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Some properties of e-symmetric rings

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Abstract: In this paper, we first give some characterizations of e-symmetric rings. We prove that R is an e-symmetric ring if and only if $a_1a_2a_3 = 0$ implies that $a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}e = 0$, where σ is any transformation of $\{1, 2, 3\}$. With the help of the Bott–Duffin inverse, we show that for $e \in ME_l(R)$, R is an e-symmetric ring if and only if for any $a \in R$ and $g \in E(R)$, if a has a Bott–Duffin (e, g)-inverse, then $g = eg$. Using the solution of the equation $axe = c$, we show that for $e \in ME_l(R)$, R is an e-symmetric ring if and only if for any $a, c \in R$, if the equation $axe = c$ has a solution, then $c = ec$. Next, we study the properties of e-symmetric $*$-rings. Finally we discuss when the upper triangular matrix ring $T_2(R)$ (resp. $T_3(R, I)$) becomes an e-symmetric ring, where $e \in E(T_2(R))$ (resp. $e \in E(T_3(R, I))$).

Key words: e-Symmetric ring, $*$-ring, left semicentral, left min-abel ring, Bott–Duffin inverse, upper triangular matrix ring

1. Introduction

Throughout this paper, all rings are associative with unity. For a ring R, $T_2(R)$ denotes the 2×2 upper triangular matrix ring over R, and $E(R)$, $U(R)$, $Z(R)$, and $N(R)$ denote the set of all idempotents, the set of all invertible elements, the center of R, and the set of all nilpotent elements of R, respectively. An element $e \in E(R)$ is called left minimal idempotent of R if Re is a minimal left ideal of R. Write $ME_l(R)$ to denote the set of all left minimal idempotents of R. An idempotent e of a ring R is called left (right) semicentral $ae = eae$ $(ea = eae)$ for each $a \in R$. A ring R is called (strongly) left min-abel [10] if either $ME_l(R) = \emptyset$ or every element e in $ME_l(R)$ is (right) left semicentral.

A ring R is symmetric [5] if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. The study of symmetric rings also can be found in [6]. Symmetric rings were generalized by Ouyang and Chen to weak symmetric rings in [8]; that is, a ring R is said to be weak symmetric if for all $a, b, c \in R$, if $abc \in N(R)$, then $acb \in N(R)$. Following [3], a ring R is called central symmetric if for any $a, b, c \in R, abc = 0$ implies $bac \in Z(R)$. Central symmetric rings are another form of generalization of symmetric rings. In [11], Wei introduced generalized weakly symmetric rings, which further generalized the concept of symmetric rings. In [7], a ring R is called (strongly) e-symmetric if for any $a, b, c \in R$, $abc = 0$ implies $(aceb = 0)$ $acbe = 0$, where $e \in E(R)$. It is shown that a ring R is e-symmetric if and only if e is left semicentral and eRe is symmetric [7, Theorem 2.2]. In [7, Theorem 3.1], it was shown that a ring R is strongly e-symmetric if and only if e is central and eRe

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is symmetric. Also, using e-symmetric rings, we gave some new characterizations of left min-abel rings in [10] and [11].

This paper is organized as follows. In Section 2, we first discuss many properties of e-symmetric rings and strongly e-symmetric rings. Then, with the help of e-symmetric rings, we give some characterizations of left min-abel rings. In Section 3, we study the e-symmetricity of $*$-rings. We show that for a $*$-ring R, if R is e-symmetric and $1+(e^*-e)^*(e^*-e)\in U(R)$, then R is strongly e-symmetric and e is a projection. In Section 4, we discuss when the upper triangular matrix ring $T_2(R)$ (resp. $T_3(R,I)$) becomes an e-symmetric ring, where $e\in E(T_2(R))$ (resp. $e\in E(T_3(R,I))$).

2. Some characterizations of e-symmetric rings

Proposition 2.1 *The following conditions are equivalent for a ring R:*
(1) R is an e-symmetric ring;
(2) $abc=0$ implies $bace=0$ for all $a,b,c\in R$.

Proof (1) $\Rightarrow$ (2) Since R is an e-symmetric ring, by [7, Theorem 2.2], e is left semicentral. Let $a,b,c\in R$ and satisfy $abc=0$. Then we have $1a(bc)=0$, $1bcae=0$; that is, $bcae=0$. Again, the e-symmetricity of R gives that $b(ae)ce=0$. Noting that e is left semicentral, then we get $bace=0$.

(2) $\Rightarrow$ (1) Let $x\in R$. We have $xe(1-e)e=0$; by hypothesis, one obtains $(1-e)xeee=0$, and it follows that $(1-e)Re=0$. Thus, e is left semicentral. By (2) we know that eRe is a symmetric ring. By [7, Theorem 2.2], R is an e-symmetric ring. □

By Proposition 2.1, we get the following corollaries.

Corollary 2.2 *Let R be an e-symmetric ring. If $abc=0$, then we have*
(1) $bace=0$; (2) $cabe=0$; (3) $cbae=0$.

Corollary 2.3 *R is an e-symmetric ring if and only if for any $a_1,a_2,a_3\in R$, $a_1a_2a_3=0$ implies that $a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}e=0$, where σ is any transformation of $\{1,2,3\}$.*

Let $e,g\in E(R)$. If $Re\cong Rg$ as left R-modules, then we say e and g are left isomorphic. Similarly, if $eR\cong gR$ as right R-modules, then we say e and g are right isomorphic.

Theorem 2.4 *Let R be an e-symmetric ring.*
(1) If g and e are left isomorphic, then R is a g-symmetric ring.
(2) If g and e are right isomorphic, then R is a g-symmetric ring.
(3) If g and e are left isomorphic, then $eR=gR$.

Proof Since R is an e-symmetric ring, by [7, Theorem 2.2], e is left semicentral.

(1) Let $\sigma:Re\to Rg$ be the left R-module isomorphism and $g=\sigma(xe)$ where $x\in R$; then $eg=e\sigma(xe)=\sigma(exe)=\sigma(xe)=g$. Let $a,b,c\in R$ and satisfy $abc=0$. Then $acbe=0$ (since R is an e-symmetric ring), so we have $acbg=acbeg=0$. Thus, R is a g-symmetric ring.

(2) Let $\tau:eR\to gR$ be the right R-module isomorphism. Then there exist $x,y\in R$ such that $\tau(e)=gx$ and $\tau(ey)=g$, so we have $g=\tau(e)y=gxy$. Let $f=ygx$. Then

$$f^2=ygxygx=yg^2x=ygx=f,$$

$$ef = eygx = \tau^{-1}(g)gx = \tau^{-1}(gx) = e,$$

$$fe = ygxe = y\tau(e)e = y\tau(e) = ygx = f,$$

and so $Re = Rf$, and e and f are left isomorphic. By (1), R is an f-symmetric ring. Then by [7, Theorem 2.2], f is left semicentral. Therefore,

$$g = g^2 = gxygxy = gxfy = fgxfy = fg.$$

Let $a, b, c \in R$ and satisfy $abc = 0$; then $acbf = 0$ (since R is an f-symmetric ring). We have $acbg = acbfg = 0$. Thus, R is a g-symmetric ring.

(3) Since g and e are left isomorphic, by (1), R is a g-symmetric ring. Hence, g is left semicentral by [7, Theorem 2.2]. Observing the proof of (1), we have $e = ge$ and $g = eg$, and this gives $eR = gR$. □

Corollary 2.5 *Let R be a strongly e-symmetric ring.*
(1) If g and e are left isomorphic, then $e = g$.
(2) If g and e are right isomorphic, then $e = g$.

Proof Since R is a strongly e-symmetric ring, by [7, Theorem 3.1], e is a central element and R is e-symmetric.

(1) If g and e are left isomorphic, then $eR = gR$ by Theorem 2.4(3). Hence, $g = eg$ and $ge = e$. Noting that e is central, then $g = ge = e$.

(2) If g and e are right isomorphic, then the proof of Theorem 2.4(2) implies that $eR = gR$, and by (1), we know that $e = g$. □

Let R be a ring and $a \in R$ and $e, f \in E(R)$. If there exists an element $y \in R$ satisfying

$$y = ey = yf,\ yae = e,\ fay = f,$$

then y is called a Bott–Duffin (e, f)-inverse of a (see [2]). If y exists, then it is unique. Denote it by $a_{BD}^{(e,f)}$.

Proposition 2.6 *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a has a Bott–Duffin (e, f)-inverse y, then:*

(1) R is f-symmetric and $eR = fR$;

(2) $y_{BD}^{(e,f)} = eaf$.

Proof (1) Since a has a Bott–Duffin $(e, f)-$inverse y, $y = ey = yf$, $yae = e$, and $fay = f$. Noting that R is e-symmetric, then e is left semicentral by [7, Theorem 2.2], so $f = fay = fa(ey) = e(faey) = ef$, and this implies that R is f-symmetric. Hence, f is left semicentral, and it follows that $e = yae = (yf)ae = (fyf)ae = f(yfae) = fe$. Therefore, $eR = fR$.

(2) Noting that e and f are left semicentral, then $eafye = eafeye = eaeye = eaye = (fe)aye = faye = fe = e$ and $fyeaf = feyeaf = eyeaf = yeaf = yefaf = yfaf = yaf = yaef = ef = f$. Then $y_{BD}^{(e,f)} = eaf$. □

Proposition 2.7 *Let R be an e-symmetric ring and $f \in E(R)$. If R satisfies one of the following conditions, then R is f-symmetric:*

(1) $eR + (1-f)R = R$*;*

(2) $ea + 1 - f \in U(R)$ *for some* $a \in R$*;*

(3) $Re + R(1-f) = R$*;*

(4) $ae + 1 - f \in U(R)$ *for some* $a \in R$*.*

Proof (1) Since R is e-symmetric, e is left semicentral by [7, Theorem 2.2]. Noting that $eR+(1-f)R = R$, then $fR = feR = efeR \subseteq eR$, and it follows that $f = ef$. The proof of Theorem 2.4(1) implies that R is f-symmetric.

(2) Set $ea + 1 - f = u \in U(R)$. Then $fu = fea$ and one obtains $f = feau^{-1}$. Noting that e is left semicentral, then $f = ef$, and this gives that R is f-symmetric.

(3) If $Re + R(1-f) = R$, then $Rf = Ref$. Set $f = cef$ for some $c \in R$. Then $f = ecef = ef$ because e is left semicentral. Therefore, R is f-symmetric.

(4) Set $ae + 1 - f = v \in U(R)$. Then $fv = fae$ and one obtains $f = faev^{-1}$. Noting that e is left semicentral, then $f = ef$, so R is f-symmetric. □

Proposition 2.8 *A ring* R *is a strongly left min-abel ring if and only if for* $e \in ME_l(R)$ *and* $a, b \in R$*,* $e = eab$ *implies that* $e = eba$*.*

Proof (⇒) Assume that R is strongly left min-abel and $e = eab$. Then e is central, and it follows that $e = ee = eabeab = ea(eba)b$. This implies that $eba \neq 0$, and one has $Re = Reba$. Set $e = ceba$ for some $c \in R$. Noting that $e = eab$, then $be = beab = ebab$. This gives $eba = bea = ebaba$, so $e = ceba = (ceba)ba = eba$.

(⇐) Let $e \in ME_l(R)$ and $x \in R$. Set $g = e + ex(1-e)$. Then $eg = g$, $ge = e$, and $g^2 = g \in ME_l(R)$. Since $e = ege$, by hypothesis, $e = eeg = eg = g$. Thus, $ex(1-e) = g - e = 0$ for each $x \in R$, and this gives that e is right semicentral. By [7, Lemma 3.3], e is central. Hence, R is strongly left min-abel. □

Theorem 2.9 *Let* $e \in ME_l(R)$*. Then* R *is an* e*-symmetric ring if and only if for any* $a \in R$ *and* $g \in E(R)$*, if* a *has a Bott–Duffin* (e, g)*-inverse, then* $g = eg$*.*

Proof (⇒) Let R be an e-symmetric ring. Then e is left semicentral. Assume that $a \in R$ and $g \in E(R)$ and a has a Bott–Duffin (e, g)-inverse. Letting $a_{BD}^{(e,g)} = y$, then $y = ey = yg$ and

$$g = gay = gaey = egaey = eg.$$

(⇐) First we prove that e is left semicentral. For any $x \in R$, set $g = e + (1-e)xe$; then $eg = e, ge = g, g^2 = g$. Obviously, e is a Bott–Duffin (e, g)-invertible element and $e_{BD}^{(e,g)} = e$. By hypothesis $g = eg = e$, and then $(1-e)xe$ for any $x \in R$. Thus, e is left semicentral.

Next, we prove that eRe is a symmetric ring. Any $a, b, c \in eRe$ satisfy $abc = 0$. Assuming that $acb \neq 0$, then $a \neq 0$ and $b \neq 0$, and so we have $Ra = Re = Rb$. Let $e = ra = sb$ for some $r, s \in R$; then $acb = aecb = asbcb = asebcb = asrabcb = 0$, which is a contradiction. Thus, eRe is a symmetric ring, and hence R is an e-symmetric ring by [7, Theorem 2.2]. □

Proposition 2.10 *Let* $e \in ME_l(R)$*. Then* R *is an* e*-symmetric ring if and only if for any* $a \in R$ *and* $g \in E(R)$*, if* a *has a Bott–Duffin* (e, g)*-inverse, then* $e = ge$*.*

Proof ($\Rightarrow$) From Theorem 2.9, we know $g = eg$. Since $e \in ME_l(R)$, $g \in ME_l(R)$, and R is a g-symmetric ring, g is left semicentral. Thus, $e = yae = ygae = gygae = ge$, where $y = a_{BD}^{(e,g)}$.

($\Leftarrow$) The proof is similar to Theorem 2.9. □

Let R be a ring and $a \in R$. If there exists $b \in R$ such that $a = aba$, then a is called a regular element of R and b is called an inner inverse. Clearly, if b exists, it is not unique. We denote by $a\{1\}$ the set of all inner inverses of a regular element a. Let $b \in a\{1\}$. Then $ab, ba \in E(R)$.

Proposition 2.11 *Let a be a regular element of R and $b \in a\{1\}$. If R is ab-symmetric, then R is ba-symmetric.*

Proof Since $b \in a\{1\}$, we have $a = aba$. Let $e = ab$ and $g = ba$; then $e, g \in E(R)$ and $ea = a = ag$. Denote $\sigma : Re \to Ra$ by $\sigma(re) = rea$ for any $r \in R$. It is easy to prove that σ is a left R-module isomorphism. Since $Ra = Rg$, we have $Re \cong Rg$ as left R-modules. By hypothesis, R is an e-symmetric ring, and thus R is a g-symmetric ring by Theorem 2.4. That is, R is a ba-symmetric ring. □

Lemma 2.12 *Let $a, b \in R$ and $e \in ME_l(R)$ satisfy $abe = e$. If e is left semicentral, then $e = bae$.*

Proof Since $abe = e$ and e is left semicentral, we have $e = aebe$. Then $Re = Rae$. Letting $e = cae$ for some $c \in R$, then $ce = c(abe) = caebe = ebe = be$ and $bae = beae = ceae = cae = e$. □

Lemma 2.13 *Let $e \in ME_l(R)$. If e is left semicentral, then eRe is a symmetic ring.*

Proof Let $a, b, c \in eRe$ and satisfy $abc = 0$. If $acb \neq 0$, then $Racb = Re$, so $e = dacbe$ for some $d \in R$. By Lemma 2.12, $e = bdace = cbdae$. Thus, $e = dacbe = daecbe = da(bdace)cbe = dabedacecbe = dab(cbdae)dacecbe = d(abc)bdaedacecbe = 0$, which is a contradiction. Hence, $acb = 0$ and so eRe is a symmmetic ring. □

Proposition 2.14 *Let $e \in ME_l(R)$. Then R is an e-symmetric ring if and only if for any $a \in R$ either $aRe = 0$ or the equation $axe = e$ has a solution.*

Proof ($\Rightarrow$) Since R is an e-symmetric ring, e is left semicentral. Let $a \in R$. If $aRe \neq 0$, then $abe \neq 0$ for some $b \in R$. Thus, $Rabe = Re$. Set $e = dabe$ for some $d \in R$. By Lemma 2.12, $e = abde$. Hence, $x = bd$ is a solution of the equation $axe = e$.

($\Leftarrow$) Let $e \in ME_l(R)$. If $(1-e)Re \neq 0$, then by hypothesis we know $(1-e)xe = e$ has a solution. However, $(1-e)xe = e$ does not have a solution and that is a contradiction. Thus, $(1-e)Re = 0$, e is left semicentral. By Lemma 2.13, eRe is a symmetric ring. Hence, R is an e-symmetric ring by [7, Theorem 2.2]. □

Theorem 2.15 *Let $e \in ME_l(R)$. Then R is an e-symmetric ring if and only if for any $a, c \in R$, if the equation $axe = c$ has a solution, then $c = ec$.*

Proof ($\Rightarrow$) Since R is an e-symmetric ring, e is left semicentral. If the equation $axe = c$ has a solution $x = b$, then $c = abe = eabe = ec$.

($\Leftarrow$) For any $a \in R$, denote $h = (1-e)ae$. If $h \neq 0$, then $Rh = Re$. Let $e = ch$ for some $c \in R$. Then $h(ch)e = hch = he = h$. Thus, the equation $hxe = h$ has a solution, and then $h = eh = e(1-e)ae = 0$, which is a contradiction. Then $(1-e)ae = 0$ for any $a \in R$. Hence, e is left semicentral. By Lemma 2.13, eRe is a symmetric ring. Hence, R is an e-symmetric ring. □

3. Symmetricity of $*$-rings

An involution $a \longmapsto a^*$ in a ring R is an antiisomorphism of degree 2; that is,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*.$$

A ring R with an involution $*$ is called a $*$-ring (see [1]).

Let R be a $*$-ring and $e \in E(R)$. If $e^* = e$, then e is called projection.

Let R be a ring and $e \in E(R)$. R is called left e-reflexive if $aRe = 0$ implies $eRa = 0$ for any $a \in R$.

Proposition 3.1 *(1) R is strongly e-symmetric if and only if R is e-symmetric and left e-reflexive.*

(2) If e is a projection element of a $*$-ring R, then R is strongly e-symmetric if and only if R is e-symmetric.

Proof (1) ($\Rightarrow$) Assume that $aRe = 0$. Since R is a strongly e-symmetric ring, by [7, Theorem 3.1], e is a central element. Then we get $eRa = eRae = 0$. Thus, R is left e-reflexive.

($\Leftarrow$) Suppose that R is e-symmetric and left e-reflexive; by [7, Theorem 2.2], e is left semicentral. Then we have $(1-e)Re = 0$. Since R is left e-reflexive, we have $eR(1-e) = 0$, so e is a central element. Thus, R is a strongly e-symmetric ring by [7, Theorem 3.1].

(2) Noting that a projection element e in a $*$-ring R is left semicentral if and only if it is central, (2) holds. □

Proposition 3.2 *Let R be a $*$-ring and $e \in E(R)$. If R is an e-symmetric ring, then:*

(1) e^*e is an idempotent element;

(2) the following conditions are equivalent:

(a) R is e^*e-symmetric,

(b) for each $x \in R$, $e^*xe = xe^*e$,

(c) e^*e is central,

(d) $ee^*e = e^*e$.

Proof (1) Since R is an e-symmetric ring, e is left semicentral, and it follows that $(e^*e)^2 = e^*ee^*e = e^*e^*e = e^*e$.

(2) $(a) \Longrightarrow (c)$ By (1), we know that e^*e is a projection. Since R is e^*e-symmetric, by Proposition 3.1(2), R is strongly e^*e-symmetric, and by [7, Theorem 3.1], e^*e is central.

$(c) \Longrightarrow (b)$ For each $x \in R$, by (c), we have $e^*ex = xe^*e$, and this gives $xe^*e = e^*exe$. Noting that e is left semicentral, $xe^*e = e^*xe$.

$(b) \Longrightarrow (d)$ Choose $x = e$; we are done.

$(d) \Longrightarrow (a)$ Let $a, b, c \in R$ and satisfy $abc = 0$. Since R is e-symmetric, $acbe = 0$, and this leads to $acbe^*e = acbee^*e = 0$. Hence, R is e^*e-symmetric. □

Proposition 3.3 *Let R be a $*$-ring. If R is e-symmetric, then the following conditions are equivalent:*

(1) $ee^* \in E(R)$;

(2) $xee^* = e^*xe$ for each $x \in R$;

(3) $ee^* = e^*e$;

(4) ee^* is central.

Proof (1) $\Longrightarrow$ (2) Since R is an e-symmetric ring and $ee^* \in E(R)$, R is ee^*-symmetric, and it follows that ee^* is left semicentral. Hence, $xee^* = ee^*xee^*$ for each $x \in R$. Noting that e is left semicentral and e^* is right semicentral, then $xee^* = e^*xe$.

(2) $\Longrightarrow$ (3) Choose $x = e$. Then, by (2), we have $ee^* = e^*e$.

(3) $\Longrightarrow$ (4) Since R is e-symmetric and $ee^* = e^*e$, R is e^*e-symmetric, and by Proposition 3.2(2), e^*e is central. Hence, ee^* is central.

(4) $\Longrightarrow$ (1) Trivial. □

Theorem 3.4 *Let R be a $*$-ring and an e-symmetric ring. If $1 + (e^* - e)^*(e^* - e) \in U(R)$, then R is a strongly e-symmetric ring and e is a projection.*

Proof Set $u = 1 + (e^* - e)^*(e^* - e)$ and $v = u^{-1}$. Then $u^* = u$, $eu = ee^*e = ue$, and it follows that $ev = ve$ and $v^* = v$, so $e^*v = ve^*$. Choose $f = ee^*v = vee^*$. Then $f^2 = (vee^*)(ee^*v) = v(ee^*e)e^*v = veue^*v = evue^*v = ee^*v = f$ and $f^* = f$, and this gives that f is a projection. Since R is e-symmetric and $f = ef$, R is f-symmetric. By Proposition 3.1(2), R is strongly f-symmetric, so f is central and it follows that $f = ef = fe = vee^*e = vue = e$. Hence, e is projection and R is a strongly e-symmetric ring. □

Let R be a $*$-ring and $e \in E(R)$. $p \in R$ is said to be a range projection [4] if p is a projection satisfying $pe = e$ and $ep = p$. The range projection of e is denoted by $e^{\perp}$.

Proposition 3.5 *Let R be a $*$-ring. If R is e-symmetric, then the following conditions are equivalent:*

(1) $1 + (e^* - e)^*(e^* - e) \in U(R)$;

(2) $e + e^* - 1 \in U(R)$;

(3) $e^{\perp}$ exists.

Proof (1) $\Longrightarrow$ (2) By Theorem 3.4, e is a projection. Hence, $e + e^* - 1 = 2e - 1 \in U(R)$.

(2) $\Longrightarrow$ (3) Follows from [4, Theorem 2.1].

(3) $\Longrightarrow$ (1) Let $p = e^{\perp}$. Then $ep = p$. Noting that R is e-symmetric, then R is $p-$symmetric, and by Proposition 3.1, p is central. It follows that $e = pe = ep = p$. Hence, $1 + (e^* - e)^*(e^* - e) = 1 \in U(R)$. □

An element $a^{\dagger}$ in a $*$-ring R is called the Moore–Pensor inverse (or MP-inverse) of a [9] if

$$aa^{\dagger}a = a,\ a^{\dagger}aa^{\dagger} = a^{\dagger},\ aa^{\dagger} = (aa^{\dagger})^*,\ a^{\dagger}a = (a^{\dagger}a)^*.$$

In this case, we call a MP-invertible in R. The set of all MP-invertible elements of R is denoted by $R^{\dagger}$.

Corollary 3.6 *Let R be a $*$-ring and an e-symmetric ring. Then $e \in R^{\dagger}$ if and only if e is a projection.*

Proof ($\Rightarrow$) Assume that $e \in R^{\dagger}$. By [4, Theorem 3.1], $e + e^* - 1 \in U(R)$. By Proposition 3.5, $1 + (e^* - e)^*(e^* - e) \in U(R)$, and by Theorem 3.4, e is a projection.

($\Leftarrow$) Suppose that e is a projection. Then $e + e^* - 1 = 2e - 1 \in U(R)$. By [4, Theorem 3.1], $e \in R^{\dagger}$. □

Theorem 3.7 *Let R be a $*$-ring and $a \in R^{\dagger}$. If R is $aa^{\dagger}$-symmetric, then a is EP.*

Proof Note that R is $aa^{\dagger}$-symmetric and $aa^{\dagger}$ is projection. Hence, by Proposition 3.1(2), R is strongly $aa^{\dagger}$-symmetric, and it follows that $aa^{\dagger}$ is central from [7, Theorem 3.1]. This gives that $a = (aa^{\dagger})a = a(aa^{\dagger}) = a^2a^{\dagger}$. Noting that $Ra = R(a^{\dagger}a)$ and $Raa^{\dagger} \cong Ra$ as left R-module, then $Raa^{\dagger} \cong Ra^{\dagger}a$ as left R-module. Since R is $aa^{\dagger}$-symmetric, it follows that R is $a^{\dagger}a$-symmetric from Theorem 2.4. Noting that $a^{\dagger}a$- is projection, then $a^{\dagger}a$ is central, which implies that $a = a(a^{\dagger}a) = a^{\dagger}a^2$. Hence, $a \in a^2R \cap Ra^2$, and one obtains that $a \in R^{\sharp}$ and so $a^{\sharp}$ exists. Now we have $a^{\sharp}a = a^{\sharp}a^2a^{\dagger} = aa^{\dagger}$; hence, a is EP. □

4. Upper triangular matrix ring

Proposition 4.1 *Let R be a ring and $e \in E(R), r \in R$. Then we have the following results:*

(1) $T_2(R)$ is a $\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}$-symmetric ring if and only if R is a symmetric ring.

(2) $T_2(R)$ is a $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$-symmetric ring if and only if R is an e-symmetric ring.

(3) $T_2(R)$ is a $\begin{pmatrix} e & e \\ 0 & 0 \end{pmatrix}$-symmetric ring if and only if R is an e-symmetric ring.

Proof (1) ($\Rightarrow$) Let $a, b, c \in R$ and satisfy $abc = 0$. Then we have

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $T_2(R)$ is a $\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}$-symmetric ring, we have

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

this is $\begin{pmatrix} acb & acbr \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then we get $acb = 0$, and so R is a symmetric ring.

($\Leftarrow$) Let $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$, $C = \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix} \in T_2(R)$, and $ABC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; this is $ABC = \begin{pmatrix} a_1a_2a_3 & * \\ 0 & c_1c_2c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and then we get $a_1a_2a_3 = c_1c_2c_3 = 0$. Since R is a symmetric ring, we have $a_1a_3a_2 = c_1c_3c_2 = 0$, so $ACB\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1a_3a_2 & a_1a_3a_2r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, $T_2(R)$ is a $\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}$-symmetric ring.

Similarly, we can prove (2) and (3). □

Let R be a ring and I an ideal of R,

$$T_3(R,I)=\{\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} | a_1,a_3,a_4,a_5,a_6 \in R \text{ } and \text{ } a_2 \in I\}.$$

Then, by the usual matrix addition and multiplication, $T_3(R,I)$ is a ring.

Proposition 4.2 *Let R be a ring, I an ideal of R, and $e \in E(R)$. Then $T_3(R,I)$ is a $\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}$-symmetric ring if and only if R is an e-symmetric ring and $Ie=0$.*

Proof Let $a \in I$, $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,I)$, $B=\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,I)$, and $C=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,I)$. Then $ABC=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since $T_3(R,I)$ is a $\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}$-symmetric ring , we have $ACB\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & ae & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence, $ae=0$, and so $Ie=0$.

Let $x,y,z \in R$ and satisfy $xyz=0$. Choose $A=\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,I)$, $B=\begin{pmatrix} y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,I)$, and $C=\begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,I)$. Then $ABC=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since $T_3(R,I)$ is a $\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}$-symmetric ring, we have $ACB\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} xzye & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence, $xzye=0$, and R is an e-symmetric ring.

Conversely, let $A=\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in T_3(R,I)$, $B=\begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in T_3(R,I)$, $C=\begin{pmatrix} c_1 & c_2 & c_3 \\ 0 & c_4 & c_5 \\ 0 & 0 & c_6 \end{pmatrix} \in T_3(R,I)$, and $ABC=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have

$$\begin{pmatrix} a_1b_1c_1 & * & * \\ 0 & a_4b_4c_4 & * \\ 0 & 0 & a_6b_6c_6 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and then

$$a_1b_1c_1=a_4b_4c_4=a_6b_6c_6=0.$$

Since R is an e-symmetric ring, we get that $a_1c_1b_1e = a_4c_4b_4e = a_6c_6b_6e = 0$. Since $a_2, b_2, c_2 \in I$, $a_1c_1b_2 + a_1c_2b_4 + a_2c_4b_4 \in I$, by hypothesis $(a_1c_1b_2 + a_1c_2b_4 + a_2c_4b_4)e = 0$. Hence,

$$ACB\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1c_1b_1e & (a_1c_1b_2 + a_1c_2b_4 + a_2c_4b_4)e & 0 \\ 0 & a_4c_4b_4e & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $T_3(R, I)$ is a $\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}$-symmetric ring. □

The following corollary follows from Proposition 4.2.

Corollary 4.3 *$T_3(R, I)$ is a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$-symmetric ring if and only if R is a symmetric ring and $I = 0$.*

Example 4.4 *Let R be a symmetric ring and $I = 0$. Take $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, 0)$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, 0)$, and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, 0)$. Then $ABC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, but $ACB \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This shows that $T_3(R, 0)$ is not a symmetric ring. Similarly, we can prove that for an e-symmetric ring R, $T_3(R, 0)$ need not be a $\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}$-symmetric ring.*

Let R be a ring,

$$WV_3(R) = \{\begin{pmatrix} a & 0 & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} | a, b, c \in R\},$$

$$WT_3(R) = \{\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} | a, b, c, d \in R\}.$$

Then by the usual matrix addition and multiplication, $WV_3(R)$ and $WT_3(R)$ are rings. Obviously, $WV_3(R)$ and $WT_3(R)$ are subrings of $T_3(R, I)$. Similarly, we can prove that Proposition 4.2 and Corollary 4.3 also hold for $WV_3(R)$ and $WT_3(R)$.

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