Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 2400 – 2416
© TÜBİTAK
doi:10.3906/mat-1803-103

On a class of Kazdan–Warner equations

Yu FANG, Mengjie ZHANG*
Department of Mathematics, Renmin University of China, Beijing, P.R. China

Received: 22.03.2018 • **Accepted/Published Online:** 04.07.2018 • **Final Version:** 27.09.2018

Abstract: Let (Σ, g) be a compact Riemannian surface without boundary and $W^{1,2}(\Sigma)$ be the usual Sobolev space. For any real number $p > 1$ and $\alpha \in \mathbb{R}$, we define a functional

$$J_{\alpha,8\pi}(u) = \frac{1}{2}\left(\int_\Sigma |\nabla_g u|^2 dv_g - \alpha(\int_\Sigma |u|^p dv_g)^{2/p}\right) - 8\pi \log \int_\Sigma he^u dv_g$$

on a function space $\mathcal{H} = \left\{u \in W^{1,2}(\Sigma) : \int_\Sigma u dv_g = 0\right\}$, where h is a positive smooth function on Σ. Denote

$$\lambda_{1,p}(\Sigma) = \inf_{u\in\mathcal{H},\, \int_\Sigma |u|^p dv_g=1} \int_\Sigma |\nabla_g u|^2 \mathrm{d}v_g.$$

If $\alpha < \lambda_{1,p}(\Sigma)$ and $J_{\alpha,8\pi}$ has no minimizer on $\mathcal{H}$, then we obtain the exact value of $\inf_{\mathcal{H}} J_{\alpha,8\pi}$ by using a method of blow-up analysis. Hence, if $\inf_{\mathcal{H}} J_{\alpha,8\pi}$ is not equal to that value, then $J_{\alpha,8\pi}|_{\mathcal{H}}$ has a critical point that satisfies a Kazdan–Warner equation. This recovers a recent result of Yang and Zhu (DOI: 10.1007/s11425-017-9086-6).

Key words: Trudinger–Moser inequality, blow-up analysis, Kazdan–Warner equation

1. Introduction and main results

Let (Σ, g) be a compact Riemannian surface without boundary, ∇_g and Δ_g be its respective gradient operator and Laplace–Beltrami operator, dv_g be its volume element, and $W^{1,2}(\Sigma)$ be the usual Sobolev space. We define a function space

$$\mathcal{H} = \left\{u \in W^{1,2}(\Sigma) : \int_\Sigma u dv_g = 0\right\}. \tag{1}$$

Let h be a positive smooth function on Σ and $J_\beta\colon W^{1,2}(\Sigma) \to \mathbb{R}$ be a fuctional defined by

$$J_\beta(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dv_g - \beta \log \int_\Sigma he^u dv_g. \tag{2}$$

In view of a manifold version of the Trudinger–Moser inequality [5], one can see that J_β has lower bound on the space $\mathcal{H}$ for all $\beta \le 8\pi$. Note that critical points of J_β on $\mathcal{H}$ are solutions to Kazdan–Warner equations [6]. In [4], Ding et al. proved that $J_{8\pi}$ must have a minimizer on $\mathcal{H}$ if

$$\inf_{u\in\mathcal{H}} J_{8\pi} \neq -8\pi - 8\pi \log \pi - 4\pi \max_{p\in\Sigma}(A_p + 2\log h(p)), \tag{3}$$

*Correspondence: zhangmengjie@ruc.edu.cn
2010 *AMS Mathematics Subject Classification:* 58J05

where $A_p = \lim_{x\to p}(G_p + 4\log r)$ is a constant, r denotes the geodesic distance between x and p, and G_p is a Green function satisfying

$$\begin{cases} \Delta_g G_p = 8\pi\delta_p - \frac{8\pi}{\mathrm{Vol}_g(\Sigma)} \\ \int_\Sigma G_p dv_g = 0, \end{cases}$$

where δ_p is the usual Dirac measure. Moreover they gave a geometric hypothesis that guarantees (3). Clearly the minimizer of J_β satisfies the following Kazdan–Warner equation:

$$\Delta_g u = \frac{8\pi h e^u}{\int_\Sigma h e^u dv_g} - \frac{8\pi}{\mathrm{Vol_g}(\Sigma)}. \tag{4}$$

There are extensions of Ding et al.'s result. Among these, we mention [10], [11], [19] and [17]. Recently, motivated by a series of works concerning Trudinger–Moser inequalities, some works [1, 9, 12–16], [18] considered the functionals

$$J_{\alpha,\beta}(u) = \frac{1}{2}\left(\int_\Sigma |\nabla_g u|^2 dv_g - \alpha\int_\Sigma u^2 dv_g\right) - \beta\log\int_\Sigma h e^u dv_g \tag{5}$$

and proved that if $\alpha < \lambda_1(\Sigma)$, the first eigenvalue of the Laplace–Beltrami operator with respect to the mean value zero condition, and

$$\inf_{u\in\mathcal{H}} J_{\alpha,8\pi} \neq -8\pi - 8\pi\log\pi - 4\pi\max_{p\in\Sigma}(A_p + 2\log h(p)), \tag{6}$$

where $A_p = \lim_{x\to p}(G_p + 4\log r)$ is a constant, r denotes the geodesic distance between x and p, and G_p is a Green function satisfying

$$\begin{cases} \Delta_g G_p - \alpha G_p = 8\pi\delta_p - \frac{8\pi}{\mathrm{Vol}_g(\Sigma)}, \\ \int_\Sigma G_p dv_g = 0, \end{cases}$$

then $\inf_{u\in\mathcal{H}} J_{\alpha,8\pi}$ can be attained by some function $u \in \mathcal{H}$ satisfying

$$\Delta_g u - \alpha u = \frac{8\pi h e^u}{\int_\Sigma h e^u dv_g} - \frac{8\pi}{\mathrm{Vol_g}(\Sigma)}. \tag{7}$$

Motivated by (5) and [8], we now consider the minimizing problem for the functional $J_{\alpha,\beta}$ defined by

$$J_{\alpha,\beta}(u) = \frac{1}{2}\left(\int_\Sigma |\nabla_g u|^2 dv_g - \alpha(\int_\Sigma |u|^p dv_g)^{2/p}\right) - \beta\log\int_\Sigma h e^u dv_g, \tag{8}$$

where $p > 1$ is a real number. To do this, we define $\lambda_{1,p}(\Sigma)$ by

$$\lambda_{1,p}(\Sigma) = \inf_{u\in\mathcal{H},\, \int_\Sigma |u|^p dv_g = 1} \int_\Sigma |\nabla_g u|^2 \mathrm{d}v_g. \tag{9}$$

By the Poincaré–Sobolev inequality, when $\alpha < \lambda_{1,p}(\Sigma)$, the norm

$$\|u\|_{\alpha,p} = \left(\int_\Sigma |\nabla_g u|^2 dv_g - \alpha(\int_\Sigma |u|^p dv_g)^{2/p}\right)^{1/2} \tag{10}$$

is an equivalent Sobolev norm on $\mathcal{H}$. Our main result reads as follows.

Theorem 1 *Let* (Σ, g) *be a compact Riemannian surface without boundary,* h *be a positive smooth function on* Σ*,* $p > 1$ *be a real number, and* $\mathcal{H}$*,* $\lambda_{1,p}(\Sigma)$ *and* $J_{\alpha,\beta}$ *be defined as in (1), (9), and (8). Then, for any* $\alpha < \lambda_{1,p}(\Sigma)$*, if* $J_{\alpha,8\pi}|_{\mathcal{H}}$ *has no minimizer, then there holds*

$$\inf_{u\in\mathcal{H}} J_{\alpha,8\pi} = -8\pi - 8\pi\log\pi - 4\pi\max_{x_0\in\Sigma}(A_{x_0} + 2\log h(x_0)),$$

where $A_{x_0} = \lim_{x\to x_0}(G_{x_0} + 4\log r)$ *is a constant,* r *denotes the geodesic distance between* x *and* x_0*, and* G_{x_0} *is a Green function satisfying*

$$\begin{cases} \Delta_g G_{x_0} + \zeta(G_{x_0}) = 8\pi\delta_{x_0} + \overline{\zeta(G_{x_0})} - \frac{8\pi}{\mathrm{Vol}_g(\Sigma)}, \\ \int_\Sigma G_{x_0} dv_g = 0, \end{cases} \tag{11}$$

where

$$\zeta(f(x)) = -\alpha\|f(x)\|_p^{2-p}|f(x)|^{p-2}f(x), \tag{12}$$

and $\overline{\zeta(f)} = 1/\mathrm{Vol}_g(\Sigma)\int_\Sigma \zeta(f)(x)dv_g$ *is its integral average on* Σ*.*

For the proof of Theorem 1, we follow the lines of [18], and thereby closely follow [4]. The difference is that in the case $p \neq 2$, the Euler–Lagrange equation u_ϵ satisfies being nonlinear. If this case happens, the maximum principle will become invalid. Fortunately, we can use the capacity estimate to calculate the infimum of $J_{\alpha,8\pi}$. This method was originally used in [7] and then in [11]. However, we cannot extend our results to higher eigenfunction space cases because of the nonlinearity of the Euler–Lagrange equation u_ϵ being satisfied.

An interesting application of Theorem 1 reads as follows.

Corollary 2 *For any* $\alpha < \lambda_{1,p}(\Sigma)$*, if*

$$\inf_{u\in\mathcal{H}} J_{\alpha,\beta}(u) \neq -8\pi - 8\pi\log\pi - 4\pi\max_{x_0\in\Sigma}(A_{x_0} + 2\log h(x_0)), \tag{13}$$

then the Kazdan–Warner equation

$$\triangle_g u + \zeta(u) = \frac{8\pi h e^u}{\int_\Sigma h e^u dv_g} + \overline{\zeta(u)} - \frac{8\pi}{\mathrm{Vol}_g(\Sigma)} \tag{14}$$

has a solution on u_0 *in* $\mathcal{H}$*, where* $\zeta(u)$ *is defined as in (12).*

In the remaining part of this paper, we prove Theorem 1. Throughout this paper, we do not distinguish between sequence and subsequence.

2. Proof of Theorem 1

For some fixed $\alpha < \lambda_{1,p}$, the proof of Theorem 1 will be divided into several steps. For simplicity, we assume $\mathrm{Vol}_g(\Sigma) = \int_\Sigma dv_g = 1$.

Step 1. Minimizers for subcritical functionals.

In this step we shall prove that $\inf_{u\in\mathcal{H}} J_{\alpha,8\pi-\epsilon}(u)$ can be attained for any small $\epsilon > 0$. Precisely, we have for any $0 < \epsilon < 8\pi$ that there exists some function $u_\epsilon \in \mathcal{H}\cap C^1(\Sigma)$ such that

$$J_{\alpha,8\pi-\epsilon}(u_\epsilon) = \inf_{u\in\mathcal{H}} J_{\alpha,8\pi-\epsilon}(u). \tag{15}$$

The proof is based on a direct method in the calculus of variations. However, for proving (15), we should introduce a Trudinger–Moser inequality.

Lemma 3 *Let* (Σ, g) *be a compact Riemannian surface without boundary and* $p > 1$ *be a real number. Then, for any* $0 \leq \alpha < \lambda_{1,p}(\Sigma)$ *and* $0 < \gamma < 4\pi$, *we have the supremum*

$$\sup_{u \in W^{1,2}(\Sigma),\, \int_\Sigma u dv_g = 0,\, \|u\|_{\alpha,p} \leq 1} \int_\Sigma e^{\gamma u^2} dv_g < \infty, \tag{16}$$

where $\lambda_{1,p}(\Sigma)$ *is defined by (9) and the norm* $\|\cdot\|_{\alpha,p}$ *is defined by (10).*

Proof. We refer readers to an argument of [14, p. 3168] to understand the proof of Lemma 3, and we omit the proof here.

□

For any fixed $0 < \epsilon < 8\pi$, we take a sequence of functions $u_j \in \mathcal{H}$ satisfying that

$$J_{\alpha,8\pi-\epsilon}(u_j) \to \inf_{u \in \mathcal{H}} J_{\alpha,8\pi-\epsilon}(u) \tag{17}$$

as $j \to \infty$.

It follows from Young's inequality that

$$\int_\Sigma h e^{u_j} dv_g \leq \int_\Sigma h e^{(4\pi-\epsilon/4)\frac{u_j^2}{\|u_j\|_{\alpha,p}^2} + \frac{\|u_j\|_{\alpha,p}^2}{16\pi-\epsilon}} dv_g. \tag{18}$$

(18) together with (16) and (17) shows that

$$\begin{aligned}
\inf_{u \in \mathcal{H}} J_{\alpha,8\pi-\epsilon}(u) + o_j(1) &= \frac{1}{2}\|u_j\|_{\alpha,p}^2 - (8\pi-\epsilon)\log\int_\Sigma h e^{u_j} dv_g \\
&\geq \frac{1}{2}\|u_j\|_{\alpha,p}^2 - \frac{8\pi-\epsilon}{16\pi-\epsilon}\|u_j\|_{\alpha,p}^2 - (8\pi-\epsilon)\log\int_\Sigma h e^{(4\pi-\epsilon/4)\frac{u_j^2}{\|u_j\|_{\alpha,p}^2}} dv_g \\
&\geq \frac{\epsilon}{32\pi}\|u_j\|_{\alpha,p}^2 - C.
\end{aligned}$$

Therefore, u_j is bounded in $\mathcal{H}$. Thus, we can assume $u_j \rightharpoonup u_\epsilon$ weakly in $\mathcal{H}$, $u_j \to u_\epsilon$ strongly in $L^q(\Sigma)$ for any $q > 0$, and $u_j \to u_\epsilon$ a.e. in Σ. Weak convergence implies

$$\|u_\epsilon\|_{\alpha,p}^2 \leq \lim_{j\to\infty} \|u_j\|_{\alpha,p}^2. \tag{19}$$

An analog of (18) shows that $e^{|u_j|}$ is bounded in $L^s(\Sigma)$ for any $s > 0$, and then we have

$$\lim_{j\to\infty} \int_\Sigma h e^{u_j} dv_g = \int_\Sigma h e^{u_\epsilon} dv_g. \tag{20}$$

(20) together with (19) shows (15). Moreover, by using the method of Lagrange multiplier, we obtain that the

Euler–Lagrange equation u_ϵ satisfies the following:

$$\begin{cases} \Delta_g u_\epsilon - \alpha \|u_\epsilon\|_p^{2-p}|u_\epsilon|^{p-2}u_\epsilon = (8\pi-\epsilon){\lambda_\epsilon}^{-1}he^{u_\epsilon} + \mu_\epsilon \quad \text{in}\,\Sigma, \\ \lambda_\epsilon = \int_\Sigma he^{u_\epsilon}dv_g, \\ \mu_\epsilon = -(8\pi-\epsilon) - \alpha\int_\Sigma \|u_\epsilon\|_p^{2-p}|u_\epsilon|^{p-2}u_\epsilon dv_g, \\ \int_\Sigma u_\epsilon dv_g = 0. \end{cases} \tag{21}$$

Applying elliptic estimates to (21), we have $u_\epsilon \in C^1(\Sigma)$ immediately. We also have $\liminf_{\epsilon\to 0}\lambda_\epsilon > 0$ by using Jensen's inequality. Notice that for any $\gamma > 0$, there exists some $u_\gamma \in \mathcal{H}$ such that

$$\inf_{u\in\mathcal{H}} J_{\alpha,8\pi}(u) + \gamma > J_{\alpha,8\pi}(u_\gamma) = \lim_{\epsilon\to 0} J_{\alpha,8\pi-\epsilon}(u_\gamma) \geq \lim_{\epsilon\to 0} J_{\alpha,8\pi-\epsilon}(u_\epsilon) = \lim_{\epsilon\to 0}\inf_{u\in\mathcal{H}} J_{\alpha,8\pi-\epsilon}(u).$$

Since $\gamma > 0$ is arbitrary, we have

$$\lim_{\epsilon\to 0}\inf_{u\in\mathcal{H}} J_{\alpha,8\pi-\epsilon}(u) \leq \inf_{u\in\mathcal{H}} J_{\alpha,8\pi}(u) \leq \inf_{u\in\mathcal{H}} J_{\alpha,8\pi}(u_\epsilon) = \lim_{\gamma\to 0}\inf_{u\in\mathcal{H}} J_{\alpha,8\pi-\gamma}(u_\epsilon).$$

Extracting a diagonal sequence, we obtain

$$\lim_{\epsilon\to 0}\inf_{u\in\mathcal{H}} J_{\alpha,8\pi-\epsilon}(u) = \inf_{u\in\mathcal{H}} J_{\alpha,8\pi}(u). \tag{22}$$

Denote

$$c_\epsilon = u_\epsilon(x_\epsilon) = \max_\Sigma u_\epsilon.$$

At the end of this step, we prove that if λ_ϵ or c_ϵ is bounded, then $J_{\alpha,8\pi}$ has a minimizer in $\mathcal{H}$. Supposing that λ_ϵ is bounded, then

$$\begin{aligned} \frac{1}{2}\|u_\epsilon\|_{\alpha,p}^2 &= J_{\alpha,8\pi-\epsilon}(u_\epsilon) + (8\pi-\epsilon)\log\int_\Sigma he^{u_\epsilon}dv_g \\ &\leq J_{\alpha,8\pi-\epsilon}(0) + (8\pi-\epsilon)\log\lambda_\epsilon \\ &\leq 8\pi|\log\int_\Sigma h dv_g| + (8\pi-\epsilon)\log\lambda_\epsilon \\ &\leq C. \end{aligned}$$

Therefore, u_ϵ is bounded in $\mathcal{H}$. By the Sobolev embedding theorem and Lemma 3, we know u_ϵ is bounded in $L^r(\Sigma)\,(\forall r > 1)$ and $e^{|u_\epsilon|}$ is bounded in $L^s(\Sigma)\,(\forall s > 1)$. Noting that $\liminf_{\epsilon\to 0}\lambda_\epsilon > 0$ and

$$|\mu_\epsilon| \leq 8\pi + \alpha|\int_\Sigma \|u_\epsilon\|_p^{2-p}|u_\epsilon|^{p-2}u_\epsilon \mathrm{d}v_g| \leq 8\pi + C\|u_\epsilon\|_p,$$

applying an elliptic estimate to (21), we obtain $u_\epsilon \to u_0$ in $\mathcal{H}\cap C^1(\Sigma)$. It follows from (22) that

$$J_{\alpha,8\pi}(u_0) = \lim_{\epsilon\to 0} J_{\alpha,8\pi-\epsilon}(u_\epsilon) = \lim_{\epsilon\to 0}\inf_{u\in\mathcal{H}} J_{\alpha,8\pi-\epsilon}(u) = \inf_{u\in\mathcal{H}} J_{\alpha,8\pi}(u).$$

Hence, we already have that u_0 is a minimizer of $J_{\alpha,8\pi}$. Supposing that u_ϵ is bounded, multiplying equation (21) by u_ϵ, the Sobolev embedding theorem together with $\liminf_{\epsilon\to 0}\lambda_\epsilon > 0$ tells us that

$$\|u_\epsilon\|_{\alpha,p}^2 \leq C\int_\Sigma |u_\epsilon| dv_g \leq C\|u_\epsilon\|_{\alpha,p}. \tag{23}$$

Thus, u_ϵ is bounded in $\mathcal{H}$. By a series of analyses, the same as λ_ϵ being bounded, we can find a u_0 in $\mathcal{H}\cap C^1(\Sigma)$ as a minimizer of $J_{\alpha,8\pi}$. Therefore, if we assume that $J_{\alpha,8\pi}$ has no minimizer on $\mathcal{H}$, there must hold

$$\lambda_\epsilon \to +\infty, \quad c_\epsilon \to +\infty. \tag{24}$$

We will precisely describe the converge of u_ϵ in the next step by using the method of blow-up analysis.

Step 2. Blow-up analysis for $\mathbf{u}_\epsilon$.

Assume $x_\epsilon \to x_0$ in Σ. We set

$$r_\epsilon = \frac{\sqrt{\lambda_\epsilon}}{\sqrt{(8\pi-\epsilon)h(x_0)}} e^{-c_\epsilon/2}. \tag{25}$$

It follows that for any $\eta < 1/2$, there holds $r_\epsilon^2 e^{\eta c_\epsilon} \to 0$. In particular, we have for any $s > 0$ the following:

$$r_\epsilon c_\epsilon^s \to 0, \tag{26}$$

where r_ϵ is defined in (25). The proof is an analogy of [18, Lemma 2.7]; we multiply both sides of equation (21) by u_ϵ and obtain

$$\|u_\epsilon\|_{\alpha,p}^2 = \frac{(8\pi-\epsilon)}{\lambda_\epsilon}\int_\Sigma h u_\epsilon e^{u_\epsilon} dv_g \le (8\pi-\epsilon)c_\epsilon. \tag{27}$$

(27) together with the Trudinger–Moser inequality (16) leads to

$$\begin{aligned} \int_\Sigma h e^{u_\epsilon} dv_g &\le C\int_\Sigma e^{(4\pi-\epsilon/2)\frac{u_\epsilon^2}{\|u_\epsilon\|_{\alpha,p}^2}+\frac{\|u_\epsilon\|_{\alpha,p}^2}{16\pi-2\epsilon}} dv_g \\ &\le Ce^{\frac{\|u_\epsilon\|_{\alpha,p}^2}{16\pi-2\epsilon}} \le Ce^{\frac{1}{2}c_\epsilon}. \end{aligned}$$

We then obtain

$$r_\epsilon^2 = \frac{\int_\Sigma h e^{u_\epsilon} dv_g}{(8\pi-\epsilon)h(x_0)} e^{-c_\epsilon} \le Ce^{-\frac{1}{2}c_\epsilon}. \tag{28}$$

This demonstrates the correctness of $r_\epsilon c_\epsilon^s \to 0$ for any $s>0$. Define two blow-up functions,

$$\varphi_\epsilon(y) = u_\epsilon\left(\exp_{x_\epsilon}(r_\epsilon y)\right) - c_\epsilon \tag{29}$$

and

$$\psi_\epsilon(y) = c_\epsilon^{-1} u_\epsilon\left(\exp_{x_\epsilon}(r_\epsilon y)\right). \tag{30}$$

For $y \in \mathbb{B}_{\delta r_\epsilon^{-1}}(0)$, where $0<\delta<i_g(\Sigma)$ is fixed and $i_g(\Sigma)$ is the injectivity radius of (Σ, g), set

$$g_\epsilon(y) = \left(\exp_{x_\epsilon}^* g\right)(r_\epsilon y). \tag{31}$$

As $\epsilon \to 0$, $g_\epsilon \to g_0$, the standard Euclidean metric. Note that $\psi_\epsilon(y) \le \psi_\epsilon(0) = 1$ and $\varphi_\epsilon(y) \le 0$. Combining (29)–(31) with (21), by a direct computation, we have

$$\triangle_{g_\epsilon}\psi_\epsilon = \alpha r_\epsilon^2 \|c_\epsilon^{-1}u_\epsilon\|_p^{2-p}\psi_\epsilon^{p-1} + c_\epsilon^{-1}\frac{h(\exp_{x_\epsilon}(r_\epsilon y))}{h(x_0)} e^{u_\epsilon\left(\exp_{x_\epsilon}(r_\epsilon y)\right)-c_\epsilon} + \mu_\epsilon r_\epsilon^2 c_\epsilon^{-1} \tag{32}$$

and

$$\triangle_{g_\epsilon}\varphi_\epsilon=\alpha r_\epsilon^2 c_\epsilon\|c_\epsilon^{-1}u_\epsilon\|_p^{2-p}\psi_\epsilon^{p-1}+\frac{h(\exp_{x_\epsilon}(r_\epsilon y))}{h(x_0)}e^{\varphi_\epsilon(y)}-\mu_\epsilon r_\epsilon^2. \tag{33}$$

For any fixed $R>0$, by a change of variable we have

$$\|\psi_\epsilon^{p-1}\|_{L^{\frac{p}{p-1}}(\mathbb{B}_R(0))}=(1+o_\epsilon(1))r_\epsilon^{-2+\frac{2}{p}}\|c_\epsilon^{-1}u_\epsilon\|_{L^p(B_{Rr_\epsilon}(x_\epsilon))}^{p-1}\leq r_\epsilon^{-2+\frac{2}{p}}\|c_\epsilon^{-1}u_\epsilon\|_{L^p(\Sigma)}^{p-1}.$$

This together with $0\leq c_\epsilon^{-1}u_\epsilon\leq 1$ gives

$$\|\alpha r_\epsilon^2\|c_\epsilon^{-1}u_\epsilon\|_p^{2-p}\psi_\epsilon^{p-1}\|_{L^{\frac{p}{p-1}}(\mathbb{B}_R(0))}\leq \alpha r_\epsilon^{\frac{2}{p}}\mathrm{Vol}_g(\Sigma)^{\frac{1}{p}}.$$

It follows that

$$\|-\triangle_{g_\epsilon}\psi_\epsilon\|_{L^{\frac{p}{p-1}}(\mathbb{B}_R(0))}\to 0\quad\text{as}\quad \epsilon\to 0.$$

Using elliptic estimates, we can find some continuous function ψ such that $\psi_\epsilon\to\psi$ in $C^0(\mathbb{B}_{R/2}(0))$. Since $R>0$ is arbitrary, we have

$$\psi_\epsilon\to\psi\quad\text{in}\quad C^0_{\mathrm{loc}}(\mathbb{R}^2). \tag{34}$$

When $1<p\leq 2$, it is easy to see that

$$\|c_\epsilon^{-1}u_\epsilon\|_p^{2-p}\leq \mathrm{Vol}_g(\Sigma)^{\frac{2}{p}-1}.$$

When $p>2$, we have for any fixed $R>0$ the following:

$$\begin{aligned} r_\epsilon^2\|c_\epsilon^{-1}u_\epsilon\|_{L^p(\Sigma)}^{2-p} &\leq r_\epsilon^2\|c_\epsilon^{-1}u_\epsilon\|_{L^p(B_{Rr_\epsilon}(x_\epsilon))}^{2-p}\\ &= r_\epsilon^{\frac{4}{p}}(\|\psi\|_{L^p(\mathbb{B}_R(0))}^{2-p}+o_\epsilon(1)). \end{aligned} \tag{35}$$

Note that $\|\psi\|_{L^p(\mathbb{B}_R(0))}>0$ since $\psi(0)=\lim_{\epsilon\to 0}\psi_\epsilon(0)=1$ and ψ is continuous. Therefore, we conclude the following:

$$r_\epsilon^2\|c_\epsilon^{-1}u_\epsilon\|_{L^p(\Sigma)}^{2-p}=o_\epsilon(1),\quad \forall p>1. \tag{36}$$

Applying elliptic estimates to (32) and (33) again, and combining with (36), we obtain

$$\psi_\epsilon\to\psi\quad\text{in}\quad C^1_{\mathrm{loc}}(\mathbb{R}^2), \tag{37}$$

where ψ is a bounded harmonic function as in (34) and satisfies $\psi(0)=1=\sup_{\mathbb{R}^2}\psi$. The Liouville theorem immediately leads to

$$\psi\equiv 1\quad\text{in}\quad \mathbb{R}^2.$$

It follow from (37) that

$$\psi_\epsilon\to 1\quad\text{in}\quad C^1_{\mathrm{loc}}(\mathbb{R}^2).$$

(35) inspires that

$$\begin{aligned} r_\epsilon^2 c_\epsilon \|c_\epsilon^{-1} u_\epsilon\|_{L^p(\Sigma)}^{2-p} &\leq r_\epsilon^2 c_\epsilon \|c_\epsilon^{-1} u_\epsilon\|_{L^p(B_{Rr_\epsilon}(x_\epsilon))}^{2-p} \\ &= r_\epsilon^{\frac{4}{p}} c_\epsilon \|\psi_\epsilon\|_{L^p(\mathbb{B}_R(0))}^{2-p} \\ &= r_\epsilon^{\frac{4}{p}} c_\epsilon (\|\psi\|_{L^p(\mathbb{B}_R(0))}^{2-p} + o_\epsilon(1)) \to 0. \end{aligned} \tag{38}$$

The last line of (38) comes from (26). Applying elliptic estimates to (32) and (33), we have

$$\varphi_\epsilon \to \varphi \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^2),$$

where φ satisfies

$$\begin{cases} \triangle_{g_0} \varphi = -e^{\varphi(y)} \quad \text{in} \quad \mathbb{R}^2, \\ \varphi(0) = 0 = \sup_{\mathbb{R}^2} \varphi, \\ \int_{\mathbb{R}^2} e^{\varphi(y)} dy < \infty. \end{cases}$$

By a result of [3], φ can be written as follow:

$$\varphi(y) = -2 \log(1 + |y|^2/8). \tag{39}$$

Moreover,

$$\int_{\mathbb{R}^2} e^{\varphi(y)} dy = 8\pi. \tag{40}$$

To understand the convergence behavior away from the blow-up point x_0, we shall next figure out how u_ϵ converges. First, we will prove that $\lambda_\epsilon^{-1} h e^{u_\epsilon} \rightharpoonup \delta_{x_0}$ in the sense of measure, where δ_{x_0} is the usual Dirac measure centered at x_0. For fixed $R > 0$, in view of (25), we obtain

$$\begin{aligned} \frac{(8\pi - \epsilon)}{\lambda_\epsilon} \int_{B_{Rr_\epsilon}(x_\epsilon)} h e^{u_\epsilon} dv_g &= r_\epsilon^2 \frac{(8\pi - \epsilon)}{\lambda_\epsilon} \int_{\mathbb{B}_{R(0)}} e^{\varphi_\epsilon(y)} dy \\ &= (1 + o_\epsilon(1)) \int_{\mathbb{B}_{R(0)}} e^{\varphi} dy. \end{aligned}$$

Combining this with (39) and (40), we have

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \int_{B_{Rr_\epsilon}(x_\epsilon)} {\lambda_\epsilon}^{-1} h e^{u_\epsilon} dv_g = 1. \tag{41}$$

Therefore,

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \int_{\Sigma \setminus B_{Rr_\epsilon}(x_\epsilon)} {\lambda_\epsilon}^{-1} h e^{u_\epsilon} dv_g = 1 - \lim_{R\to\infty} \lim_{\epsilon\to 0} \int_{B_{Rr_\epsilon}(x_\epsilon)} {\lambda_\epsilon}^{-1} h e^{u_\epsilon} dv_g = 0. \tag{42}$$

For any $\nu \in C^0(\Sigma)$, we have

$$\lim_{\epsilon\to 0} {\lambda_\epsilon}^{-1} \int_\Sigma \nu h e^{u_\epsilon} dv_g = \nu(x_0). \tag{43}$$

Secondly, we need to prove that $\|u_\epsilon\|_{L^p(\Sigma)}$ is bounded and demonstrate that u_ϵ is bounded in $W^{1,q}(\Sigma)$ for all $1 < q < 2$. In fact, if we assume $\|u_\epsilon\|_{L^p(\Sigma)} \to \infty$, we can construct $\tilde{u}_\epsilon = u_\epsilon/\|u_\epsilon\|_{L^p(\Sigma)} \in \mathcal{H}$ verifying that $||\tilde{u}_\epsilon\|_{L^p(\Sigma)} = 1$ and

$$\triangle_g \tilde{u}_\epsilon - \alpha|\tilde{u}_\epsilon|^{p-2}\tilde{u}_\epsilon = \frac{(8\pi-\epsilon)he^{u_\epsilon}}{\lambda_\epsilon\|u_\epsilon\|_{L^p(\Sigma)}} - \frac{(8\pi-\epsilon)}{\|u_\epsilon\|_{L^p(\Sigma)}} - \alpha\int_\Sigma \|u_\epsilon\|_p^{1-p}|u_\epsilon|^{p-2}u_\epsilon dv_g := f_\epsilon^{(1)}.$$

It is not difficult to see that $\int_\Sigma \|u_\epsilon\|_p^{1-p}|u_\epsilon|^{p-2}u_\epsilon dv_g$ is a uniformly bounded sequence. Assume

$$b = \lim_{\epsilon\to 0}\left(-\alpha\int_\Sigma \|u_\epsilon\|_p^{1-p}|u_\epsilon|^{p-2}u_\epsilon dv_g\right). \tag{44}$$

We get that $\|f_\epsilon^{(1)}\|_{L^1(\Sigma)}$ is bounded by letting $\nu = 1$ in (43). Applying [18, Lemma 2.10], we obtain that $\tilde{u}_\epsilon$ is bounded in $W^{1,q}(\Sigma)$ for any $1 < q < 2$ with $2q/(2-q) > p$. We assume the following:

$$\begin{aligned} &\tilde{u}_\epsilon \rightharpoonup \tilde{u}_0 \quad \text{weakly in} \quad W^{1,q}(\Sigma), \\ &\tilde{u}_\epsilon \to \tilde{u}_0 \quad \text{strongly in} \quad L^s(\Sigma) \quad \text{for any } 0 < s < \frac{2q}{2-q}, \\ &\tilde{u}_\epsilon \to \tilde{u}_0 \quad \text{a.e. in} \quad \Sigma. \end{aligned} \tag{45}$$

Moreover, $\tilde{u}_0$ is a weak solution to the equation

$$\begin{cases} \triangle_g w - \alpha|w|^{p-2}w = b, \\ \int_\Sigma w dv_g = 0. \end{cases}$$

This lead to $\tilde{u}_0 \equiv 0$, which is contradictory to (45) that $\|\tilde{u}_0\|_{L^p(\Sigma)} = \lim_{\epsilon\to 0}\|\tilde{u}_\epsilon\|_{L^p(\Sigma)} = 1$. Therefore, $\|u_\epsilon\|_{L^p(\Sigma)}$ is bounded. Then, for

$$\triangle_g u_\epsilon = \alpha\|u_\epsilon\|_p^{2-p}|u_\epsilon|^{p-2}u_\epsilon + (8\pi-\epsilon){\lambda_\epsilon}^{-1}he^{u_\epsilon} + \mu_\epsilon := f_\epsilon^{(2)}, \tag{46}$$

we immediately obtain that $f_\epsilon^{(2)}$ is bounded in $L^1(\Sigma)$. Applying [18, Lemma 2.10] again, $\|\nabla u_\epsilon\|_{L^q(\Sigma)} \le C$. Since u_ϵ is bounded in $W^{1,q}(\Sigma)$ for all $1 < q < 2$, there exists some G_{x_0} such that u_ϵ converges to G_{x_0} weakly in $W^{1,q}(\Sigma)$, strongly in $L^s(\Sigma)$ for any $0 < s < \frac{2q}{2-q}$, and almost everywhere in Σ. One can check that G_{x_0} is the distributional solution to the equation

$$\begin{cases} \Delta_g G_{x_0} - \alpha\|G_{x_0}\|_p^{2-p}|G_{x_0}|^{p-2}G_{x_0} = 8\pi\delta_{x_0} + b - 8\pi, \\ \int_\Sigma G_{x_0} dv_g = 0, \end{cases} \tag{47}$$

where b is the constant shown in (44). Integrating by (47) shows

$$b = -\alpha\int_\Sigma \|G_{x_0}\|_p^{2-p}|G_{x_0}|^{p-2}G_{x_0} dv_g.$$

Applying an elliptic estimate to (47), G_{x_0} can be locally represented by

$$G_{x_0}(x) = -4\log r + A_{x_0} + \vartheta_\alpha(x), \tag{48}$$

where r denotes the geodesic distance between x and x_0, A_{x_0} is a real number depending only on x_0 and α, $\vartheta_\alpha(x) \in C^1(\Sigma)$, and $\vartheta_\alpha(p) = 0$. Moreover,

$$u_\epsilon \to G_{x_0} \quad \text{in} \quad C^1_{\text{loc}}(\Sigma \backslash \{x_0\}). \tag{49}$$

For any domain $\Sigma' \subset\subset \Sigma \setminus \{x_0\}$, set $u_\epsilon = u_\epsilon^{(1)} + u_\epsilon^{(2)}$, $u_\epsilon^{(1)}$ to be a solution to

$$\triangle_g u_\epsilon^{(1)} - \alpha \|u_\epsilon^{(1)}\|_p^{2-p} |u_\epsilon^{(1)}|^{p-2} u_\epsilon^{(1)} = b - 8\pi \quad \text{on } \Sigma' \tag{50}$$

and $u_\epsilon^{(2)}$ to be a solution to

$$\begin{cases} \triangle_g u_\epsilon^{(2)} = \alpha \|u_\epsilon^{(2)}\|_p^{2-p} |u_\epsilon^{(2)}|^{p-2} u_\epsilon^{(2)} + \lambda_\epsilon^{-1} h e^{u_\epsilon} & \text{in } \Sigma' \\ u_\epsilon^{(2)} = 0, & \text{on } \partial\Sigma', \end{cases} \tag{51}$$

In view of (50), referring to the proof of $\|u_\epsilon\|_{L^p(\Sigma)}$ being bounded, we have that $\|u_\epsilon^{(1)}\|_{L^s(\Sigma)}$ is bounded for any $s > 0$. For any $\Sigma'' \subset\subset \Sigma'$, we obtain that $u_\epsilon^{(1)}$ is uniformly bounded in Σ'' by applying elliptic estimate to (50). Then $\alpha \|u_\epsilon^{(2)}\|_p^{2-p} |u_\epsilon^{(2)}|^{p-2} u_\epsilon^{(2)}$ is bounded in $L^1(\Sigma')$. This together with $\lambda_\epsilon^{-1} h e^{u_\epsilon} \to 0$ in $L^1(\Sigma')$, and a result of [2], implies that for any $s > 0$, there is a constant C such that

$$\|e^{|u_\epsilon^{(2)}|}\|_{L^s(\Sigma')} \le C.$$

Combining this with the uniform boundedness of $u_\epsilon^{(1)}$ in Σ'', we have that $f_\epsilon^{(2)}$ in (46) is bounded in $L^s(\Sigma'')$ for any $s > 2$. By applying an elliptic estimate to (46), we get $u_\epsilon \to G_{x_0}$ in $C^1_{\text{loc}}(\Sigma'')$. Since the choice of Σ' and Σ'' is arbitrary, we obtain (49).

Step 3. The lower bound estimate.

In this step we get the lower bound estimate by using the method of capacity estimate; the idea comes from [7, 11]. We first introduce a new functional: $K_{\alpha,\beta} : W^{1,2}(\Sigma) \to \mathbb{R}$,

$$K_{\alpha,\beta}(v) = \frac{1}{2}\left(\int_\Sigma |\nabla_g v|^2 dv_g - \alpha\left(\int_\Sigma |v - \bar{v}|^p dv_g\right)^{2/p}\right) + \beta \int_\Sigma v dv_g - \beta \log \int_\Sigma h e^v dv_g,$$

where h is the same as in (8) and $\bar{v} = 1/\text{Vol}_g(\Sigma) \int_\Sigma v dv_g$ is the integral average of v. It is not difficult to verify that $v - \bar{v} \in \mathcal{H}$ and

$$K_{\alpha,\beta}(v) = J_{\alpha,\beta}(v - \bar{v}).$$

Most importantly, for any real number a, there holds

$$K_{\alpha,\beta}(v + a) = K_{\alpha,\beta}(v). \tag{52}$$

For any $v \in W^{1,2}(\Sigma)$, we can choose $u = v - \bar{v} \in \mathcal{H}$ to make $K_{\alpha,\beta}(v) = J_{\alpha,\beta}(u)$ stand, which leads to

$$\inf_{v \in W^{1,2}(\Sigma)} K_{\alpha,\beta}(v) \ge \inf_{u \in \mathcal{H}} J_{\alpha,\beta}(u). \tag{53}$$

On the other hand, for any $u \in \mathcal{H}$ we can still choose $v = u \in W^{1,2}(\Sigma)$ to make $J_{\alpha,\beta}(u) = K_{\alpha,\beta}(v)$, which shows

$$\inf_{u\in\mathcal{H}} J_{\alpha,\beta}(u) \geq \inf_{v\in W^{1,2}(\Sigma)} K_{\alpha,\beta}(v). \tag{54}$$

Combining (53) with (54), we obtain

$$\inf_{v\in W^{1,2}(\Sigma)} K_{\alpha,\beta}(v) = \inf_{u\in\mathcal{H}} J_{\alpha,\beta}(u).$$

Let v_ϵ be the minimizer for subcritical functional $K_{\alpha,8\pi-\epsilon}(v)$. In view of (52), we might assume v_ϵ in a function space

$$L_h = \left\{ v \in W^{1,2}(\Sigma) : \int_\Sigma h e^v dv_g = 1 \right\}. \tag{55}$$

Let

$$m_\epsilon = v(x_\epsilon) = \max_\Sigma v_\epsilon.$$

Then redefine

$$r_\epsilon = \frac{1}{\sqrt{(8\pi-\epsilon)h(x_0)}} e^{-m_\epsilon/2}. \tag{56}$$

Analogous to step 2, if $K_{\alpha,8\pi}$ has no minimizer on L_h, we have $\varphi_\epsilon(y) = v_\epsilon\left(\exp_{x_\epsilon}(r_\epsilon y)\right) - m_\epsilon \to -2\log(1+|y|^2/8)$, and $v_\epsilon - \bar{v}_\epsilon \to G_{x_0}$, as $\epsilon \to 0$. Take any fixed $R > 0$ and small δ such that $2\delta < i_g(\Sigma)$.

Set

$$i_\epsilon = \inf_{\partial B_{Rr_\epsilon}(x_\epsilon)} v_\epsilon, \qquad s_\epsilon = \sup_{\partial B_\delta(x_\epsilon)} v_\epsilon.$$

We let

$$i_\epsilon - s_\epsilon = m_\epsilon + d_\epsilon - \bar{v}_\epsilon.$$

As $\epsilon \to 0$,

$$\begin{aligned} d_\epsilon = i_\epsilon - s_\epsilon - m_\epsilon + \bar{v}_\epsilon &= \inf_{\partial B_{Rr_\epsilon}(x_\epsilon)} (v_\epsilon - m_\epsilon) - \sup_{\partial B_\delta(x_\epsilon)} (v_\epsilon - \bar{v}_\epsilon) \\ &= \inf_{\partial B_{Rr_\epsilon}(x_\epsilon)} (v_\epsilon - \bar{v}_\epsilon - c_\epsilon) - \sup_{\partial B_\delta(x_\epsilon)} (v_\epsilon - \bar{v}_\epsilon) \\ &\to \varphi(R) - \sup_{\partial B_\delta(x_\epsilon)} G_{x_0}. \end{aligned}$$

Define a function space

$$\mathscr{W}_\epsilon(a,b) = \left\{ v \in W^{1,2}(B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)) : v\mid_{\partial B_\delta(x_\epsilon)} = a, v\mid_{\partial B_{Rr_\epsilon}(x_\epsilon)} = b \right\}.$$

Clearly $\inf_{v\in\mathscr{W}_\epsilon(s_\epsilon,i_\epsilon)} \int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)} |\nabla_g v|^2 dv_g$ is attained by $l(x)$ verifying

$$\begin{cases} \triangle_g l = 0 \quad \text{in} \quad B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon), \\ l\mid_{\partial B_\delta(x_\epsilon)} = s_\epsilon, \\ l\mid_{\partial B_{Rr_\epsilon}(x_\epsilon)} = i_\epsilon. \end{cases}$$

Denote r as the geodesic distance between x and x_ϵ; then

$$l = \frac{i_\epsilon - s_\epsilon}{\log Rr_\epsilon - \log\delta}\log r - \frac{i_\epsilon\log\delta - s_\epsilon\log r_\epsilon}{\log Rr_\epsilon - \log\delta},$$

and

$$\inf_{v\in\mathscr{W}_\epsilon(s_\epsilon,i_\epsilon)}\int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)}|\nabla_g v|^2dv_g = \int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)}|\nabla_g l|^2dv_g = \frac{2\pi(s_\epsilon - i_\epsilon)^2}{\log\delta - \log Rr_\epsilon}.$$

Denote

$$\tilde{v}_\epsilon = \max\left\{s_\epsilon, \min\{v_\epsilon, i_\epsilon\}\right\}.$$

Then, if ϵ is sufficiently small, $\tilde{v}\in\mathscr{W}_\epsilon(s_\epsilon,i_\epsilon)$ and $|\triangle\tilde{v}|\le|\triangle v|$ a.e. in $B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}$. Therefore,

$$\begin{aligned}
\int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)}|\nabla_g v_\epsilon|^2dv_g &\ge \int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)}|\nabla_g\tilde{v}_\epsilon|^2dv_g \ge \int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)}|\nabla_g l|^2dv_g\\
&= \frac{4\pi(m_\epsilon + d_\epsilon - \bar{v}_\epsilon)^2}{\log\delta^2 - \log R^2r_\epsilon^2} = \frac{4\pi(m_\epsilon + d_\epsilon - \bar{v}_\epsilon)^2}{m_\epsilon + \log\delta^2 - \log R^2 + \log(8\pi-\epsilon)h(x_0)}\\
&\ge 4\pi\frac{(m_\epsilon + d_\epsilon - \bar{v}_\epsilon)^2}{m_\epsilon}\left(1 + \frac{\log R^2 - \log\delta^2 + \log(8\pi-\epsilon)h(x_0)}{m_\epsilon} + \frac{C}{m_\epsilon^2}\right)\\
&\ge 4\pi\frac{(m_\epsilon - \bar{v}_\epsilon)^2}{m_\epsilon} + 8\pi d_\epsilon(1 - \frac{\bar{v}_\epsilon}{m_\epsilon})\\
&\quad + 4\pi d_\epsilon\left(1 - \frac{\bar{v}_\epsilon}{m_\epsilon}\right)^2\left(\log R^2 - \log\delta^2 + \log(8\pi-\epsilon)h(x_0)\right) + \frac{C'\bar{v}_\epsilon}{m_\epsilon^2},
\end{aligned}$$

where C and C' are constants depending only on δ and R. We then have those estimates by (39):

$$\begin{aligned}
\frac{1}{2}\int_{B_{Rr_\epsilon}(x_\epsilon)}|\nabla_g v_\epsilon|^2dv_g &= \frac{1}{2}\int_{\mathbb{B}_R(0)}|\nabla\varphi|^2dv_g + o_\epsilon(1)\\
&= 8\pi\log(1+\frac{R^2}{8}) - 8\pi + o_\epsilon(1) + O\left(\frac{1}{R^2}\right),
\end{aligned} \tag{57}$$

and by (47) and (49)

$$-\frac{\alpha}{2}(\int_\Sigma|v-\bar{v}|^pdv_g)^{2/p} = -\frac{\alpha}{2}\|G_{x_0}\|_p^2 + o_\epsilon(1). \tag{58}$$

Thus, we conclude

$$\begin{aligned}
&\frac{1}{2}\int_{B_\delta(x_\epsilon)\setminus B_{Rr_\epsilon}(x_\epsilon)}|\nabla_g v_\epsilon|^2 dv_g - \frac{\alpha}{2}(\int_\Sigma |v_\epsilon-\bar{v}_\epsilon|^p dv_g)^{2/p} + (8\pi-\epsilon)\bar{v}.\\
\geq\ & 2\pi\frac{(m_\epsilon-\bar{v}_\epsilon)^2}{m_\epsilon} + 2\pi d_\epsilon\left(1-\frac{\bar{v}_\epsilon}{m_\epsilon}\right)^2\left(\log R^2-\log\delta^2+\log(8\pi-\epsilon)h(x_0)\right)\\
&+4\pi d_\epsilon(1-\frac{\bar{v}_\epsilon}{m_\epsilon}) + \frac{C'\bar{v}_\epsilon}{m_\epsilon^2} - \frac{\alpha}{2}\|G_{x_0}\|_p^2 + (8\pi-\epsilon)\bar{v} + o_\epsilon(1)\\
=\ & 2\pi(1+\frac{\bar{v}_\epsilon}{m_\epsilon})^2 + 2\pi d_\epsilon\left(1-\frac{\bar{v}_\epsilon}{m_\epsilon}\right)^2\left(\log R^2-\log\delta^2+\log(8\pi-\epsilon)h(x_0)\right)\\
&+4\pi d_\epsilon(1-\frac{\bar{v}_\epsilon}{m_\epsilon}) + \frac{C'\bar{v}_\epsilon}{m_\epsilon^2} - \frac{\alpha}{2}\|G_{x_0}\|_p^2 + o_\epsilon(1).
\end{aligned}$$

Set $a_\epsilon = 1+\bar{v}/m_\epsilon$. There holds

$$K_{\alpha,8\pi-\epsilon}(v_\epsilon) \geq 2\pi m_\epsilon\left(a_\epsilon + O\left(\frac{1}{m_\epsilon}\right)\right)^2 + C,$$

for any fixed R and δ. On the other hand, $K_{\alpha,8\pi-\epsilon}(v_\epsilon)\leq C$, and we obtain $a_\epsilon = O\left(\frac{1}{\sqrt{m_\epsilon}}\right)\to 0$. Thus,

$$\begin{aligned}
&\frac{1}{2}\int_{B_\delta(x_\epsilon)}|\nabla_g v_\epsilon|^2 dv_g - \frac{\alpha}{2}(\int_\Sigma |v_\epsilon-\bar{v}_\epsilon|^p dv_g)^{2/p} + (8\pi-\epsilon)\bar{v}_\epsilon\\
\geq\ & \frac{1}{2}\int_{\mathbb{B}_R(0)}|\nabla\varphi_\epsilon|^2 dv_g - \frac{\alpha}{2}\|G_{x_0}\|_p^2 + 8\pi d_\epsilon\\
&+8\pi(\log R^2-\log\delta^2+\log(8\pi-\epsilon)h(x_0)) + o_\epsilon(1)\\
=\ & \frac{1}{2}\int_{\mathbb{B}_R(0)}|\nabla\varphi_\epsilon|^2 dv_g - \frac{\alpha}{2}\|G_{x_0}\|_p^2 + 8\pi\varphi(R) - 8\pi\sup_{\partial B_\delta(p)} G_{x_0}(\cdot)\\
&+8\pi(\log R^2-\log\delta^2+\log(8\pi-\epsilon)h(x_0)) + o_\epsilon(1)\\
=\ & \frac{1}{2}\int_{\mathbb{B}_R(0)}|\nabla\varphi_\epsilon|^2 dv_g - \frac{\alpha}{2}\|G_{x_0}\|_p^2 + 8\pi\log(\frac{R^2}{1+R^2/8}) - 8\pi\log(1+R^2/8)\\
&+\log 8\pi + \log h(x_0) + 8\pi\log\delta - 8\pi A_{x_0} + o_\epsilon(1) + o_\delta(1).
\end{aligned} \tag{59}$$

It also follows from (47) and (49) that

$$\begin{aligned}
\frac{1}{2}\int_{\Sigma\setminus B_\delta(x_\epsilon)}|\nabla_g v_\epsilon|^2 dv_g &= \frac{1}{2}\int_{\Sigma\setminus B_\delta(x_\epsilon)}|\nabla_g G_{x_0}(\cdot)|^2 dv_g + o_\epsilon(1)\\
&= \frac{\alpha}{2}(\int_{\Sigma\setminus B_\delta(x_\epsilon)}|\nabla_g G_{x_0}(\cdot)|^p dv_g)^{2/p} - \frac{1}{2}\int_{\partial B_\delta(x_\epsilon)} G_{x_0}\frac{\partial G_{x_0}}{\partial n} ds_g + o(1)\\
&= \frac{\alpha}{2}\|G_{x_0}\|_p^2 - 16\pi\log\delta + 4\pi A_{x_0} + o(1).
\end{aligned}$$

This together with (59) implies

$$
\begin{aligned}
K_{\alpha,8\pi-\epsilon}(v_\epsilon) &= \frac{1}{2}\int_{B_\delta(x_\epsilon)} |\nabla_g v_\epsilon|^2 dv_g - \frac{\alpha}{2}(\int_\Sigma |v_\epsilon - \bar{v_\epsilon}|^p dv_g)^{2/p} + (8\pi-\epsilon)\bar{v_\epsilon} \\
&\geq -8\pi - 8\pi\log\pi - 8\pi\log h(x_0) - 4\pi A_{x_0} + o(1) + O\left(\frac{1}{R^2}\right).
\end{aligned}
$$

Hence, when $\epsilon \to 0$ and $R \to +\infty$ respectively,

$$
\begin{aligned}
\inf_{u\in\mathcal{H}} J_{\alpha,8\pi}(u) &= \inf_{v\in W^{1,2}(\Sigma)} K_{\alpha,8\pi}(v) \\
&\geq -8\pi - 8\pi\log\pi - 4\pi\max_{x_0\in\Sigma}\left(A_{x_0} + 2\log h(x_0)\right). \qquad (60)
\end{aligned}
$$

Step 4. Existence of extremal functions.

In this step we aim to construct a sequence of function $(\phi_\epsilon)_{\epsilon>0}$ satisfying

$$
\lim_{\epsilon\to 0} J_{\alpha,8\pi}(\phi_\epsilon - \bar{\phi}_\epsilon) = -8\pi - 8\pi\log\pi - 4\pi\max_{x\in\Sigma}\left(A_x + 2\log h(x)\right). \qquad (61)
$$

Assume $A_{\tilde{x}} + 2\log h(\tilde{x}) = 4\pi\max_{x\in\Sigma}(A_x + 2\log h(x))$. Denote r to be the geodesic distance between $\tilde{x}$ and x. Combining (60) with (61), we complete the proof of Theorem 1.

Set

$$
\phi_\epsilon = \begin{cases} c - 2\log(1 + \frac{r^2}{8\epsilon^2}) & \text{for} \quad x \in B_{R\epsilon}(\tilde{x}), \\ G - \eta\vartheta_\alpha & \text{for} \quad x \in B_{2R\epsilon}(\tilde{x}) \setminus B_{R\epsilon}(\tilde{x}), \\ G_{\tilde{x}} & \text{for} \quad x \in \Sigma \setminus B_{2R\epsilon}(\tilde{x}), \end{cases} \qquad (62)
$$

Here, ϑ_α is the function in (48), $\eta \in C_0^\infty(B_{2R\epsilon}(x_0))$ is a cut-off function verifying that $\eta = 1$ on $B_{R\epsilon}(x_0)$, and $\|\nabla_g \eta\|_{L^\infty(B_{2R\epsilon}(\tilde{x}))} = O(\frac{1}{R\epsilon})$. It is clear that $\bar{\phi}_\epsilon = o_\epsilon(1)$ and $\|\phi_\epsilon - \bar{\phi}_\epsilon\|_p^2 = \|G_{\tilde{x}}\|_p^2 + o_\epsilon(1)$.
c is defined by

$$
c = 2\log(1 + R^2/8) - 4\log R - 4\log\epsilon + A_{\tilde{x}},
$$

where $R = R(\epsilon)$ satisfies $R \to \infty$ and $(R\epsilon)^2\log R \to 0$ as $\epsilon \to 0$. It is easy to see that

$$
\frac{1}{2}\int_{B_{R\epsilon}(\tilde{x})} |\nabla_g\phi_\epsilon|^2 dv_g = 8\pi\log(1 + R^2/8) - 8\pi + o_\epsilon(1) \qquad (63)
$$

and

$$\begin{aligned}\frac{1}{2}\int_{\Sigma\setminus B_{R\epsilon}(\tilde{x})}|\nabla_g\phi_\epsilon|^2dv_g &= \frac{1}{2}\int_{\Sigma\setminus B_{R\epsilon}(\tilde{x})}|\nabla_g G_{\tilde{x}}|^2dv_g+\frac{1}{2}\int_{B_{2R\epsilon}(\tilde{x})\setminus B_{R\epsilon}(\tilde{x})}|\nabla_g(\eta\vartheta_\alpha)|^2dv_g\\ &\quad -\int_{B_{2R\epsilon}(\tilde{x})\setminus B_{R\epsilon}(\tilde{x})}\nabla_g G_{\tilde{x}}\nabla_g(\eta\vartheta_\alpha)dv_g\\ &= \frac{1}{2}\int_{\Sigma\setminus B_{R\epsilon}(\tilde{x})}G_{\tilde{x}}\triangle_g G_{\tilde{x}}dv_g-\frac{1}{2}\int_{\partial B_{R_\epsilon}(\tilde{x})}G_{\tilde{x}}\frac{\partial G_{\tilde{x}}}{\partial n}ds_g\\ &\quad +\frac{1}{2}\int_{B_{2R\epsilon}(\tilde{x})\setminus B_{R\epsilon}(\tilde{x})}|\nabla_g(\eta\vartheta_\alpha)|^2dv_g\\ &\quad -\int_{B_{2R\epsilon}(\tilde{x})\setminus B_{R\epsilon}(\tilde{x})}(\eta\vartheta_\alpha)\triangle_g G_{\tilde{x}}dv_g+\int_{B_{R\epsilon}(\tilde{x})}(\eta\vartheta_\alpha)\frac{\partial G_{\tilde{x}}}{\partial n}ds_g. \end{aligned}\tag{64}$$

We obtain by (48)

$$-\frac{1}{2}\int_{\partial B_{R_\epsilon}(\tilde{x})}G_{\tilde{x}}\frac{\partial G_{\tilde{x}}}{\partial n}ds_g=-16\pi\log(R\epsilon)+4\pi A_{\tilde{x}}+o_\epsilon(1).\tag{65}$$

It follows from (47) and (48) that

$$\frac{1}{2}\int_{\Sigma\setminus B_{R\epsilon}(\tilde{x})}G_{\tilde{x}}\triangle_g G_{\tilde{x}}dv_g=\frac{\alpha}{2}\|G_{\tilde{x}}\|_p^2+o_\epsilon(1).\tag{66}$$

It is not difficult to see that the other three terms on the right-hand side of (64) converge to 0 as $\epsilon\to 0$. This together with (65) and (66) shows

$$\frac{1}{2}\int_{\Sigma\setminus B_\delta(x_\epsilon)}|\nabla_g\phi_\epsilon|^2dv_g=-16\pi\log(R\epsilon)+4\pi A_{\tilde{x}}+\frac{\alpha}{2}\|G_{\tilde{x}}\|_p^2+o_\epsilon(1).\tag{67}$$

We obtain the following by (63) and (67):

$$\frac{1}{2}\int_{\Sigma\setminus B_\delta(x_\epsilon)}|\nabla_g\phi_\epsilon|^2dv_g-\frac{\alpha}{2}\|\phi_\epsilon-\bar{\phi_\epsilon}\|_p^2=-16\pi\log\epsilon-8\pi\log 8-8\pi+4\pi A_{\tilde{x}}+o_\epsilon(1).\tag{68}$$

Then we need to give a estimate of $\int_\Sigma he^{\phi_\epsilon}dv_g$. Choosing some $\delta>0$ sufficiently small such that $G_{\tilde{x}}$ has analogous local repression to (48) in $B_\delta(\tilde{x})$ gives

$$\begin{aligned}\int_\Sigma he^{\phi_\epsilon}dv_g &= h(\tilde{x})\int_{B_{R\epsilon}(\tilde{x})}e^{\phi_\epsilon}dv_g+\int_{B_{R\epsilon}(\tilde{x})}(h-h(\tilde{x}))e^{\phi_\epsilon}dv_g\\ &\quad +\int_{B_\delta(\tilde{x})\setminus B_{R\epsilon}(\tilde{x})}he^{\phi_\epsilon}dv_g+\int_{\Sigma\setminus B_\delta(\tilde{x})}he^{\phi_\epsilon}dv_g.\end{aligned}\tag{69}$$

A straightforward calculation shows

$$h(\tilde{x})\int_{B_{R\epsilon}(\tilde{x})}e^{\phi_\epsilon}dv_g=8\pi h(\tilde{x})e^{-2\log 8-2\log\epsilon+A_{\tilde{x}}+o_\epsilon(1)}\tag{70}$$

and

$$\int_{B_{R\epsilon}(\tilde{x})}(h-h(\tilde{x}))e^{\phi_\epsilon}dv_g=\epsilon^{-2}o_\epsilon(1). \tag{71}$$

It also holds that

$$\begin{aligned} 0<\int_{B_\delta(\tilde{x})\setminus B_{R\epsilon}(\tilde{x})}he^{\phi_\epsilon}dv_g &\le C(\max_\Sigma h)\int_{B_\delta(\tilde{x})\setminus B_{R\epsilon}(\tilde{x})}e^{G_{\tilde{x}}}dv_g \\ &\le C(\max_\Sigma h)\left(\frac{1}{(R\epsilon)^2}-\frac{1}{\delta^2}\right) \end{aligned} \tag{72}$$

and

$$\int_{\Sigma\setminus B_\delta(\tilde{x})}he^{\phi_\epsilon}dv_g\le C(\max_\Sigma h)\int_{\Sigma\setminus B_\delta(\tilde{x})}e^{G_{\tilde{x}}}dv_g. \tag{73}$$

(69)–(73) show that

$$\int_\Sigma he^{\phi_\epsilon}dv_g=(1+o_\epsilon(1))8\pi h(\tilde{x})e^{-2\log 8-2\log\epsilon+A_{\tilde{x}}},$$

which leads to

$$\log\int_\Sigma he^{\phi_\epsilon}dv_g=-\log 8+\log(\pi h(\tilde{x}))-2\log\epsilon+A_{\tilde{x}}+o_\epsilon(1). \tag{74}$$

Combining (74) and (67), we have

$$J_{\alpha,8\pi}(\phi_\epsilon-\bar{\phi}_\epsilon)=-8\pi-8\pi\log\pi-4\pi(2\log h(\tilde{x})+A_{\tilde{x}})+o_\epsilon(1),$$

which gives (61) by letting $\epsilon\to 0$.

References

[1] Adimurthi, Druet O. Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality. Comm Part Diff Equ 2004; 29: 295-322.

[2] Brezis H, Merle F. Uniform estimates and blow-up behavior for solutions of $-\triangle u=V(x)e^u$ in two dimensions. Comm Part Diff Equ 1991; 16: 1223-1253.

[3] Chen W, Li C. Classification of solutions of some nonlinear elliptic equations. Duke Math J 1991; 63: 615-622.

[4] Ding W, Jost J, Li J,Wang G. The differential equation $-\Delta u=8\pi-8\pi he^u$ on a compact Riemann surface. Asian J Math 1997; 1: 230-248.

[5] Fontana L. Sharp borderline Sobolev inequalities on compact Riemannian manifolds. Comm Math Helv 1993; 68: 415-454.

[6] Kazdan J, Warner F. Curvature functions for compact 2-manifolds. Ann Math 1974; 99: 14-47.

[7] Li J, Li Y. Solutions for Toda systems on Riemann surfaces. Ann Sc Norm Super Pisa Cl Sci 2005; 4: 703-728.

[8] Lu G,Yang Y. The sharp constant and extremal functions for Moser-Trudinger inequalities involving L^p norms. Discret Contin Dyn S 2009; 25: 963-979.

[9] Tintarev C. Trudinger-Moser inequality with remainder terms. J Funct Anal 2014; 266: 55-66.

[10] Wang M. The asymptotic behavior of Chern-Simons Higgs model on a compact Riemann surface with boundary. Acta Math Sin (Engl Ser) 2012; 28: 145-170.

[11] Wang M, Liu Q. The equation $\Delta u + \nabla\phi \cdot \nabla u = 8\pi c(1 - he^u)$ on a Riemann surface. J Part Diff Equ 2012; 25: 335-355.

[12] Yang Y. A sharp form of Moser-Trudinger inequality in high dimension. J Funct Anal 2006; 239: 100-126.

[13] Yang Y. A sharp form of the Moser-Trudinger inequality on a compact Riemannian surface. T Am Math Soc 2007; 359: 5761-5776.

[14] Yang Y. Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two. J Diff Equ 2015; 258: 3161-3193.

[15] Yang Y. A Trudinger-Moser inequality on a compact Riemannian surface involving Gaussian curvature. J Geom Anal 2016; 26: 2893-2913.

[16] Yang Y, Zhu X. An improved Hardy-Trudinger-Moser inequality. Ann Global Anal Geom 2016; 49: 23-41.

[17] Yang Y, Zhu X. A remark on a result of Ding-Jost-Li-Wang. P Am Math Soc 2017; 145: 3953-3959.

[18] Yang Y, Zhu X. Existence of solutions to a class of Kazdan-Warner equations on compact Riemannian surface. Sci China Math 2018; 61: 1109-1128.

[19] Zhou C. Existence of solution for mean field equation for the equilibrium turbulence. Nonlinear Anal 2008; 69: 2541-2552.