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Some subclasses of analytic functions of complex order

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Abstract: In this paper, we introduce and investigate two new subclasses of analytic functions in the open unit disk in the complex plane. Several interesting properties of the functions belonging to these classes are examined. Here, sufficient, and necessary and sufficient, conditions for the functions belonging to these classes, respectively, are also given. Furthermore, various properties like order of starlikeness and radius of convexity of the subclasses of these classes and radii of starlikeness and convexity of these subclasses are examined.

Key words: Analytic function, coefficient bound, starlike function, convex function

1. Introduction and preliminaries

Let A be the class of analytic functions $f(z)$ in the open unit disk $U = \{z \in \mathrm{C} : \ |z| < 1\}$, normalized by $f(0) = 0 = f'(0) - 1$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathrm{C}, \tag{1.1}$$

and S denote the class of all functions in A that are univalent in U.

Also, let us define by T the subclass of all functions $f(z)$ in A of the form

$$f(z) = z - a_2 z^2 - a_3 z^3 - \cdots - a_n z^n - \cdots = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.2}$$

Furthermore, we will denote by $S^*(\alpha)$ and $C(\alpha)$ the subclasses of S that are, respectively, starlike and convex functions of order $\alpha \, (\alpha \in [0,1))$. By definition (see for details [4,5] and also [9]),

$$S^*(\alpha) = \left\{ f \in S : \ \mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in U \right\}, \ \alpha \in [0,1)\,, \tag{1.3}$$

and

$$C(\alpha) = \left\{ f \in S : \ \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in U \right\}, \ \alpha \in [0,1)\,. \tag{1.4}$$

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For convenience, $S^* = S^*(0)$ and $C = C(0)$ are, respectively, starlike and convex functions in U. It is easy to verify that $C \subset S^* \subset S$. For details on these classes, one could refer to the monograph by Goodman [5].

Note that we will use $TS^*(\alpha) = T \cap S^*(\alpha)$, $TC(\alpha) = T \cap C(\alpha)$, and in the special case we have $TS^* = T \cap S^*$, $TC = T \cap C$ for $\alpha = 0$.

An interesting unification of the function classes $S^*(\alpha)$ and $C(\alpha)$ is provided by the class $S^*C(\alpha, \beta)$ of functions $f \in S$, which also satisfies the following condition:

$$Re\left\{\frac{zf'(z) + \beta z^2 f''(z)}{\beta zf'(z) + (1-\beta)f(z)}\right\} > \alpha, \alpha \in [0,1), \beta \in [0,1], \ z \in U.$$

In the special case, for $\beta = 0$ and $\beta = 1$, respectively, we have $S^*C(\alpha, 0) = S^*(\alpha)$ and $S^*C(\alpha, 1) = C(\alpha)$ in terms of the simpler classes $S^*(\alpha)$ and $C(\alpha)$, defined by (1.3) and (1.4), respectively. Also, we will use $TS^*C(\alpha, \beta) = T \cap S^*C(\alpha, \beta)$.

The class $TS^*C(\alpha, \beta)$ and various other subclasses of T were studied by Altintaş et al. [2,3], Irmak et al. [6], Altintaş [1], Mustafa [7], and Silverman [8].

Inspired by the aforementioned works, we define a subclass of analytic functions as follows.

Definition 1.1 *A function $f \in S$ given by (1.1) is said to be in the class $S^*C(\alpha, \beta; \tau)$, $\alpha \in [0,1)$, $\beta \in [0,1]$, $\tau \in \mathrm{C}^* = \mathrm{C} - \{0\}$ if the following condition is satisfied:*

$$Re\left\{1 + \frac{1}{\tau}\left[\frac{zf'(z) + \beta z^2 f''(z)}{\beta zf'(z) + (1-\beta)f(z)} - 1\right]\right\} > \alpha, z \in U, \alpha \in [0,1), \beta \in [0,1], \tau \in \mathrm{C}^*.$$

In the special case, we have $S^*C(\alpha, \beta; 1) = S^*C(\alpha, \beta)$ for $\tau = 1$. Note that we will use $TS^*C(\alpha, \beta; \tau) = T \cap S^*C(\alpha, \beta; \tau)$. Also, we have $TS^*C(\alpha, \beta; 1) = TS^*C(\alpha, \beta)$ and $TS^*C(\alpha, 0; \tau) = T \cap S^*(\alpha; \tau) = TS^*(\alpha; \tau)$, $TS^*C(\alpha, 1; \tau) = T \cap C(\alpha; \tau) = TC(\alpha; \tau)$.

In this paper, two new subclasses, $S^*C(\alpha, \beta; \tau)$ and $TS^*C(\alpha, \beta; \tau)$, of the analytic functions in the open unit disk are introduced. Various characteristic properties of the functions belonging to these classes are examined. Sufficient conditions for the analytic functions belonging to the class $S^*C(\alpha, \beta; \tau)$, and necessary and sufficient conditions for those belonging to the class $TS^*C(\alpha, \beta; \tau)$, are also given. Furthermore, various properties like order of starlikeness and radius of convexity of the subclasses $TC(\alpha; \tau)$ and $TS^*(\alpha; \tau)$, respectively, and radii of starlikeness and convexity of the subclasses $S^*C(\alpha, \beta; \tau)$ and $TS^*C(\alpha, \beta; \tau)$ are examined.

2. Coefficient bounds for the classes $S^*C(\alpha, \beta; \tau)$ and $TS^*C(\alpha, \beta; \tau)$

In this section, we will examine some inclusion results of the subclasses $S^*C(\alpha, \beta; \tau)$ and $TS^*C(\alpha, \beta; \tau)$ of analytic functions in the open unit disk. Also, we give coefficient bound estimates for the functions belonging to these classes.

A sufficient condition for the functions in class $S^*C(\alpha, \beta; \tau)$ is given by the following theorem.

Theorem 2.1 *Let $f \in A$. Then the function $f(z)$ belongs to the class $S^*C(\alpha, \beta; \tau)$, $\alpha \in [0,1)$, $\beta \in [0,1]$, $\tau \in \mathrm{C}^* = \mathrm{C} - \{0\}$ if the following condition is satisfied:*

$$\sum_{n=2}^{\infty} [n + (1-\alpha)|\tau| - 1]\,[1 + (n-1)\beta]\,|a_n| \le (1-\alpha)|\tau|.$$

The result is sharp for the functions

$$f_n(z) = z + \frac{(1-\alpha)\,|\tau|}{[n + (1-\alpha)\,|\tau| - 1]\,[1 + (n-1)\beta]} z^n, \; z \in U, \; n = 2, 3, \ldots .$$

Proof According to Definition 1.1, a function $f(z)$ is in the class $S^*C(\alpha, \beta; \tau)$, $\alpha \in [0,1)$, $\beta \in [0,1]$, $\tau \in \mathrm{C}^* = \mathrm{C} - \{0\}$ if and only if

$$Re\left\{1 + \frac{1}{\tau}\left[\frac{zf'(z) + \beta z^2 f''(z)}{\beta z f'(z) + (1-\beta) f(z)} - 1\right]\right\} > \alpha.$$

It suffices to show that

$$\left|\frac{1}{\tau}\left[\frac{zf'(z) + \beta z^2 f''(z)}{\beta z f'(z) + (1-\beta) f(z)} - 1\right]\right| < 1 - \alpha. \tag{2.1}$$

Considering (1.1), by simple computation, we write

$$\begin{aligned} \left|\frac{1}{\tau}\left[\frac{zf'(z) + \beta z^2 f''(z)}{\beta z f'(z) + (1-\beta) f(z)} - 1\right]\right| &= \left|\frac{1}{\tau}\frac{\sum_{n=2}^{\infty}(n-1)\,[1+\beta(n-1)]\,a_n z^n}{z + \sum_{n=2}^{\infty}[1+\beta(n-1)]\,a_n z^n}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}(n-1)\,[1+\beta(n-1)]\,|a_n|}{|\tau|\,\{1 - \sum_{n=2}^{\infty}[1+\beta(n-1)]\,|a_n|\}}. \end{aligned}$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}(n-1)\,[1+\beta(n-1)]\,|a_n| \leq |\tau|\,(1-\alpha)\left\{1 - \sum_{n=2}^{\infty}[1+\beta(n-1)]\,|a_n|\right\},$$

which is equivalent to

$$\sum_{n=2}^{\infty}[n + (1-\alpha)\,|\tau| - 1]\,[1 + (n-1)\beta]\,|a_n| \leq (1-\alpha)\,|\tau|\,. \tag{2.2}$$

Hence, inequality (2.1) is true if condition (2.2) is satisfied.

Thus, the proof of Theorem 2.1 is completed. □

Setting $\tau = 1$ in Theorem 2.1, we arrive at the following corollary.

Corollary 2.2 *The function $f(z)$ by definition by (1.1) belongs to the class $S^*C(\alpha, \beta)$, $\alpha \in [0,1)$, $\beta \in [0,1]$ if the following condition is satisfied:*

$$\sum_{n=2}^{\infty}(n-\alpha)\,[1+\beta(n-1)]\,|a_n| \leq 1 - \alpha.$$

The result is sharp for the functions

$$f_n(z) = z + \frac{1-\alpha}{(n-\alpha)\,[1+\beta(n-1)]} z^n, \; z \in U, \; n = 2, 3, \ldots .$$

Remark 2.3 *The result obtained in Corollary 2.2 verifies Corollary 2.1 in [8].*

Choosing $\beta = 0$ in Corollary 2.2, we have the following result.

Corollary 2.4 *(see [8, p. 110, Theorem 1]) The function $f(z)$ by definition by (1.1) belongs to the class $S^*(\alpha)$, $\alpha \in [0,1)$ if the following condition is satisfied:*

$$\sum_{n=2}^{\infty}(n-\alpha)\left|a_n\right| \leq 1-\alpha.$$

The result is sharp for the functions

$$f_n(z) = z + \frac{1-\alpha}{n-\alpha}z^n,\ z \in U,\ n = 2,3,\dots .$$

Taking $\beta = 1$ in Corollary 2.2, we arrive at the following result.

Corollary 2.5 *(see [8, p. 110, Corollary of Theorem1]) The function $f(z)$ by definition by (1.1) belongs to the class $C(\alpha)$, $\alpha \in [0,1)$ if the following condition is satisfied:*

$$\sum_{n=2}^{\infty}n(n-\alpha)\left|a_n\right| \leq 1-\alpha.$$

The result is sharp for the functions

$$f_n(z) = z + \frac{1-\alpha}{n(n-\alpha)}z^n,\ z \in U,\ n = 2,3,\dots .$$

Remark 2.6 *The results obtained in Corollary 2.4 and 2.5 verify Corollary 2.3 and 2.4 in [7], respectively.*

From the following theorem, we see that for the functions in the class $TS^*C(\alpha,\beta;\tau), \alpha \in [0,1), \beta \in [0,1], \tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$ the converse of Theorem 2.1 is also true.

Theorem 2.7 *Let $f \in T$. Then the function $f(z)$ belongs to the class $TS^*C(\alpha,\beta;\tau)$, $\alpha \in [0,1)$, $\beta \in [0,1]$, $\tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$ if and only if*

$$\sum_{n=2}^{\infty}\left[n + (1-\alpha)\left|\tau\right| - 1\right]\left[1 + (n-1)\beta\right]a_n \leq (1-\alpha)\left|\tau\right|.$$

The result is sharp for the functions

$$f_n(z) = z - \frac{(1-\alpha)\left|\tau\right|}{\left[n + (1-\alpha)\left|\tau\right| - 1\right]\left[1 + (n-1)\beta\right]}z^n,\ z \in U,\ n = 2,3,\dots ..$$

Proof The proof of the sufficiency of the theorem can be proved similarly to the proof of Theorem 2.1. Therefore, we will prove only the necessity of the theorem.

Assume that $f \in TS^*C(\alpha,\beta;\tau)$, $\alpha \in [0,1), \beta \in [0,1], \tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$; that is,

$$Re\left\{1 + \frac{1}{\tau}\left[\frac{zf'(z) + \beta z^2 f''(z)}{\beta z f'(z) + (1-\beta)f(z)} - 1\right]\right\} > \alpha,\ z \in U. \tag{2.3}$$

Using (1.2) and (2.3), we can easily show that

$$Re\left\{\frac{-\sum_{n=2}^{\infty}(n-1)\left[1+\beta(n-1)\right]a_n z^n}{\tau\left\{z-\sum_{n=2}^{\infty}\left[1+\beta(n-1)\right]a_n z^n\right\}}\right\}>\alpha-1.$$

The expression

$$\frac{-\sum_{n=2}^{\infty}(n-1)\left[1+\beta(n-1)\right]a_n z^n}{\tau\left\{z-\sum_{n=2}^{\infty}\left[1+\beta(n-1)\right]a_n z^n\right\}}$$

is real if we choose z real.

Thus, from the previous inequality, letting $z\to 1$ through real values, we have

$$\frac{-\sum_{n=2}^{\infty}(n-1)\left[1+\beta(n-1)\right]a_n}{\tau\left\{1-\sum_{n=2}^{\infty}\left[1+\beta(n-1)\right]a_n\right\}}\geq\alpha-1. \tag{2.4}$$

We will examine the last inequality depending on the different cases of the sign of parameter τ as follows.

Let $\tau>0$. Then, from (2.4), we have

$$-\sum_{n=2}^{\infty}(n-1)\left[1+\beta(n-1)\right]a_n\geq(\alpha-1)\tau\left\{1-\sum_{n=2}^{\infty}\left[1+\beta(n-1)\right]a_n\right\},$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left[n+(1-\alpha)\tau-1\right]\left[1+\beta(n-1)\right]a_n\leq(1-\alpha)\tau. \tag{2.5}$$

Now, let $\tau<0$. Then, since $\tau=-|\tau|$, from (2.4), we get

$$\frac{\sum_{n=2}^{\infty}(n-1)\left[1+\beta(n-1)\right]a_n}{|\tau|\left\{1-\sum_{n=2}^{\infty}\left[1+\beta(n-1)\right]a_n\right\}}\geq\alpha-1;$$

that is,

$$\sum_{n=2}^{\infty}(n-1)\left[1+\beta(n-1)\right]a_n\geq(\alpha-1)\left|\tau\right|\left\{1-\sum_{n=2}^{\infty}\left[1+\beta(n-1)\right]a_n\right\}.$$

Therefore,

$$\sum_{n=2}^{\infty}\left[n+(\alpha-1)\left|\tau\right|-1\right]\left[1+\beta(n-1)\right]a_n\geq-(1-\alpha)\left|\tau\right|.$$

Since $\alpha<1$ (or $1-\alpha>\alpha-1$), from the last inequality, we have

$$\sum_{n=2}^{\infty}\left[n+(1-\alpha)\left|\tau\right|-1\right]\left[1+\beta(n-1)\right]a_n\geq-(1-\alpha)\left|\tau\right|. \tag{2.6}$$

Thus, from (2.6) and (2.7), the proof of the necessity of theorem and the proof of theorem are completed. □

The special case of Theorem 2.7 was proved by Altintaş et al. [2], $\tau=1$ (there $p=n=1$).

Setting $\tau=1$ in Theorem 2.7, we arrive at the following corollary.

Corollary 2.8 *The function $f(z)$ by definition by (1.2) belongs to the class $TS^*C(\alpha,\beta)$, $\alpha \in [0,1)$, $\beta \in [0,1]$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)\left[1+\beta(n-1)\right]a_n \leq 1-\alpha.$$

Remark 2.9 *The result obtained in Corollary 2.8 verifies Theorem 1 in [2].*

Taking $\beta = 0$ in Corollary 2.8, we have the following result.

Corollary 2.10 *(see [8, p. 110, Theorem 2]) The function $f(z)$ by definition by (1.2) belongs to the class $TS^*(\alpha)$, $\alpha \in [0,1)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n \leq 1-\alpha.$$

Choosing $\beta = 1$ in Corollary 2.8, we arrive at the following result.

Corollary 2.11 *(see [8, p. 111, Corollary 2]) The function $f(z)$ by definition by (1.2) belongs to the class $TC(\alpha)$, $\alpha \in [0,1)$ if and only if*

$$\sum_{n=2}^{\infty}n(n-\alpha)a_n \leq 1-\alpha.$$

Remark 2.12 *The results obtained in Corollary 2.10 and 2.11 verify Corollary 2.7 and 2.8 in [7], respectively.*

Corollary 2.13 *The function $f(z)$ by definition by (1.2) belongs to the class $TS^*(\alpha;\tau)$, $\alpha \in [0,1), \tau \in \mathrm{R}^*$ if and only if*

$$\sum_{n=2}^{\infty}\left[n+(1-\alpha)\left|\tau\right|-1\right]a_n \leq (1-\alpha)\left|\tau\right|.$$

Corollary 2.14 *The function $f(z)$ by definition by (1.2) belongs to the class $TC(\alpha;\tau)$, $\alpha \in [0,1), \tau \in \mathrm{R}^*$ if and only if*

$$\sum_{n=2}^{\infty}n\left[n+(1-\alpha)\left|\tau\right|-1\right]a_n \leq (1-\alpha)\left|\tau\right|.$$

On the coefficient bound estimates of the functions belonging in the class $TS^*C(\alpha,\beta;\tau)$, we give the following theorem.

Theorem 2.15 *Let the function $f(z)$ by definition by (1.2) belong to the class $TS^*C(\alpha,\beta;\tau)$, $\alpha \in [0,1), \beta \in [0,1], \tau \in \mathrm{R}^* = \mathrm{R}-\{0\}$. Then,*

$$\sum_{n=2}^{\infty}\left|a_n\right| \leq \frac{(1-\alpha)\left|\tau\right|}{(1+\beta)\left[1+(1-\alpha)\left|\tau\right|\right]} \tag{2.7}$$

and

$$\sum_{n=2}^{\infty}n\left|a_n\right| \leq \frac{2(1-\alpha)\left|\tau\right|}{(1+\beta)\left[1+(1-\alpha)\left|\tau\right|\right]}. \tag{2.8}$$

Proof Using Theorem 2.7, we obtain

$$[1+(1-\alpha)\left|\tau\right|](1+\beta)\sum_{n=2}^{\infty}\left|a_n\right| \le \sum_{n=2}^{\infty}[n+(1-\alpha)\left|\tau\right|-1]\left[1+(n-1)\beta\right]\left|a_n\right| \le (1-\alpha)\left|\tau\right|.$$

From here, we can easily show that (2.7) is true.

Similarly, we write

$$(1+\beta)\sum_{n=2}^{\infty}[n+(1-\alpha)\left|\tau\right|-1]\left|a_n\right| \le \sum_{n=2}^{\infty}[n+(1-\alpha)\left|\tau\right|-1]\left[1+(n-1)\beta\right]\left|a_n\right| \le (1-\alpha)\left|\tau\right|;$$

that is,

$$(1+\beta)\sum_{n=2}^{\infty}n\left|a_n\right| \le (1-\alpha)\left|\tau\right| + [1-(1-\alpha)\left|\tau\right|](1+\beta)\sum_{n=2}^{\infty}\left|a_n\right|.$$

Using (2.7) in the last inequality, we obtain

$$(1+\beta)\sum_{n=2}^{\infty}n\left|a_n\right| \le \frac{2(1-\alpha)\left|\tau\right|}{1+(1-\alpha)\left|\tau\right|},$$

which immediately yields inequality (2.8).

Thus, the proof of Theorem 2.15 is completed. □

Setting $\tau = 1$ in Theorem 2.15, we obtain the following corollary.

Corollary 2.16 *Let the function* $f(z)$ *by definition by* (1.2) *belong to the class* $TS^*C(\alpha,\beta)$, $\alpha \in [0,1)$, $\beta \in [0,1]$. *Then,*

$$\sum_{n=2}^{\infty}\left|a_n\right| \le \frac{1-\alpha}{(2-\alpha)(1+\beta)} \quad \textit{and} \quad \sum_{n=2}^{\infty}n\left|a_n\right| \le \frac{2(1-\alpha)}{(2-\alpha)(1+\beta)}.$$

Remark 2.17 *The result obtained in Corollary 2.13 verifies Lemma 2 (with* $n = p = 1$ *) of [2].*

From Theorem 2.7, for the coefficient bound estimates, we arrive at the following result.

Corollary 2.18 *Let* $f \in TS^*C(\alpha,\beta;\tau)$, $\alpha \in [0,1)$, $\beta \in [0,1]$, $\tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$. *Then,*

$$\left|a_n\right| \le \frac{(1-\alpha)\left|\tau\right|}{[n+(1-\alpha)\left|\tau\right|-1]\left[1+(n-1)\beta\right]}, \; n = 2,3,\dots\,.$$

Remark 2.19 *Numerous consequences of Corollary 2.14 can indeed be deduced by specializing the various parameters involved. Many of these consequences were proved by earlier workers on the subject (see, for example, [1, 8, 10]).*

3. Order of starlikeness and radius of convexity for the classes $TC(\alpha;\tau)$ and $TS^*(\alpha;\tau)$

In this section, we will examine some properties like order of starlikeness and radius of convexity of the subclasses $TC(\alpha;\tau)$ and $TS^*(\alpha;\tau)$. On this, we can give the following theorem.

Theorem 3.1 *Let $f \in TC(\alpha;\tau)$, $\alpha \in [0,1)$, $\tau \in \mathrm{R}^*$. Then the function $f(z)$ belongs to the class $TS^*(\alpha_0;\tau)$, where $\alpha_0 = \frac{2+(1-\alpha)(|\tau|-1)}{2+(1-\alpha)|\tau|}$; that is, $f \in TS^*(\alpha_0;\tau)$. The result is sharp for the functions*

$$f_n(z) = z - \frac{(1-\alpha)|\tau|}{n(n+(1-\alpha)|\tau|-1)} z^n,\ z \in U,\ n = 2,3,\dots .$$

Proof In view of Corollary 2.13 and Corollary 2.14, we must prove that

$$\sum_{n=2}^{\infty} \frac{n-1+\left[1-\frac{2+(1-\alpha)(|\tau|-1)}{2+(1-\alpha)|\tau|}\right]|\tau|}{\left[1-\frac{2+(1-\alpha)(|\tau|-1)}{2+(1-\alpha)|\tau|}\right]|\tau|} a_n \le 1 \tag{3.1}$$

if

$$\sum_{n=2}^{\infty} \frac{n[n-1+(1-\alpha)|\tau|]}{(1-\alpha)|\tau|} a_n \le 1.$$

It suffices to show that

$$\frac{n[n-1+(1-\alpha)|\tau|]}{(1-\alpha)|\tau|} \ge \frac{n-1+\left[1-\frac{2+(1-\alpha)(|\tau|-1)}{2+(1-\alpha)|\tau|}\right]|\tau|}{\left[1-\frac{2+(1-\alpha)(|\tau|-1)}{2+(1-\alpha)|\tau|}\right]|\tau|} \tag{3.2}$$

for all $n = 2,3,\dots$.

The last inequality is equivalent to

$$\frac{n[n-1+(1-\alpha)|\tau|]-(1-\alpha)|\tau|}{1-\alpha} \ge (n-1)[2+(1-\alpha)|\tau|].$$

Taking into account that $\alpha \ge 0$ $\left(\text{or } \frac{1}{1-\alpha} \ge 1\right)$, it suffices to show that

$$n[n-1+(1-\alpha)|\tau|]-(1-\alpha)|\tau| \ge (n-1)[2+(1-\alpha)|\tau|],$$

which is equivalent to $n(n-1) \ge 2(n-1)$; that is, $n \ge 2$. Thus, inequality (3.2) is provided for all $n = 2,3,\dots$.

With this the proof of Theorem 3.1 is completed. □

Corollary 3.2 *(see [8, p. 113, Theorem 7]) If $f \in TC(\alpha)$, $\alpha \in [0,1)$, then the function $f(z)$ belongs to the class $TS^*(\alpha_0)$, where $\alpha_0 = \frac{2}{3-\alpha}$; that is, $f \in TS^*\left(\frac{2}{3-\alpha}\right)$. The result is sharp for the functions*

$$f_n(z) = z - \frac{1-\alpha}{n(n-\alpha)} z^n,\ z \in U,\ n = 2,3,\dots .$$

Note 3.1. *There is no converse to Theorem 3.1. That is, a function in $TS^*(\alpha;\tau)$ need not be convex. To show this, we need only find coefficients a_n, $n=2,3,...$ for which*

$$\sum_{n=2}^{\infty}\frac{n-1+(1-\alpha)\left|\tau\right|}{(1-\alpha)\left|\tau\right|}a_n\leq 1 and \sum_{n=2}^{\infty}\frac{n\left(n-1+\left|\tau\right|\right)}{\left|\tau\right|}a_n\geq 1. \tag{3.3}$$

Note that the functions $f_n(z)=z-\frac{(1-\alpha)|\tau|}{n-1+(1-\alpha)|\tau|}z^n$ for $n\geq\left[\left[\frac{1}{(1-\alpha)|\tau|}\right]\right]+1$ all satisfy both inequalities of (3.3), where $[\![x]\!]$ is the exact value of number x.

By considering the above note, we now determine the radius of convexity for functions in $TS^*(\alpha;\tau)$. The following theorem is about this.

Theorem 3.3 *Let $f\in TS^*(\alpha;\tau)$, $\alpha\in[0,1)$, $\tau\in \mathrm{R}^*$. Then the radius of convexity of the function $f(z)$ is $r^c(f)=r(\alpha;\tau)=\inf\left\{\left[\frac{n-1+(1-\alpha)|\tau|}{n^2(1-\alpha)|\tau|}\right]^{1/(n-1)}:n=2,3,...\right\}$; that is, the function $f(z)$ is convex in the disk $U_{r(\alpha;\tau)}=\{z:\ |z|<r(\alpha;\tau)\}$.*

Proof *Proof.* Let $f\in TS^*(\alpha;\tau)$, $\alpha\in[0,1)$, $\tau\in\mathrm{R}^*$. It suffices that $|zf''(z)/f'(z)|\leq 1$ for $|z|<r(\alpha;\tau)$.

By simple computation, we have

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{-\sum_{n=2}^{\infty}n(n-1)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}na_nz^{n-1}}\right|\leq\frac{\sum_{n=2}^{\infty}n(n-1)a_n\left|z\right|^{n-1}}{1-\sum_{n=2}^{\infty}na_n\left|z\right|^{n-1}}.$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}n(n-1)a_n\left|z\right|^{n-1}\leq 1-\sum_{n=2}^{\infty}na_n\left|z\right|^{n-1},$$

which is equivalent to

$$\sum_{n=2}^{\infty}n^2a_n\left|z\right|^{n-1}\leq 1. \tag{3.4}$$

According to Corollary 2.14,

$$\sum_{n=2}^{\infty}\frac{n-1+(1-\alpha)\left|\tau\right|}{(1-\alpha)\left|\tau\right|}a_n\leq 1.$$

Hence, inequality (3.4) will be true if

$$n^2\left|z\right|^{n-1}\leq\frac{n-1+(1-\alpha)\left|\tau\right|}{(1-\alpha)\left|\tau\right|},\ n=2,3,....$$

Solving the last inequality for $|z|$, we obtain

$$|z|\leq\left[\frac{n-1+(1-\alpha)\left|\tau\right|}{n^2\left(1-\alpha\right)\left|\tau\right|}\right]^{1/(n-1)},\ n=2,3,....$$

From here, we obtain the desired result. The proof of the theorem is completed. □

Corollary 3.4 *(see [8, p. 113, Theorem 8]) If* $f \in TS^*(\alpha)$, $\alpha \in [0,1)$, *then the radius of convexity of the function* $f(z)$ *is* $r^c(f) = r(\alpha) = \inf\left\{ \left[\frac{n-\alpha}{n^2(1-\alpha)}\right]^{1/(n-1)} : n = 2,3,... \right\}$ *; that is, the function* $f(z)$ *is convex in the disk* $U_{r(\alpha;\tau)} = \{z : \ |z| < r(\alpha)\}$.

4. Radii of convexity and starlikeness of the class $TS^*C(\alpha,\beta;\tau)$

In this section, we determine the radius of convexity and starlikeness for the class $TS^*C(\alpha,\beta;\tau)$.

Theorem 4.1 *Let* $f \in TS^*C(\alpha,\beta;\tau), \alpha \in [0,1), \beta \in [0,1], \tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$. *Then the radius of convexity of function* $f(z)$ *is*

$$r^c(f) = r^c(\alpha,\beta;\tau) = \inf\left\{ \left\{ \frac{[n-1+(1-\alpha)|\tau|]\,[1+(n-1)\beta]}{n^2(1-\alpha)|\tau|} \right\}^{1/(n-1)}, \ n = 2,3,... \right\};$$

that is, the function $f(z)$ *is convex in the disk* $U_{r^c(f)} = \{z : \ |z| < r^c(f)\}$.

Proof Let $f \in TS^*C(\alpha,\beta;\tau), \alpha \in [0,1), \beta \in [0,1], \tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$. It suffices to show that $|zf''(z)/f'(z)| \leq 1$ for $|z| < r(\alpha,\beta;\tau)$. In view of the proof of Theorem 3.3, we must prove that (3.4) is true for $|z| < r(\alpha,\beta;\tau)$. From the result of Theorem 2.7, it is easy to see that inequality (3.4) will be true if

$$n^2 |z|^{n-1} \leq \frac{[n-1+(1-\alpha)|\tau|]\,[1+(n-1)\beta]}{(1-\alpha)|\tau|}, \ n = 2,3,....$$

Solving the last inequality for $|z|$, we obtain

$$|z| \leq \left\{ \frac{[n-1+(1-\alpha)|\tau|]\,[1+(n-1)\beta]}{n^2(1-\alpha)|\tau|} \right\}^{1/(n-1)}, \ n = 2,3,....$$

Thus, the proof of Theorem 4.1 is completed. □

Corollary 4.2 *If* $f \in TS^*C(\alpha,\beta), \alpha \in [0,1), \beta \in [0,1]$, *then the radius of convexity of function* $f(z)$ *is*

$$r^c(f) = r^c(\alpha,\beta) = \inf\left\{ \left\{ \frac{(n-\alpha)\,[1+(n-1)\beta]}{n^2(1-\alpha)} \right\}^{1/(n-1)}, \ n = 2,3,... \right\};$$

that is, the function $f(z)$ *is convex in the disk* $U_{r^c(f)} = \{z : \ |z| < r^c(f)\}$.

Theorem 4.3 *Let* $f \in TS^*C(\alpha,\beta;\tau), \alpha \in [0,1), \beta \in [0,1], \tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$. *Then the radius of starlikeness of function* $f(z)$ *is*

$$r^*(f) = r^*(\alpha,\beta;\tau) = \inf\left\{ \left\{ \frac{[n-1+(1-\alpha)|\tau|]\,[1+(n-1)\beta]}{n(1-\alpha)|\tau|} \right\}^{1/n}, \ n = 2,3,... \right\};$$

that is, the function $f(z)$ *is starlike in the disk* $U_{r^*(f)} = \{z : \ |z| < r^*(f)\}$.

Proof It suffices to show that $|zf'(z)/f(z)-1|\leq 1$ for all $|z|\leq r^*(f)$. By simple computation we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{\sum_{n=2}^{\infty}(n-1)\,a_n z^n}{z-\sum_{n=2}^{\infty}a_n z^n}\right|\leq\frac{\sum_{n=2}^{\infty}(n-1)\,a_n\,|z|^n}{1-\sum_{n=2}^{\infty}a_n\,|z|^n}.$$

From this, we can easily see that $|zf'(z)/f(z)-1|\leq 1$ if

$$\sum_{n=2}^{\infty}(n-1)\,a_n\,|z|^n\leq 1-\sum_{n=2}^{\infty}a_n\,|z|^n\,,$$

which is equivalent to

$$\sum_{n=2}^{\infty}na_n\,|z|^n\leq 1. \tag{4.1}$$

According to Theorem 2.7, it is clear that inequality (4.1) will be true if

$$n\,|z|^n\leq\frac{[n+(1-\alpha)\,|\tau|-1]\,[1+(n-1)\,\beta]}{(1-\alpha)\,|\tau|},n=2,3,....$$

From this,

$$|z|\leq\left\{\frac{[n+(1-\alpha)\,|\tau|-1]\,[1+(n-1)\,\beta]}{n\,(1-\alpha)\,|\tau|}\right\}^{1/n},n=2,3,...;$$

that is,

$$|z|\leq\inf\left\{\left\{\frac{[n+(1-\alpha)\,|\tau|-1]\,[1+(n-1)\,\beta]}{n\,(1-\alpha)\,|\tau|}\right\}^{1/n},n=2,3,...\right\}.$$

Thus, inequality (4.1) is provided for $|z|\leq r^*(f)=r^*(\alpha,\beta;\tau)$, where

$$r^*(\alpha,\beta;\tau)=\inf\left\{\left\{\frac{[n+(1-\alpha)\,|\tau|-1]\,[1+(n-1)\,\beta]}{n\,(1-\alpha)\,|\tau|}\right\}^{1/n},n=2,3,...\right\}.$$

With this, the proof of Theorem 4.3 is completed. □

Corollary 4.4 *If $f\in TS^*C(\alpha,\beta),\alpha\in[0,1)\,,\beta\in[0,1]$, then the radius of starlikeness of function $f(z)$ is*

$$r^*(f)=r^*(\alpha,\beta)=\inf\left\{\left\{\frac{(n-\alpha)\,[1+(n-1)\beta]}{n\,(1-\alpha)}\right\}^{1/n},\ n=2,3,...\right\};$$

that is, the function $f(z)$ is starlike in the disk $U_{r^*(f)}=\{z:\ |z|<r^*(f)\}$.

5. Integral transforms of the function class $TS^*C(\alpha,\beta;\tau)$

In this section, we consider integral transforms of the function class $TS^*C(\alpha,\beta;\tau)$ of the type

$$F\,(f,z)=(2-c)\int_0^1\frac{f(zx)}{x^c}dx,\ c\in(0,2)\,. \tag{5.1}$$

In this section our purpose is to investigate some geometric properties of integral transforms (5.1). We will give the following theorem on the fact that the integral transform $F(f,z)$ of the function $f(z)$ belongs to the class $TS^{*}C(\alpha,\beta;\tau)$.

Theorem 5.1 *Let the function* $f \in T$ *defined by formula* (1.2) *be in the class* $TS^{*}C(\alpha,\beta;\tau)$, $\alpha \in [0,1)$, $\beta \in [0,1]$, $\tau \in \mathrm{R}^{*} = \mathrm{R} - \{0\}$. *Then the integral transform* (5.1) *belongs to the class* $TS^{*}C(\gamma,\beta;\tau)$, $\gamma \in [0,\gamma_0]$, *where* $\gamma_0 = 1 - \frac{(1-\alpha)(2-c)}{3-c+(1-\alpha)|\tau|}$. *The result of the theorem is sharp.*

Proof Let $f \in TS^{*}C(\alpha,\beta;\tau)$, $\alpha \in [0,1)$, $\beta \in [0,1]$, $\tau \in \mathrm{R}^{*} = \mathrm{R} - \{0\}$. Then, by simple computation, we can write

$$F(f,z) = z - \sum_{n=2}^{\infty} \frac{2-c}{n+1-c}\, a_n z^n. \tag{5.2}$$

It is clear that $F \in T$. Now we need to show that $F \in TS^{*}C(\gamma,\beta;\tau)$. For this, according to Theorem 2.7, we need to find the values of γ for which the following inequality is satisfied:

$$\sum_{n=2}^{\infty} \frac{2-c}{n+1-c} \frac{[n+(1-\gamma)|\tau|-1][1+(n-1)\beta]}{(1-\gamma)|\tau|}\, a_n \le 1. \tag{5.3}$$

In view of Theorem 2.7, since $f \in TS^{*}C(\alpha,\beta;\tau)$, $\alpha \in [0,1)$, $\beta \in [0,1]$, $\tau \in \mathrm{R}^{*} = \mathrm{R} - \{0\}$, we have

$$\sum_{n=2}^{\infty} \frac{[n+(1-\alpha)|\tau|-1][1+(n-1)\beta]}{(1-\alpha)|\tau|}\, a_n \le 1. \tag{5.4}$$

It is easy to see that inequality (5.3) will be true if

$$\frac{2-c}{n+1-c} \frac{[n+(1-\gamma)|\tau|-1][1+(n-1)\beta]}{(1-\gamma)|\tau|} \le \frac{[n+(1-\alpha)|\tau|-1][1+(n-1)\beta]}{(1-\alpha)|\tau|},$$

which is equivalent to

$$\frac{2-c}{n+1-c} \frac{n-1+(1-\gamma)|\tau|}{1-\gamma} \le \frac{n-1+(1-\alpha)|\tau|}{1-\alpha}.$$

Solving the last inequality for γ, we obtain

$$\gamma \le 1 - \frac{(1-\alpha)(2-c)}{n+1-c+(1-\alpha)|\tau|},\ n = 2,3,....$$

From the last inequality, we can write

$$\gamma \le \inf\left\{1 - \frac{(1-\alpha)(2-c)}{n+1-c+(1-\alpha)|\tau|},\ n = 2,3,...\right\}. \tag{5.5}$$

By simple computation, we can easily show that the function $\varphi : \mathrm{N}_2 \to \mathrm{R}$ defined by

$$\varphi(n) = 1 - \frac{(1-\alpha)(2-c)}{n+1-c+(1-\alpha)|\tau|}$$

is an increasing function, where $\mathrm{N}_2 = \{2, 3, ...\} = \mathrm{N} - \{1\}$. Using this, from (5.5), we obtain

$$\gamma \leq 1 - \frac{(1-\alpha)(2-c)}{3 - c + (1-\alpha)|\tau|}.$$

Thus, the proof of Theorem 5.1 is completed. □

Corollary 5.2 *If $f \in TS^*C(\alpha, \beta; \tau)$, $\alpha \in [0, 1), \beta \in [0, 1], \tau \in \mathrm{R}^* = \mathrm{R} - \{0\}$, then the integral transform*

$$F(f, z) = \int_0^1 \frac{f(zx)}{x}\, dx$$

is in the class $TS^*C(\gamma, \beta; \tau)$, $\gamma \in [0, \gamma_0]$, where $\gamma_0 = 1 - \frac{1-\alpha}{2+(1-\alpha)|\tau|}$. The result of the theorem is sharp.

Corollary 5.3 *If $f \in TS^*C(\alpha, \beta)$, $\alpha \in [0, 1), \beta \in [0, 1]$, then the integral transform*

$$F(f, z) = \int_0^1 \frac{f(zx)}{x}\, dx$$

is in the class $TS^*C(\gamma, \beta)$, $\gamma \in \left[0, \frac{2}{3-\alpha}\right]$. The result of the theorem is sharp.

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