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On the summability methods of logarithmic type and the Berezin symbol

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Abstract: We prove by means of the Berezin symbols some theorems for the (L)-summability method for sequences and series. Also, we prove a new Tauberian type theorem for (L)-summability.

Key words: (L)-summability, Berezin symbol, (e)-convergence, compact operator, Tauberian type theorem, Dirichlet space, diagonal operator

1. Introduction

In this article, by applying a new functional analytic approach based on the so-called the Berezin symbol technique, we prove the following results (see [3, 4]). Also, we give a new Tauberian type theorem for (L)-summable sequences of complex numbers.

Recall that a sequence $(a_n)_{n\geq 0}$ of complex numbers a_n is said to be summable to a finite number ζ by the logarithmic method (L) (or (L)-summable to ζ) if

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

converges in the open interval $(0,1)$ and

$$\lim_{x\to 1^-} -\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \zeta.$$

The series $\sum\limits_{n=0}^{\infty} a_n$ is (L)-summable to ζ if the sequence of partial sums $s := (s_n)_{n\geq 0}$ (where $s_n = \sum\limits_{k=0}^{n} a_k$) is (L)-summable to ζ.

Theorem 1 *If $(a_k)_{k\geq 0}$ converges to ζ, then $(a_k)_{k\geq 0}$ (L)-converges to ζ.*

Theorem 2 *If the series $\sum\limits_{k=0}^{\infty} a_k$ converges to ζ, then $\sum\limits_{k=0}^{\infty} a_k$ is (L)-summable to ζ.*

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Before beginning the presentation, we recall some basic definitions and notations.

Recall that in [6], Karaev introduced the notions of an (e)-convergent sequence and (e)-convergent series for the complex numbers as follows.

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space on some suitable set Ω with reproducing kernel

$$k_{\mathcal{H},\lambda}(z) := \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z), \tag{1}$$

where $\{e_n(z)\}_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}$. Let $(a_n)_{n\geq 0}$ be any sequence of complex numbers.

(1) We say that the sequence $(a_n)_{n\geq 0}$ is (e)-*convergent* to l if $\sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2$ is convergent for all $\lambda \in \Omega$ and

$$\lim_{\lambda\to\zeta} \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2 = l$$

for every $\zeta \in \partial\Omega$.

(2) We say that the series $\sum_{n=0}^{\infty} a_n$ is (e)-*summable* to l if $\sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2$ converges for all $\lambda \in \Omega$ and

$$\lim_{\lambda\to\zeta} \sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2 = l$$

for every $\zeta \in \partial\Omega$.

It was shown that the definition of (e)-convergence of sequence and series coincides with the classical Abel convergence and the Borel convergence of sequence and series for the Hardy space $\mathcal{H}(\Omega) = H^2(\mathbb{D})$ and the Fock space $\mathcal{H}(\Omega) = \mathcal{F}(\mathbb{C})$ (see more details in [2, 5], and also [7] for related problems), respectively. One of our aims in the present article is to show in detail that (e)-summability implies the classical (L)-summability for $\mathcal{H}(\Omega) = \mathcal{D}(\mathbb{D})$, which is the Dirichlet space of analytic functions on $\mathbb{D}$, and thus to show once again the universality of the (e)-summability notion.

The associated diagonal operator D_a on $\mathcal{H}$ for any bounded sequence $(a_n)_{n\geq 0}$ of complex numbers is defined by the formula $D_a e_n(z) := a_n e_n(z)$, $n = 0, 1, 2, \ldots$, with respect to the orthonormal basis $(e_n(z))_{n\geq 0}$ of $\mathcal{H}$. An elementary calculus shows by virtue of formula (1) that

$$\widetilde{D}_a(\lambda) = \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2, \quad \lambda \in \Omega. \tag{2}$$

Following Nordgren and Rosenthal [9], we say that RKHS $\mathcal{H}(\Omega)$ is standard if the underlying set Ω is a subset of a topological space and the boundary $\partial\Omega$ is nonempty and has the property that $(k_{\mathcal{H},\lambda_n})_n$ converges weakly to 0 whenever $(\lambda_n)_n$ is a sequence in Ω that converges to a point in $\partial\Omega$. The prototypical standard RKHSs are, for example, the Hardy–Hilbert space $H^2(\mathbb{D})$, the Bergman–Hilbert space $L_a^2(\mathbb{D})$, the Fock–Hilbert space $\mathcal{F}(\mathbb{C})$, and the Dirichlet–Hilbert space $\mathcal{D}(\mathbb{D})$.

Recall that [8] the Dirichlet space $\mathcal{D}$ is the Hilbert space of analytic functions $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on the unit disk $\mathbb{D}$ with $\int\limits_{\mathbb{D}} \left|f'(z)\right|^2 dA/\pi = \sum\limits_{n=0}^{\infty} (n+1)|a_n|^2 < \infty$, where dA denotes the usual Lebesgue measure on $\mathbb{D}$.

For any bounded linear operator A on $\mathcal{D}$, the Berezin symbol of A is the function $\widetilde{A}$ defined by (see [1, 9])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{D}} \ (\lambda \in \Omega),$$

where $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of the space $\mathcal{D}$. Since the sequence $\left\{z^n/\sqrt{n+1} : n \geq 0\right\}$ is an orthonormal basis of the Dirichlet space, the reproducing kernel of $\mathcal{D}$ is given by formula (1).

$$k_\lambda(z) = \sum_{n=0}^{\infty} \frac{\left(\overline{\lambda} z\right)^n}{n+1} = \frac{1}{\overline{\lambda} z} \log \frac{1}{1 - \overline{\lambda} z}.$$

2. The results

First, we characterize the (L)-summability method in terms of the Berezin symbol.

Theorem 3 *Let $(a_n)_{n\geq 0}$ be a bounded sequence of complex numbers, and let D_a be the diagonal operator on the Dirichlet space $\mathcal{D}$ with diagonal elements a_n, $n \geq 0$, with respect to the orthonormal basis $\left\{z^n/\sqrt{n+1}\right\}_{n\geq 0}$ of $\mathcal{D}$. Then the sequence $(a_n)_{n\geq 0}$ is (L)-summable to ζ if and only if*

$$\lim_{x \to 1^-} \widetilde{D}_a\left(\sqrt{x}\right) = \zeta.$$

Proof Since $(a_n)_{n\geq 0}$ is the bounded sequence, D_a is a bounded operator on $\mathcal{D}$. If $\widehat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{D}$, then we obtain by using formula (2) for all $\lambda \in \mathbb{D}$ that

$$\begin{aligned}
\widetilde{D}_a(\lambda) &= \frac{1}{\sum_{n=0}^{\infty} \frac{\left(|\lambda|^2\right)^n}{n+1}} \sum_{n=0}^{\infty} a_n \frac{\left(|\lambda|^2\right)^n}{n+1} = \frac{1}{\frac{1}{|\lambda|^2} \log \frac{1}{1-|\lambda|^2}} \sum_{n=0}^{\infty} a_n \frac{\left(|\lambda|^2\right)^n}{n+1} \\
&= -\frac{|\lambda|^2}{\log\left(1 - |\lambda|^2\right)} \sum_{n=0}^{\infty} a_n \frac{\left(|\lambda|^2\right)^n}{n+1} = -\frac{1}{\log\left(1 - |\lambda|^2\right)} \sum_{n=0}^{\infty} a_n \frac{\left(|\lambda|^2\right)^{n+1}}{n+1},
\end{aligned}$$

and therefore $\widetilde{D}_a$ is a radial function on $\mathbb{D}$; that is, $\widetilde{D}_a(\lambda) = \widetilde{D}_a(|\lambda|)$.

Let $|\lambda|^2 = x$. Then

$$\widetilde{D}_a\left(\sqrt{x}\right) = -\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}. \tag{3}$$

We therefore get the desired assertions from (3), which proves the theorem. □

Now we are ready to prove the results.

Proof [**Proof of Theorem 1**]Let us define the diagonal operator D_a on the Dirichlet space $\mathcal{D}$ as follows:

$$D_a \frac{z^n}{\sqrt{n+1}} = a_n \frac{z^n}{\sqrt{n+1}}, \ n = 0, 1, 2, \ldots \ .$$

Since (a_k) is the bounded sequence, D_a is a bounded operator on $\mathcal{D}$. Then we get (see (3)):

$$\widetilde{D}_a\left(\sqrt{x}\right) = -\frac{1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1},\ 0 < x < 1. \tag{4}$$

Thus, we have from (4):

$$\begin{aligned}-\frac{1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1} &= -\frac{1}{\log(1-x)}\sum_{n=0}^{\infty}(a_n-\zeta)\frac{x^{n+1}}{n+1}\\ &+\zeta\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\\ &= \widetilde{D}_{a_k-\zeta}\left(\sqrt{x}\right)+\zeta.\end{aligned}$$

Since $a_k-\zeta \to 0$ as $n\to\infty$ by the condition of the theorem, we have that $D_{a_k-\zeta}$ is a compact operator on $\mathcal{D}$. Hence, its Berezin symbol vanishes on the boundary, i.e.

$$\lim_{x\to 1^-}\widetilde{D}_{a_k-l}\left(\sqrt{x}\right)=0.$$

Then we conclude from the last equality

$$\lim_{x\to 1^-} -\frac{1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}=\zeta,$$

which finishes the proof. □

Proof of Theorem 2 By using the argument to prove Theorem 1, it can easily be modified to prove the equality

$$\widetilde{D}_s\left(\sqrt{x}\right)=\lim_{x\to 1^-} -\frac{1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}, \tag{5}$$

where D_s denotes the diagonal operator on $\mathcal{D}$ with diagonal elements $s_n,\ n\geq 0$. Formula (5) means that the series $\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$ for all $0<x<1$ is convergent. On the other hand,

$$D_s = LI + D_{s-L},$$

where the diagonal operator D_{s-L} is compact, since by the hypothesis of the theorem, $s_k-\zeta\to 0$ as $n\to\infty$, and hence from (5) we get

$$\begin{aligned}\lim_{x\to 1^-} -\frac{1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1} &= \lim_{x\to 1^-}\widetilde{D}_s\left(\sqrt{x}\right)\\ &=\lim_{x\to 1^-}\left(L+\widetilde{D}_{s-L}\left(\sqrt{x}\right)\right)\\ &=\zeta+\lim_{x\to 1^-}\widetilde{D}_{s-\zeta}\left(\sqrt{x}\right)=\zeta,\end{aligned}$$

which means that the series $\sum\limits_{k=0}^{\infty}a_k$ is (L)-summable to ζ. The theorem is proved. □

Let ℓ_1^2 denote the unit sphere of the sequences space ℓ^2 :

$$\ell_1^2 := \left\{ (x_m)_{m\geq 0} \in \ell^2 : \|(x_m)\|_{\ell^2} = 1 \right\}.$$

Now we will prove a Tauberian theorem for (L)-summable sequences of complex numbers by applying a result due to Nordgren and Rosenthal [9, Corollary 2.8], which means that an operator A on a standard RKHS $\mathcal{H}(\Omega)$ is compact if and only if all the Berezin symbols of unitary orbits $U^{-1}AU$, where U is unitary on $\mathcal{H}(\Omega)$, of the operator A vanish on the boundary.

Theorem 4 *Let $(a_n)_{n\geq 0}$ be a bounded sequence of complex numbers such that $(a_n)_{n\geq 0}$ (L)-converges to ζ. Suppose that*

$$\sum_{m=0}^{\infty} a_m \left| \sum_{n=0}^{\infty} \overline{x_m^{(n)}} \frac{\lambda^n}{\sqrt{n+1}} \right|^2 = o\left(- \frac{\log\left(1-|\lambda|^2\right)}{|\lambda|^2} \right) \tag{6}$$

for every double sequence $(x_m^{(n)})_{m,n=0}^{\infty}$ with $(x_m^{(n)})_{m\geq 0} \in \ell_1^2$ $(\forall n \geq 0)$ and $(x_m^{(n)})_{n\geq 0} \in \ell_1^2$ $(\forall m \geq 0)$ whenever λ tends to infinity. Then $a_n \to 0$ as $n \to \infty$.

Proof Since (L)-$a_n \to \zeta$ if and only if (L)-$(a_n - \zeta) \to 0$, we assume without loss of generality that $\zeta = 0$. We will use the same method as in [7] for the proof of the theorem. Let $U : \mathcal{D} \to \mathcal{D}$ be an arbitrary unitary operator of the Dirichlet space $\mathcal{D}$. Then

$$U\left(\frac{z^n}{\sqrt{n+1}}\right) = \sum_{m=0}^{\infty} b_m^{(n)} \frac{z^m}{\sqrt{m+1}}$$

with $\left(b_m^{(n)}\right)_{m\geq 0} \in \ell_1^2$ for every $n \geq 0$. It is easy to see then that $\left(b_m^{(n)}\right)_{n\geq 0} \in \ell_1^2$ for every $m \geq 0$.

Then we obtain the following:

$$\begin{aligned}
\widetilde{U^{-1} D_a} U(\lambda) &= \left\langle U^{-1} D_a U \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&= -\frac{|\lambda|^2}{\log\left(1-|\lambda|^2\right)} \left\langle D_a U \sum_{n\geq 0} \frac{\overline{\lambda}^n}{\sqrt{n+1}} \frac{z^n}{\sqrt{n+1}}, U \sum_{n\geq 0} \frac{\overline{\lambda}^n}{\sqrt{n+1}} \frac{z^n}{\sqrt{n+1}} \right\rangle \\
&= -\frac{|\lambda|^2}{\log\left(1-|\lambda|^2\right)} \left\langle D_a \sum_{n\geq 0} \frac{\overline{\lambda}^n}{\sqrt{n+1}} U\left(\frac{z^n}{\sqrt{n+1}}\right), \sum_{n\geq 0} \frac{\overline{\lambda}^n}{\sqrt{n+1}} U\left(\frac{z^n}{\sqrt{n+1}}\right) \right\rangle
\end{aligned}$$

$$= -\frac{|\lambda|^2}{\log\left(1-|\lambda|^2\right)}\left\langle \sum_{n\geq 0}\frac{\overline{\lambda}^n}{\sqrt{n+1}} D_a \sum_{m\geq 0} b_m^{(n)}\frac{z^m}{\sqrt{m+1}}, \sum_{n\geq 0}\frac{\overline{\lambda}^n}{\sqrt{n+1}}\sum_{m\geq 0} b_m^{(n)}\frac{z^m}{\sqrt{m+1}}\right\rangle$$

$$= -\frac{|\lambda|^2}{\log\left(1-|\lambda|^2\right)}\left\langle \sum_{n\geq 0}\frac{\overline{\lambda}^n}{\sqrt{n+1}} \sum_{m\geq 0} b_m^{(n)} a_m\frac{z^m}{\sqrt{m+1}}, \sum_{n\geq 0}\frac{\overline{\lambda}^n}{\sqrt{n+1}}\sum_{m\geq 0} b_m^{(n)}\frac{z^m}{\sqrt{m+1}}\right\rangle$$

$$= -\frac{|\lambda|^2}{\log\left(1-|\lambda|^2\right)}\left\langle \sum_{m\geq 0} a_m \left(\sum_{n\geq 0} b_m^{(n)}\frac{\overline{\lambda}^n}{\sqrt{n+1}}\right)\frac{z^m}{\sqrt{m+1}}, \sum_{m\geq 0}\left(\sum_{n\geq 0} b_m^{(n)}\frac{\overline{\lambda}^n}{\sqrt{n+1}}\right)\frac{z^m}{\sqrt{m+1}}\right\rangle$$

$$= -\frac{|\lambda|^2}{\log\left(1-|\lambda|^2\right)}\sum_{m\geq 0} a_m \left|\sum_{n\geq 0}\overline{b_m^{(n)}}\frac{\lambda^n}{\sqrt{n+1}}\right|^2,$$

and therefore

$$\widetilde{U^{-1}D_a U}(\lambda) = -\frac{|\lambda|^2}{\log\left(1-|\lambda|^2\right)}\sum_{m\geq 0} a_m \left|\sum_{n\geq 0}\overline{b_m^{(n)}}\frac{\lambda^n}{\sqrt{n+1}}\right|^2, \quad \lambda\in\mathbb{D}. \tag{7}$$

By considering condition (6), we have from the last formula (7) that $\widetilde{U^{-1}D_a U}$ vanishes on the boundary for every unitary operator $U \in \mathcal{B}(\mathcal{D})$. Then, by the above mentioned result of Nordgren and Rosenthal [9, Corollary 2.8], we conclude that D_a is a compact operator on the Dirichlet Hilbert space $\mathcal{D}$ and as a result $\lim_{n\to\infty} a_n = 0$, which proves the theorem. □

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