(A^n) \longrightarrow H(A^n)$ be a self-adjoint projector; an element of ${}_\varepsilon L(A)$ can be written $H(A^n) - [M,\varphi]$. If $\widetilde{T}(H(A^n) - [M,\varphi]) = 0$, i.e. $H(B^n) - [M_B,\varphi_B] = 0$, then there exists $m \in \mathbb{N}$ such that $H(B^n) \oplus H(B^m)$ is isometric to $(M_B,\varphi_B) \oplus H(B^m)$; this means that $H(B^{n+m})$ is isometric to $\left(M_B \oplus H(B^m), \varphi_B \oplus \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}\right)$. Thus, up to the addition of $0 = [H(A^m)] - [H(A^m)]$ to $H(A^n) - [M,\varphi]$, we can assume that $(M_B,\varphi_B) = H(B^n)$. Using the language of projectors, we have to prove that if $p_B = p \otimes 1 : H(A^n) \otimes B = H(B^n) \longrightarrow H(A^n) \otimes B = H(B^n)$ complies with $\left(\mathrm{Im} p_B, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}_{|\mathrm{Im} p_B}\right) \overset{isometric}{\approx} H(B^m)$ for a certain $m \in \mathbb{N}$, then $\left(\mathrm{Im} p, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}_{|\mathrm{Im} p}\right)$ is isometric to $H(A^m)$. Therefore, let $p_0 : H(A^n) \longrightarrow H(A^n)$ be the self-adjoint projector such that $\mathrm{Im} p_0 = H(A^m)$ looked as a direct factor of $H(A^n)$. We will show that if $p : H(A^n) \longrightarrow H(A^n)$ is such that $\left(\mathrm{Im} p_B, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}_{|\mathrm{Im} p_B}\right)$ is isometric to $H(B^m)$, then p is isometric to p_0. By construction p_B and $(p_0)_B$ are isometric; so there is a unitary operator $\alpha \in U(H(B^n))$ such that $p_B = \alpha \circ (p_0)_B \circ \alpha^{-1}$. We put $X = H(B^n)$ and $\varphi = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$; the following diagram, in Figure 1, is commutative since $p_B = \alpha \circ (p_0)_B \circ \alpha^{-1}$ and $\alpha^{-1} = \alpha^* = \varphi^{-1} \circ ({}^t\alpha) \circ \varphi$. For every $r \in \mathbb{N}^*$, hypothesis 1 implies that the matrix group $Gl_r(A)$ and the algebra $M_r(A)$ are dense in $Gl_r(B)$ and $M_r(B)$, respectively. Let $\tau > 0$ and $\alpha' \in Gl(A^n \oplus {}^tA^n)$ such that $\|\alpha - \alpha'\| \leq \tau$. We are going to approximate α' by an $\alpha'' \in U(H(A^n))$ by using the polar decomposition of α'. We recall lemma 4.1.4 from Rickart's book [9]:

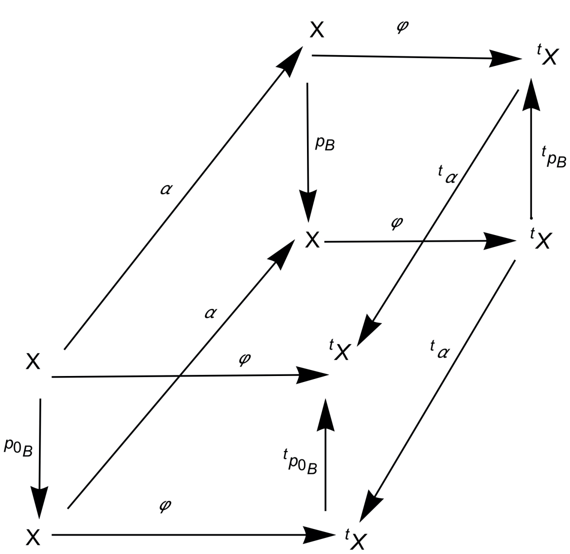

Figure 1. α is a unitary morphism of $H(B^n)$.

Lemma [9] *Let U be a Banach algebra and h an element of U with $\nu(h) = \lim_{n\longrightarrow\infty} \|h^n\|^{\frac{1}{n}} < 1$. Then there exists an element k in U such that $k^2 = h$. If U is a Banach *-algebra with a locally continuous involution and h is hermitian then k is also hermitian.*

k is written $\sqrt{h}$ and is constructed as the limit of an absolutely convergent series. In our situation, let us put $u = \alpha - \alpha'$; we have

$$\begin{aligned} \alpha'\alpha'^* &= (\alpha - u)(\alpha^* - u^*) \\ &= \alpha\alpha^* - u\alpha^* - \alpha u^* + uu^* \\ &= 1 + \eta, \end{aligned}$$

where $\eta = -u\alpha^{-1} - \alpha u^* + uu^*$. The norm of η satisfies

$$\|\eta\|_B \leq \left\|u\alpha^{-1}\right\|_B + \|\alpha u^*\|_B + \|uu^*\|_B \leq \left\|\alpha^{-1}\right\|_B \tau + \|\alpha\|_B \lambda\tau + \lambda\tau^2 \leq K\tau^2,$$

where $\|u^*\|_B \leq \lambda\tau$ for a constant $\lambda > 0$ and K is a constant independent from τ. Hence, if τ is taken small enough, $\sqrt{\alpha'\alpha'^*}$ exists ; let us put $\alpha'' = \left(\sqrt{\alpha'\alpha'^*}\right)^{-1} \circ \alpha' \in U(H(A^n))$. Since the applications $\beta \longrightarrow \beta^*, \beta \longrightarrow \beta^{-1}$, and $\beta \longrightarrow \sqrt{\beta}$ are continuous on their respective domains, α'' is a unitary approximation of α' and thus of α too; we have $\|\alpha'' - \alpha\|_B \leq L\tau$ and $\|\alpha''^* - \alpha^*\|_B = \left\|\alpha''^{-1} - \alpha^{-1}\right\|_B \leq L'\tau$, where L and L' are positive constants. If we put $p'' = \alpha'' \circ p_0 \circ \alpha''^* = \alpha'' \circ p_0 \circ \alpha''^{-1}$, then it is a self-adjoint projector on $H(A^n)$ and the following diagram shown in Figure 2, where $\beta = 1 - p - p'' + 2pp''$, is commutative: β is an isomorphism of modules but is not necessarily unitary; since $\|\beta\|$ is close to 1, $\sqrt{\beta\beta^*}$ does exist. Thus we can proceed as we did for α'; we take the polar decomposition β'' of β and we get $\beta'' \in U(H(A^n))$, which commutes with β. We get the following commutative diagram in Figure 3: which shows that p'' and p are isometric. Since $p'' = \alpha'' \circ p_0 \circ \alpha''^*$, we have shown, by transitivity, that p and p_0 are isometric.

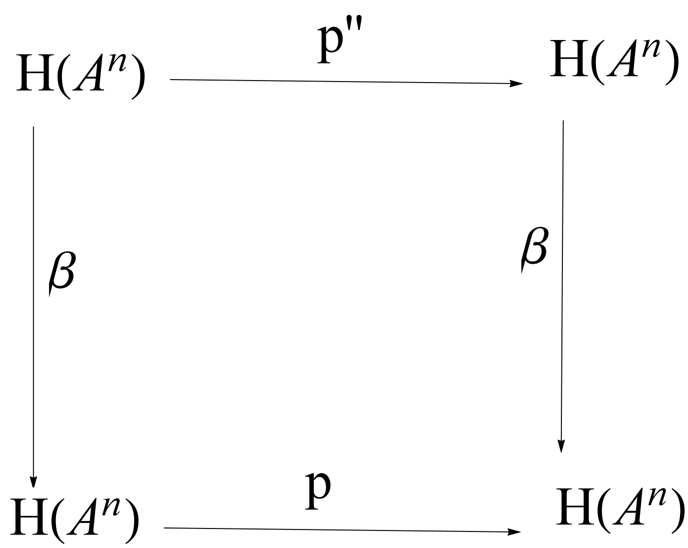

Figure 2. Commutative diagram but not unitary.

4. $\widetilde{T}$ is surjective

We assume the algebras A and B are real; if they are complex, then the following proof would go without having to complexify. The complexifications $A \underset{\mathbb{R}}{\otimes} \mathbb{C} = A \oplus iA$ and $B \underset{\mathbb{R}}{\otimes} \mathbb{C} = B \oplus iB$ are written as $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$,

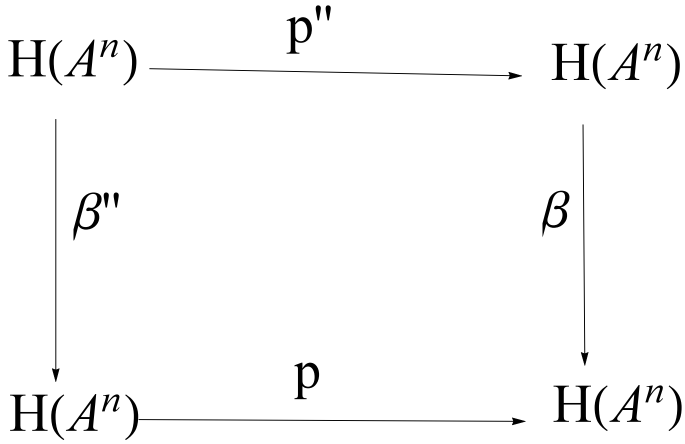

Figure 3. Commutativity implies that p and p_0 are isometric.

respectively. Let $\nu_0 : H(B^n) \longrightarrow H(B^n)$ be a self-adjoint involution, defining a generator of ${}_{\varepsilon}L(B)$; we have to find a self-adjoint involution $\nu : H(A^n) \longrightarrow H(A^n)$ such that $\nu_B \approx \nu_0$. We recall that the "spectrum" map

$$\begin{cases} sp : \mathcal{A} \longrightarrow 2^{\mathbb{C}} \\ x \rightsquigarrow sp(x) = \{z \in \mathbb{C} : (x - z.1) \text{ is not inversible}\} \end{cases}$$

where $\mathcal{A}$ is a complex Banach algebra, is upper semicontinuous. In our case, if $a \in M_n(A_{\mathbb{C}})$, then hypothesis 2 implies that the spectrum of a as an element of $M_n(A_{\mathbb{C}})$ is the same as the spectrum of a as an element of $M_n(B_{\mathbb{C}})$. We have $sp(\nu_0) = \{+1, -1\}$, where $\nu_0 \in M_n(B) \oplus i0 \subset M_n(B) \oplus iM_n(B) = M_n(B_{\mathbb{C}})$. Let us take $V = sp(\nu_0) + B(0, \varepsilon)$, where $B(0, \varepsilon)$ is the open disk in $\mathbb{C}$ centered at 0 and with positive radius ε much smaller than 1 as shown in Figure 4: Thus there is a neighborhood U of ν_0 in $End(B^n_{\mathbb{C}} \oplus {}^tB^n_{\mathbb{C}})$ such that $sp(\nu) \subset V$ for any $\nu \in U$. Let $W = sp(\nu_0) + B(0, 2\varepsilon)$ and $\partial W = \gamma_{+1} \cup \gamma_{-1}$; we consider the following application:

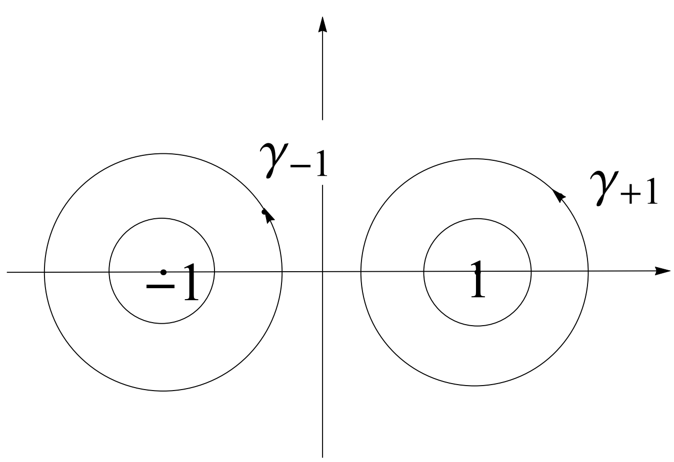

Figure 4. Special neighborhood of $sp(\nu_0)$.

$$\begin{cases} F : U \longrightarrow End\left(B_{\mathbb{C}}^{n} \oplus {}^{t}B_{\mathbb{C}}^{n}\right) \\ F\left(\nu\right) = \frac{1}{2i\pi}\int_{\gamma_{+1}\cup\gamma_{-1}} f\left(z\right)\left[z.1-\nu\right]^{-1}dz, \end{cases}$$

where

$$f\left(z\right) = \begin{cases} +1 \text{ on } (+1) + B\left(0, 2\varepsilon\right) \\ -1 \text{ on } (-1) + B\left(0, 2\varepsilon\right) \end{cases}$$

Let us note that, although $End\left(B_{\mathbb{C}}^{n} \oplus {}^{t}B_{\mathbb{C}}^{n}\right)$ is not a commutative algebra, the integral is perfectly defined since it is the limit of Riemann sums taking place in the closure of a commutative subalgebra of the latter. The function $f\left(z\right)$ is holomorphic on a neighborhood of $sp\left(\nu\right)$ for any $\nu \in U \subset End\left(B_{\mathbb{C}}^{n} \oplus {}^{t}B_{\mathbb{C}}^{n}\right)$. Hence $F\left(\nu\right)$ is well defined and, by holomorphic functional calculus, the relation $f\left(z\right)^{2} = 1$ implies $\left(F\left(\nu\right)\right)^{2} = 1$ in $End\left(B_{\mathbb{C}}^{n} \oplus {}^{t}B_{\mathbb{C}}^{n}\right)$. Thus $F\left(\nu\right)$ is an involution for any $\nu \in U$. Let us show that $F\left(\nu_{0}\right) = \nu_{0}$; we put $p_{+}\left(\nu_{0}\right) = \frac{1}{2i\pi}\int_{\gamma_{+1}}\left[z.1-\nu_{0}\right]^{-1}dz$ and $p_{-}\left(\nu_{0}\right) = \frac{1}{2i\pi}\int_{\gamma_{-1}}\left[z.1-\nu_{0}\right]^{-1}dz$. The holomorphic functional calculus shows that $p_{+}\left(\nu_{0}\right)$ and $p_{-}\left(\nu_{0}\right)$ are projection operators on the following spaces respectively:

$$E_{+} = \left\{x \in B_{\mathbb{C}}^{n} \oplus {}^{t}B_{\mathbb{C}}^{n} : \nu_{0}\left(x\right) = x\right\}$$

and

$$E_{-} = \left\{x \in B_{\mathbb{C}}^{n} \oplus {}^{t}B_{\mathbb{C}}^{n} : \nu_{0}\left(x\right) = -x\right\}.$$

Moreover, $p_{+}\left(\nu_{0}\right) + p_{-}\left(\nu_{0}\right) = 1$ in $End\left(B_{\mathbb{C}}^{n} \oplus {}^{t}B_{\mathbb{C}}^{n}\right)$; composing by ν_{0}, we get $\nu_{0} \circ p_{+}\left(\nu_{0}\right) + \nu_{0} \circ p_{-}\left(\nu_{0}\right) = \nu_{0}$ that is, $p_{+}\left(\nu_{0}\right) - p_{-}\left(\nu_{0}\right) = \nu_{0}$. On the other hand, we have $p_{+}\left(\nu_{0}\right) - p_{-}\left(\nu_{0}\right) = F\left(\nu_{0}\right)$; therefore, we get $F\left(\nu_{0}\right) = \nu_{0}$. Since F is continuous, if we choose a ball $B\left(\nu_{0}, \eta\right)$ with radius η small enough, we can find a neighborhood $U' \subset U$ of ν_{0} such that $F\left(U'\right) \subset B\left(\nu_{0}, \eta\right)$. If we put $E_{A} = End\left(A_{\mathbb{C}}^{n} \oplus {}^{t}A_{\mathbb{C}}^{n}\right)$, which is not empty, and $E_{B} = End\left(B_{\mathbb{C}}^{n} \oplus {}^{t}B_{\mathbb{C}}^{n}\right)$, then $U' \cap E_{B} \neq \varnothing$. It follows, since E_{A} is dense in E_{B}, that $U' \cap E_{A} \neq \varnothing$. Thus let us take $\nu_{1} \in U' \cap E_{A}$, assuming that it is self-adjoint (otherwise, we would take $\frac{1}{2}\left(\nu_{1} + \nu_{1}^{*}\right)$).

Lemma 6 $F\left(\nu_{1}\right) \in E_{A}$.

Proof Since the inclusion of $E_{A} \oplus iE_{A}$ in $E_{B} \oplus iE_{B}$ is continuous, the integral $F\left(\nu_{1}\right)$ is the same in $E_{A} \oplus iE_{A}$ and $E_{B} \oplus iE_{B}$; moreover, $\overline{F\left(\nu_{1}\right)} = \frac{-1}{2i\pi}\int_{\gamma_{+1}\cup\gamma_{-1}} f\left(z\right)\left[\overline{z}.1-\nu_{1}\right]^{-1}d\overline{z}$; recall that $\overline{\nu_{1}} = \nu_{1}$ since $\nu_{1} \in E_{B} + i0$. If we put $\overline{z} = \xi$ and noting that complex conjugation reverses the orientations on $\gamma_{\pm 1}$, we have $\overline{F\left(\nu_{1}\right)} = \frac{-1}{2i\pi}\int_{-\left(\gamma_{+1}\cup\gamma_{-1}\right)} f\left(\xi\right)\left[\xi.1-\nu_{1}\right]^{-1}d\xi = F\left(\nu_{1}\right)$. □

Hence $F\left(\nu_{1}\right)$ is an involution on $H\left(A^{n}\right)$; it is self-adjoint since we have, putting $\psi = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$:

$$\begin{aligned} F\left(\nu_{1}\right)^{*} &= \tfrac{1}{2i\pi}\psi^{-1} \circ \left(\int_{\gamma_{+1}\cup\gamma_{-1}} f\left(z\right)\ {}^{t}\left[z.1-\nu_{1}\right]^{-1}dz\right) \circ \psi \\ &= \tfrac{1}{2i\pi}\int_{\gamma_{+1}\cup\gamma_{-1}} f\left(z\right)\left[z.1-\psi^{-1} \circ {}^{t}\nu_{1} \circ \psi\right]^{-1}dz \end{aligned}$$

However, $\psi^{-1} \circ {}^{t}\nu_{1} \circ \psi = \nu_{1}^{*} = \nu_{1}$; so $F\left(\nu_{1}\right)^{*} = F\left(\nu_{1}\right)$.

We finish the proof by noting that, in Figure 5, $\beta = 1 + \nu_{0} \circ F\left(\nu_{1}\right)$ is an isometry and the diagram in Figure 5 is commutative:

Therefore, we have a self-adjoint involution $F\left(\nu_{1}\right)$ on $H\left(A^{n}\right)$ whose image by $\otimes B$ is isometric to the initial involution ν_{0} proving the surjectivity of $\widetilde{T}$.

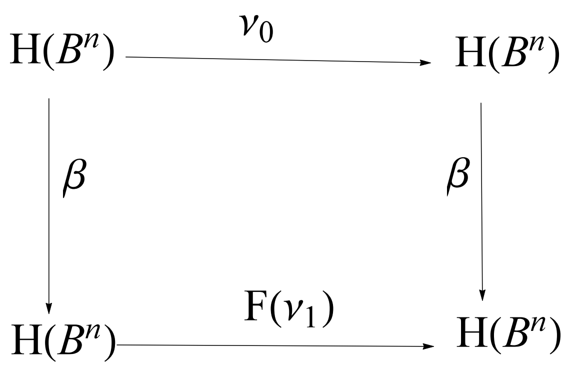

Figure 5. ν_0 and $F(\nu_1)$ are isometric.

5. Application

Let $T^n = S^1 \times \cdots \times S^1$ (n factors) be the n-dimensional torus endowed with the continuous involution $x = \left(e^{i\theta_1}, \cdots, e^{i\theta_n}\right) \rightsquigarrow \overline{x} = \left(e^{-i\theta_1}, \cdots, e^{-i\theta_n}\right)$ and $\mathcal{C} = C(T^n, \mathbb{C})$ the complex Banach algebra of continuous functions on T^n with the sup-norm. We endow $\mathcal{C}$ with the antiinvolution $\widetilde{f}(x) = \overline{f(\overline{x})}$; let us put $\mathcal{C}_0 = \left\{f \in C : \widetilde{f} = f\right\}$ and $\mathcal{C}_1 = \left\{f \in C : \widetilde{f} = -f\right\}$. $\mathcal{C}_0$ is a real Banach algebra and $\mathcal{C}_1 = i\mathcal{C}_0$ is a $\mathcal{C}_0$-module of rank 1. If $f \in \mathcal{C}$, let $f(x) = \sum_{p \in \mathbb{Z}^n} c_p e^{ip\theta}$ be its Laurent series, where $p = (p_1, \cdots p_n) \in \mathbb{Z}^n, x = e^{i\theta} = \left(e^{i\theta_1}, \cdots, e^{i\theta_n}\right)$, and $e^{ip\theta} = \left(e^{ip_1\theta_1}, \cdots, e^{ip_n\theta_n}\right)$. We have $\widetilde{f}(x) = \overline{f(\overline{x})} = \overline{\sum_{p \in \mathbb{Z}^n} c_p e^{-ip\theta}} = \sum_{p \in \mathbb{Z}^n} \overline{c_p} e^{ip\theta}$. If $f \in \mathcal{C}_0$, we must have $c_p = \overline{c_p}, \forall p \in \mathbb{Z}^n$, i.e. they are real. Let $\mathcal{C}^r$, $r \in \mathbb{N}^*$, be the complex Banach algebra of complex functions on T^n of class $\mathcal{C}^r$ with the norm $\|f\| = \sum_{1 \leq |\alpha| \leq r} \left(\|\partial_z^\alpha f\| + \|\|\partial_{\overline{z}}^\alpha f\|\|\right)$, where, for example, $\|\partial_z^\alpha f\| = \sup_{x \in T^n} \left|\frac{\partial^\alpha f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(x)\right|$. Of course, $\mathcal{C}^r$ is dense in $\mathcal{C}$; let us put $\mathcal{C}_0^r = \mathcal{C}_0 \cap \mathcal{C}^r$, which is a real Banach algebra, dense in $\mathcal{C}_0$. Let $f \in \mathcal{C}_0^r$, for $r \geq 2$; its Fourier coefficients satisfy the relations $c_p(f) = (-1)^{|\alpha|} \frac{1}{(ip)^\alpha} c_p(D^\alpha f)$, where $\alpha = (\alpha_1, \cdots \alpha_n) \in \mathbb{N}^n, |\alpha| = \sum \alpha_i = r$ and $(ip)^\alpha = \prod_{j=1,\ldots,n} (ip_j)^{\alpha_j}$. Thus, if $r \geq 2$, the Laurent series $\sum_{p \in \mathbb{Z}^n} c_p(f) z^p$ of $f \in \mathcal{C}_0^r$ is absolutely summable; let us write $\widehat{\mathbb{R}_n}$ for the real Banach algebra of absolutely summable Laurent series $\sum_{p \in \mathbb{Z}^n} c_p(f) z^p$ with real coefficients. We have a dense and continuous inclusion of $\mathcal{C}_0^r$ in $\widehat{\mathbb{R}_n}$, and the two pairs $\left(\mathcal{C}_0^r, \widehat{\mathbb{R}_n}\right)$ and $(\mathcal{C}_0^r, \mathcal{C}_0)$ satisfy the hypotheses of the density theorem; we conclude that ${}_\varepsilon L\left(\widehat{\mathbb{R}_n}\right)$ and ${}_\varepsilon L(\mathcal{C}_0)$ are isomorphic.

Further work [2], necessitating the definition of algebraic and topological Real ${}_\varepsilon L$-theories, gives the following results:

1. For $\varepsilon = 1$, for the two possible involutions, we get $\left\{ \begin{array}{l} KR(T^n) \oplus KR(T^n) \\ KO_{\mathbb{Z}_2}(T^n) \end{array} \right.$.

2. For $\varepsilon = -1$, ${}_{-1}L(A)$ and ${}_{-1}L(X, A)$ are new invariants, not expressible in term of usual K-theories.

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