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Equivalence problem for compatible bi-Hamiltonian structures on three-dimensional orientable manifolds

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Abstract: We solve the equivalence problem for compatible bi-Hamiltonian structures on three-dimensional orientable manifolds via Cartan's method of equivalence. The problem separates into two branches on total space, one of which ends up with the intransitive involutive structure equations. For the transitive case, we obtain an $\{e\}$-structure on both total and base spaces.

Key words: Bi-Hamiltonian structure, Poisson structure, Cartan's method of equivalence, intransitive structure equations, Maurer–Cartan equations

1. Introduction

The equivalence problem is whether two differential geometric objects on a manifold could be transformed into each other via a class of diffeomorphisms, which are obtained as the solution of a certain system of differential equations. The first known example of this problem is Poincare's proof of the fact that two hypersurfaces of three real dimensions in $\mathbb{C}^2$ may fail to be biholomorphically equivalent [20]. In 1932 Cartan solved the equivalence problem of two hypersurfaces by finding local invariants [2], and then this solution was developed and generalized into a method that can be applied to the solution of various equivalence problems. For a more detailed historical account of the subject we refer to [6]. Cartan's method of equivalence can be applied not only to geometric objects but also to various mathematical structures on a manifold [19], including differential equations. Cartan showed that it is possible to define a projective connection whose geodesics are the integral curves of a given ordinary differential equation [1]. More recently, the equivalence problem of second-order ordinary differential equations [5, 11], third-order ordinary differential equations [21], Riccati equations [3], second-order partial differential equations [16, 17], and certain types of Painlevé transendents [11, 14] are studied. Furthermore, Cartan's method of equivalence is also applied to variational calculus [12, 13], control theory [7], and nonholonomic geometries [4]. The equivalence problem of a vector field and a two-form on a manifold studied in [8] is closely related to our work.

In this work, the equivalence problem for autonomous dynamical systems defined by nonvanishing vector fields on orientable three-dimensional manifolds is addressed. Equivalently, one may consider a vector field whose support is a three-dimensional orientable manifold. For a manifold of arbitrary dimension, the existence of nonvanishing vector fields on a manifold depends on its first Steifel–Whitney class. Since all orientable three-

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dimensional manifolds are parallelizable, there is no obstruction to the existence of a nonvanishing vector field on the manifold. However, having a nonvanishing vector field is not sufficient for defining a nontrivial equivalence problem because any diffeomorphism on a manifold maps nonvanishing vector fields to each other. Therefore, the equivalence problem for nonvanishing vector fields is trivial and we need some further property, or more precisely another geometric structure related to a nonvanishing vector field, to define a nontrivial equivalence problem. In [10] it was shown that any nonvanishing local vector field on an orientable three-dimensional manifold admits a bi-Hamiltonian structure, i.e. if $v(x)$ is a nonvanishing vector field on a three-dimensional manifold M with an arbitrary metric g on it, then there exist two independent functions $H_1(x)$ and $H_2(x)$, and a function $\phi(x)$ such that

$$v(x) = \phi \nabla H_1 \times \nabla H_2. \tag{1}$$

As indicated above, a nonvanishing vector field possesses too general geometric information since, in principal, it admits a $GL(n,\mathbb{R})$-structure in the language of G-structures. The bi-Hamiltonian structure for a nonvanishing vector field serves a more refined geometric structure by determining the certain transformation rule imposed by the compatibility condition and the Jacobi identity for Poisson structures, which are identified with vector fields orthogonal to (1). On the other hand, existence of a bi-Hamiltonian structure for a vector field $v(x)$ in three dimensions implies the integrability of the system of equations $\dot{x}(t) = v(x(t))$. An integral curve is determined by the intersection of the level surfaces corresponding to constant values of functionally independent Hamiltonians. With these aspects, investigation of whether there exist inequivalent bi-Hamiltonian structures in three dimensions is a reasonable study and as we show in this paper the problem has a nontrivial counterpart, which is determined by certain values of the torsion coefficients, leading to an invariant coframe. In this work, the equivalence problem for compatible bi-Hamiltonian structures of nonvanishing vector fields is solved using Cartan's method of equivalence.

2. The local existence of bi-Hamiltonian structures on three-dimensional manifolds

Given a nonvanishing vector field $v(x)$ on an orientable three-dimensional manifold M, the dynamical system defined by $v(x)$ is given by

$$\dot{x}(t) = v(x(t)). \tag{2}$$

In this section, fundamental ideas of [10] related to our work are summarized. In [10] it was shown that (2) is locally bi-Hamiltonian. For this purpose first we will describe the Poisson structures in three dimensions.

Definition 1 *A Poisson structure on a manifold M is a Lie algebra structure on $C^\infty(M)$, i.e. a $\mathbb{R}$-bilinear map*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \tag{3}$$

satisfying

1. *Skew-symmetry:* $\{f,g\} = -\{g,f\}$,
2. *Jacobi identity:* $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0,$
3. *Leibniz rule:* $\{fg,h\} = f\{g,h\} + g\{f,h\}$.

By the Leibniz rule, the Poisson structure assigns a vector field, v_f, to each function f, namely

$$\{f,g\} = v_f\,(g) = \langle dg, v_f \rangle = \mathcal{J}\,(df, dg)\,, \tag{4}$$

where

$$\mathcal{J} : T^*M \to TM \tag{5}$$

is called the Poisson bivector. The Poisson bivector $\mathcal{J}$ is a skew-symmetric tensor satisfying the Jacobi identity

$$[\mathcal{J}, \mathcal{J}]_{SN}\,(df, dg, dh) = \{f, \{g,h\}\} + \{h, \{f,g\}\} + \{g, \{h,f\}\} = 0, \tag{6}$$

where $[\cdot,\cdot]_{SN}$ is the Schouten–Nijenhuis bracket. The image of a closed one-form under a Poisson bivector is called a Hamiltonian vector field, and the dynamical system (2) defined by a Hamiltonian vector field is said to have a Hamiltonian form. Then the bi-Hamiltonian structure related to the vector field is defined in [18] as follows:

Definition 2 *A dynamical system is called bi-Hamiltonian if it can be written in Hamiltonian form in two distinct ways:*

$$v = \mathcal{J}_1\,(dH_2) = \mathcal{J}_2\,(dH_1)\,, \tag{7}$$

such that $\mathcal{J}_1$ and $\mathcal{J}_2$ are nowhere multiples of each other. This bi-Hamiltonian structure is compatible if $\mathcal{J}_1 + \mathcal{J}_2$ is also a Poisson structure.

For three-dimensional manifolds, symplectic structures cannot be defined since a 3×3 skew-symmetric tensor cannot be invertible. For a detailed account of Poisson structures of dynamical systems on three-dimensional manifolds we refer to [9]. In [9], Poisson bivector fields are mapped to vector fields by the Lie algebra isomorphism $so\,(3) \simeq \mathbb{R}^3$ defined by the metric on M,

$$\mathcal{J} = \mathcal{J}^{mn} e_m \wedge e_n = \varepsilon_k^{mn} J^k e_m \wedge e_n, \tag{8}$$

and the vector field

$$J = J^k e_k \tag{9}$$

is called the Poisson vector field on M. Using (9), the Jacobi identity (6) can be expressed simply by

$$J \cdot (\nabla \times J) = 0 \tag{10}$$

and the bi-Hamiltonian structure given in (7) becomes

$$v = J_1 \times \nabla H_2 = J_2 \times \nabla H_1. \tag{11}$$

Since J_1 and J_2 are nowhere multiples of each other by definition, they are linearly independent and by (11) we have

$$J_i \cdot v = 0 \tag{12}$$

for $i = 1, 2$.

Now, defining

$$\widehat{e}_1 = \frac{v}{\|v\|} \tag{13}$$

and extending to a local orthonormal frame field $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$, we get the structure functions $\left(C_{ij}^k(x)\right)$ via the relation

$$[\widehat{e}_i, \widehat{e}_j] = C_{ij}^k(x)\, \widehat{e}_k. \tag{14}$$

Then the local existence theorem of bi-Hamiltonian systems is given in [10] as follows:

Theorem 3 *Any three-dimensional dynamical system*

$$\dot{x}(t) = v(x(t)) \tag{15}$$

has a pair of compatible Poisson structures

$$J_i = \alpha_i (\widehat{e}_2 + \mu_i \widehat{e}_3) \tag{16}$$

in which $\mu_i s$ are determined by the equation

$$\widehat{e}_1 \cdot \nabla \mu_i = -C_{31}^2 - \mu_i \left(C_{31}^3 + C_{12}^2\right) - \mu_i^2 C_{12}^3 \tag{17}$$

and $\alpha_i s$ are determined by the equation

$$\widehat{e}_1 \cdot \nabla \ln \frac{\alpha_i}{\|v\|} = C_{31}^3 + \mu_i C_{12}^3. \tag{18}$$

Furthermore, (15) is a locally bi-Hamiltonian system with a pair of local Hamiltonian functions determined by

$$J_i = (-1)^{i+1} \phi \nabla H_i, \tag{19}$$

where

$$\phi = \frac{\alpha_1 \alpha_2 (\mu_2 - \mu_1)}{\|v\|}. \tag{20}$$

By definition of the bi-Hamiltonian structure $\{\widehat{e}_1, J_1, J_2\}$ forms a local frame field, which may not be orthogonal. Since the equivalence problem is formulated by using coframes, the formulation of the basic properties of bi-Hamiltonian structures in differential forms is necessary. Letting $\boldsymbol{\Omega}$ be the volume form associated to the metric on M, the local Poisson one-form $\boldsymbol{J}$ is defined by

$$\boldsymbol{J} = \imath_{\mathcal{J}} \boldsymbol{\Omega}. \tag{21}$$

Then, (11) becomes

$$\iota_v \boldsymbol{\Omega} = \boldsymbol{J}_1 \wedge dH_2 = \boldsymbol{J}_2 \wedge dH_1 \tag{22}$$

and the Jacobi identity is given by

$$\boldsymbol{J}_i \wedge d\boldsymbol{J}_i = 0, \text{ for } i = 1, 2 \tag{23}$$

with compatibility condition

$$\boldsymbol{J}_1 \wedge d\boldsymbol{J}_2 = -\boldsymbol{J}_2 \wedge d\boldsymbol{J}_1. \tag{24}$$

Using (19) the bi-Hamiltonian form (22) can be written as

$$\iota_v \boldsymbol{\Omega} = \phi dH_1 \wedge dH_2. \tag{25}$$

Then we obtain the local coframe field $\left\{\omega^1, \omega^2, \omega^3\right\}$, which is dual to the local frame field, $\{\widehat{e}_1, J_1, J_2\}$.

3. The equivalence problem for compatible bi-Hamiltonian structures

In this section we formulate and solve the the equivalence problem of the compatible bi-Hamiltonian structures. Basically, the equivalence problem consists of associating certain coframes to the given local structure and then finding a diffeomorphism relating these coframes via pullback. We begin with an outline of Cartan's equivalence method in the following subsection.

3.1. An outline of Cartan's method of equivalence

Cartan's method of equivalence deals with the problem of the existence of a local diffeomorphism $\Phi : U \to \overline{U}$ for given coframe fields $\omega = (\omega^1, ..., \omega^n)$ and $\bar{\omega} = (\bar{\omega}^1, ..., \bar{\omega}^n)$ defined on open neighborhoods U and $\overline{U}$ of the manifolds M and $\overline{M}$, and a connected linear group $G \subset GL(n, \mathbb{R})$ such that $\Phi^*\bar{\omega} = g\omega,\ g \in G$. Here, $G \subset GL(n, \mathbb{R})$ is called the structure group of the problem. It is suitable to note that U and $\overline{U}$ would be coordinate neighborhoods of the same manifold. The idea is based on associating a canonical coframe to a geometric structure and in many cases there is no such coframe field, and the method starts with considering all admissible coframes encoding the underlying geometric structure. Accordingly, an equivalence problem is in one-to-one correspondence with the diffeomorphism $\Phi^{(1)} : U \times G \to \overline{U} \times G$ such that $\Phi^{(1)*}\bar{\theta} = \theta$, where $\bar{\theta} = g\bar{\omega}$ and $\theta = g\omega$ are the lifts of the coframes ω and $\bar{\omega}$ to $U \times G$ and $\overline{U} \times G$. The structure equations for a lifted coframe are determined by

$$d\theta = dgg^{-1} \wedge \theta + gd\omega, \tag{26}$$

where

$$\left(dgg^{-1}\right)^i_j = \pi^i_j = a^i_{j\rho}\pi^\rho \tag{27}$$

is the Maurer–Cartan matrix of right invariant forms on G. The coefficients $a^i_{j\rho}$ are the structure constants of the Lie algebra of G and π^ρ is a basis for the Maurer–Cartan forms. (26) is written in components as

$$d\theta^i = \sum_k \pi^i_k \wedge \theta^k + \frac{1}{2}\sum_{j,k} \gamma^i_{jk}(x, g)\theta^j \wedge \theta^k. \tag{28}$$

The components γ^i_{jk} are called the torsion coefficients and they depend on base and group coordinates. That is, they depend on local coordinates on M and G. In some cases the nature of the problem tells us that some of the Maurer–Cartan forms are equivalent to a zero modulo coframe on base manifold. That is, they can be expressed as a combination of the coframe elements ω^i. This process is called principal component decomposition. Now, suppose that principal component decomposition is performed. Equations in (28) do not define the Lie algebra valued differential form $(\pi^i_j) = \left(a^i_{j\rho}\pi^\rho\right)$ nor the torsion terms uniquely. By this ambiguity one can modify the components of the Lie algebra valued form as $\pi^\rho \mapsto \pi^\rho + v^\rho_k\theta^k$ and eliminate as many torsion terms as possible by solving the linear system of equations

$$\gamma^i_{jk} = a^i_{j\rho}v^\rho_k - a^i_{k\rho}v^\rho_j, \tag{29}$$

for v^ρ_k. This process is called Lie algebra compatible absorption. The remaining torsion terms after the process of absorption are called intrinsic or essential torsions. The remaining parts of Cartan's method are reduction and prolongation. The reduction process is based on normalizing an essential torsion to a suitable constant so that the structure group is reduced to a subgroup with 1 less dimension. This subgroup is identified with the isotropy group of the underlying torsion term. If all essential torsions can be normalized in this way (after one or more loops), one obtains an $\{e\}-$structure or an invariant coframe for the problem. In this case, the equivalence problem is reduced to the equivalence problem for the coframes whose solution is known. On the other hand, after all possible reductions, if there are still group parameters then Cartan's test is applied for the involutivity even if there exist nonconstant essential torsion terms that are independent from the group parameters, and such a case is called intransitive. If all the remaining essential torsion terms are constant then the structure equations are called transitive. In both cases, if Cartan's test is satisfied one obtains the involutive coframe. If the structure equations do not pass Cartan's test then the problem is prolonged to $U \times G$ with structure group $G^{(1)}$ by adding the remaining Maurer–Cartan basis to the original coframe. This structure group is parametrized by the solutions of the equations $0 = a^i_{j\rho} v^\rho_k - a^i_{k\rho} v^\rho_j$ for v^ρ_j. Then the problem is handled like the original problem for the lifted coframes on $U \times G$ with structure group $G^{(1)}$ throughout Cartan's algorithm described above. It is suitable here to note that intransitive equivalence problems are more complicated since the group independence of successive coframe derivatives of the structure invariants is also required for a solution; for details, we refer to [19, Theorem 11.16.] and preceding discussions. Also, we should note that the first branch of the problem considered in this work results in the intransitive involutive structure equations. For concrete description of Cartan's method of equivalence with applications to various problems, we refer to [6, 15, 19].

3.2. The formulation of the equivalence problem for compatible bi-Hamiltonian structures

Now, given two dynamical systems

$$\dot{x}(t) = v(x(t)) \tag{30}$$

and

$$\dot{\overline{x}}(t) = \overline{v}(\overline{x}(t)) \tag{31}$$

defined on two open neighborhoods U and $\overline{U}$ of M by nonvanishing vector fields v and $\overline{v}$, respectively, they can be associated with the coframes $\{\omega^1, \omega^2, \omega^3\}$ on U and $\{\overline{\omega}^1, \overline{\omega}^2, \overline{\omega}^3\}$ on $\overline{U}$ defined by their bi-Hamiltonian structures, respectively. Now the second step is to determine the type of the diffeomorphism

$$\Phi : U \to \overline{U} \tag{32}$$

that maps these dynamical systems and coframes associated to their bi-Hamiltonian structures to each other. Therefore, the first restriction for the diffeomorphism in question is that it should map the dynamical systems to each other because otherwise it would not be meaningful to talk about the equivalence of bi-Hamiltonian structures of dynamical systems that are not equivalent. In other words, we should have

$$\overline{v} = \Phi_* v. \tag{33}$$

Since ω^1 is defined as the dual of the vector field v, (33) implies

$$\Phi^*(\overline{\omega}^1) = \omega^1. \tag{34}$$

Once we have two equivalent dynamical systems, we should start with defining coframes representing their bi-Hamiltonian structure. Given the dynamical system (2) on an open neighborhood U of M, a compatible bi-Hamiltonian structure of (2) is uniquely represented by the local coframe field $\omega = \left\{\omega^1, \omega^2, \omega^3\right\}$ satisfying the following conditions:

1. Invariance condition:
$$\omega^1 \wedge *\omega^i = 0 \text{ for } i = 2, 3. \tag{35}$$

2. Jacobi identity:
$$\omega^i \wedge d\omega^i = 0 \text{ for } i = 2, 3. \tag{36}$$

3. Compatibility condition:
$$\omega^2 \wedge d\omega^3 + \omega^3 \wedge d\omega^2 = 0. \tag{37}$$

Then the local equivalence problem of two compatible bi-Hamiltonian structures $\omega_U = \left\{\omega^1, \omega^2, \omega^3\right\}$ and $\overline{\omega}_{\overline{U}} \left\{\overline{\omega}^1, \overline{\omega}^2, \overline{\omega}^3\right\}$ is defined as the existence of a diffeomorphism $\Phi : U \to \overline{U}$ such that the invariance condition

$$\Phi^*\left(\overline{\omega}^1\right) \wedge * \, \Phi^*\left(\overline{\omega}^i\right) = \omega^1 \wedge * \, \Phi^*\left(\overline{\omega}^i\right) = 0, \text{ for } i = 2, 3, \tag{38}$$

the Jacobi identity
$$\Phi^*\left(\overline{\omega}^i\right) \wedge d\Phi^*\left(\overline{\omega}^i\right) = 0, \text{ for } i = 2, 3, \tag{39}$$

and the compatibility condition

$$\Phi^*\left(\overline{\omega}^2\right) \wedge d\Phi^*\left(\overline{\omega}^3\right) + \Phi^*\left(\overline{\omega}^2\right) \wedge d\Phi^*\left(\overline{\omega}^3\right) = 0 \tag{40}$$

are preserved. Now it is not difficult to show that the equivalence group or the structure group of the equivalence problem determined by the pullback map Φ^*

$$\Phi^* \overline{\omega}_{\overline{U}} = g\omega_U, \; g \in G \tag{41}$$

is the subgroup G of $GL(3, \mathbb{R})$ of the form

$$G = \left\{ \left(\begin{array}{ccc} a & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{array} \right) \middle| \, a(bf - ec) \neq 0 \right\} \tag{42}$$

3.3. The solution of the equivalence problem for compatible bi-Hamiltonian structures

As indicated in the previous section, the very first step in the solution of this problem is to lift the coframe ω_U to $U \times G$,
$$\theta = g\omega_U, \qquad \theta^i = g^i_j \omega^j. \tag{43}$$

Then the structure equations on $U \times G$ have the form

$$d\theta = dgg^{-1} \wedge \theta + gd\omega_U, \tag{44}$$

where the Maurer–Cartan matrix dgg^{-1} is given by

$$dgg^{-1} = \left(\begin{array}{ccc} \frac{da}{a} & 0 & 0 \\ 0 & \frac{fdb - edc}{bf - ec} & \frac{-cdb + bdc}{bf - ec} \\ 0 & \frac{fde - edf}{bf - ec} & \frac{-cde + bdf}{bf - ec} \end{array} \right). \tag{45}$$

Using conditions (38)–(40), we get

$$\left.\begin{cases} bdc - cdb & \equiv & 0 \\ fde - edf & \equiv & 0 \\ fdb - edc - (bdf - cde) & \equiv & 0 \end{cases}\right\} \bmod \left(\theta^2, \theta^3\right). \tag{46}$$

Now the structure equation for the lifted coframe θ becomes

$$d\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T^1_{23} & T^1_{31} & T^1_{12} \\ T^2_{23} & 0 & T^2_{12} \\ T^3_{23} & T^3_{31} & 0 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{47}$$

By the compatibility condition we obtain

$$T^2_{12} = -T^3_{31}. \tag{48}$$

The torsion terms $T^1_{12}, T^1_{13}, T^2_{12}, T^2_{23}$, and T^3_{23} can be absorbed by modifying α and β by

$$\begin{aligned} \alpha &\mapsto \alpha + T^1_{12}\theta^2 - T^1_{31}\theta^3 \\ \beta &\mapsto \beta - T^2_{12}\theta^1 + T^2_{23}\theta^3 - T^3_{23}\theta^2. \end{aligned} \tag{49}$$

After absorption the structure equations become

$$d\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T^1_{23} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{50}$$

Now we have two cases depending on the value of the torsion term T^1_{23}.

3.3.1. ***Case I*** $T^1_{23} = 0$.

If $T^1_{23} = 0$, then the structure equations are reduced to

$$d\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}, \tag{51}$$

where the reduced Cartan characters are $\sigma_1 = 2$, $\sigma_2 = 0$ and the degree of indeterminacy is $\sigma = 1$, and hence Cartan's test fails. Therefore, one needs to prolong the problem. Since the structure equations are invariant under the transformation

$$\hat{\alpha} = \alpha + \lambda\theta^1, \tag{52}$$

the equivalence problem can be formulated on $U \times G \times G^{(1)}$ where $G^{(1)}$ is the structure group of the prolonged equivalence problem. Lift the coframe on $U \times G$ to $U^{(1)} = U \times G \times G^{(1)}$ by

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \alpha \\ \beta \end{pmatrix}. \tag{53}$$

From (51) we get

$$\begin{aligned} d\alpha \wedge \theta^1 &= 0, \\ d\beta \wedge \theta^2 &= 0, \\ d\beta \wedge \theta^3 &= 0. \end{aligned} \tag{54}$$

Since $\theta^1, \theta^2, \theta^3$ are nonvanishing, by Poincare's lemma we obtain

$$\begin{aligned} d\alpha &= \xi \wedge \theta^1, \\ d\beta &= \kappa\theta^2 \wedge \theta^3, \end{aligned} \tag{55}$$

for a 1-form ξ and a smooth function κ on $U^{(1)}$. Now the structure equations for the prolonged coframe become

$$\begin{aligned} d\theta^1 &= \hat{\alpha} \wedge \theta^1, \\ d\theta^2 &= \hat{\beta} \wedge \theta^2, \\ d\theta^3 &= \hat{\beta} \wedge \theta^3, \\ d\hat{\alpha} &= \rho \wedge \theta^1 + \lambda\hat{\alpha} \wedge \theta^1 + \xi \wedge \theta^1, \\ d\hat{\beta} &= \kappa\theta^2 \wedge \theta^3, \end{aligned} \tag{56}$$

where $\rho = d\lambda$ is the Maurer–Cartan form on $G^{(1)}$. We can absorb the torsion terms with the θ^1 factor by using the transformation

$$\rho \mapsto \rho - \lambda\hat{\alpha} - \xi. \tag{57}$$

After this absorption, structure equations become

$$\begin{aligned} d\theta^1 &= \hat{\alpha} \wedge \theta^1, \\ d\theta^2 &= \hat{\beta} \wedge \theta^2, \\ d\theta^3 &= \hat{\beta} \wedge \theta^3, \\ d\hat{\alpha} &= \rho \wedge \theta^1, \\ d\hat{\beta} &= \kappa\theta^2 \wedge \theta^3. \end{aligned} \tag{58}$$

Using the integrability condition

$$d^2\hat{\beta} = d\left(\kappa\theta^2 \wedge \theta^3\right) = 0, \tag{59}$$

we get

$$(d\kappa + 2\kappa\hat{\beta}) \wedge \theta^2 \wedge \theta^3 = 0, \tag{60}$$

which suggests that κ is independent of the group parameter of $G^{(1)}$. Therefore, it is not possible to normalize by the action of $G^{(1)}$, and hence κ is an invariant of the problem. This situation implies that the equivalence problem under consideration is intransitive. Notice also from (60) that $d\kappa$ is independent from $\hat{\alpha}$ and θ^1. This implies that all coframe derivatives of κ are also independent from the group parameter. From (58) it follows that the reduced Cartan characters are $\sigma_1 = 1$, $\sigma_2 = 0$ and the degree of indeterminacy is $\sigma = 1$, and so Cartan's test is satisfied. It follows from here and [19, Theorem 11.16] that we have intransitive structure equations for an involutive coframe on $U \times G$ and such a regular, analytic coframe has an infinite dimensional symmetry group depending on a single function of one variable.

3.3.2. *Case II* $T^1_{23} \neq 0$.

In this case, we first try to determine the effect of the structure group on T^1_{23} and to normalize T^1_{23} to an appropriate constant. For this purpose we compute $d^2\theta^1$, which gives

$$d\alpha \wedge \theta^1 - T^1_{23}\alpha \wedge \theta^2 \wedge \theta^3 + dT^1_{23} \wedge \theta^2 \wedge \theta^3 + 2T^1_{23}(\beta \wedge \theta^2 \wedge \theta^3 + T^2_{12}\theta^1 \wedge \theta^2 \wedge \theta^3) = 0. \tag{61}$$

Taking the wedge product of both sides of (61) with θ^1 leads to

$$\left(dT^1_{23} + 2T^1_{23}\beta - T^1_{23}\alpha\right) \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = 0, \tag{62}$$

which implies that

$$dT^1_{23} \equiv T^1_{23}(\alpha - 2\beta) \bmod (\theta^1, \theta^2, \theta^3). \tag{63}$$

This suggests that the equivalence group is acting on the T^1_{23} term by scaling, and if we normalize T^1_{23} to 1 we get

$$\alpha \equiv 2\beta \bmod (\theta^1, \theta^2, \theta^3). \tag{64}$$

After normalization, the structure equations become

$$d\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} 2\beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} 1 & q & p \\ s & 0 & h \\ k & -h & 0 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{65}$$

Now we can absorb the torsion terms s, h, and k by defining $\beta \mapsto \hat{\beta} - h\theta^1 - k\theta^2 + s\theta^3$. Then the structure equations take the form

$$d\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} 2\hat{\beta} & 0 & 0 \\ 0 & \hat{\beta} & 0 \\ 0 & 0 & \hat{\beta} \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} 1 & \tilde{q} & \tilde{p} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix}. \tag{66}$$

Since $d\theta^1 \equiv 2\hat{\beta} \wedge \theta^1 + \tilde{q}\theta^3 \wedge \theta^1 \pmod{\theta^2}$ and $d\theta^1 \equiv 2\hat{\beta} \wedge \theta^1 + \tilde{p}\theta^1 \wedge \theta^2 \pmod{\theta^3}$, one obtains the following by taking the exterior differential of both sides of these equations:

$$d\tilde{q} \equiv -\tilde{q}\hat{\beta} \bmod \theta^i, \tag{67}$$

$$d\tilde{p} \equiv -\tilde{p}\hat{\beta} \bmod \theta^i, \; i = 1, 2, 3. \tag{68}$$

From here it follows that

$$\frac{d\tilde{q}}{\tilde{q}} \equiv \frac{d\tilde{p}}{\tilde{p}} \bmod \theta^i, \text{ if } \tilde{q} \neq 0, \tilde{p} \neq 0. \tag{69}$$

These equations tell us that the structure group is acting on torsion terms by scaling and we can normalize them to 1 provided that they are nonzero. Now, before performing the normalization, it is more convenient to consider the following case:

Subcase i. $\tilde{q} = \tilde{p} = 0$. In this subcase no reduction is possible and the structure equations become

$$\begin{aligned} d\theta^1 &= 2\hat{\beta} \wedge \theta^1 + \theta^2 \wedge \theta^3, \\ d\theta^2 &= \hat{\beta} \wedge \theta^2, \\ d\theta^3 &= \hat{\beta} \wedge \theta^3. \end{aligned} \tag{70}$$

The first reduced character is $\sigma_1 = 1$ and $\sigma = 0$ and Cartan's involution test fails, and since the degree of indeterminacy is $\sigma = 0$ we should prolong the problem by computing $d\hat{\beta}$. Taking the exterior derivatives of both sides of equations (67) and (68), we get

$$d\hat{\beta} \wedge \theta^i = 0, \text{ for } i = 1, 2, 3 \tag{71}$$

and hence

$$d\hat{\beta} = 0. \tag{72}$$

Now we have an $\{e\}$-structure or an invariant coframe $\Theta = \{\theta^1, \theta^2, \theta^3, \hat{\beta}\}$ on $U \times G$:

$$\begin{array}{rcl} d\theta^1 & = & 2\hat{\beta} \wedge \theta^1 + \theta^2 \wedge \theta^3, \\ d\theta^2 & = & \hat{\beta} \wedge \theta^2, \\ d\theta^3 & = & \hat{\beta} \wedge \theta^3, \\ d\hat{\beta} & = & 0. \end{array} \tag{73}$$

This implies that for $\tilde{q} = \tilde{p} = 0$ the problem reduces the equivalence problem for coframes on $U \times G$. Since the structure functions for $\Theta = \{\theta^1, \theta^2, \theta^3, \hat{\beta}\}$ are all constant, it follows from here that the coframe Θ is a rank zero coframe and the equations in (73) are the Maurer–Cartan equations for a Lie group and all equivalent coframes must have the same structure constants with Θ. Also, (73) tells us that $\hat{\beta}$ may be used to define a unique flat connection 1-form on the tangent bundle of U provided that it can be written in terms of θ^is. Now let us consider the following subcases:

Subcase ii. $\tilde{q} = 0,\ \tilde{p} \neq 0$. In this subcase we can normalize $\tilde{p}$ to 1. This implies that $\hat{\beta} \equiv 0 \bmod \theta^i$ and hence we obtain the $\{e\}$-structure or an invariant coframe $\theta = \{\theta^1, \theta^2, \theta^3\}$ on U. In this case, $\hat{\beta}$ can be written as $\hat{\beta} = \sum_i k_i \theta^i$. Here the functions k_i are defined on U. In this case, structure equations (66) take the form

$$\begin{array}{rcl} d\theta^1 & = & (1 - 2k_2)\theta^1 \wedge \theta^2 + 2k_3\theta^3 \wedge \theta^1 + \theta^2 \wedge \theta^3, \\ d\theta^2 & = & k_1\theta^1 \wedge \theta^2 - k_3\theta^2 \wedge \theta^3, \\ d\theta^3 & = & -k_1\theta^3 \wedge \theta^1 + k_2\theta^2 \wedge \theta^3. \end{array} \tag{74}$$

The case $\tilde{p} = 0,\ \tilde{q} \neq 0$ is also managed in a similar manner.

Subcase iii. $\tilde{p} \neq 0,\ \tilde{q} \neq 0$. In this subcase either $\tilde{p}$ or $\tilde{q}$ can be used for reduction and any choice leads to the $\{e\}$-structure. If we normalize $\tilde{q}$ to 1, then we obtain $\hat{\beta} \equiv 0 \bmod \theta^i$. This implies that $\tilde{p}$ is an invariant of the problem. Structure equations (66) take the form

$$\begin{array}{rcl} d\theta^1 & = & (\tilde{p} - 2k_2)\theta^1 \wedge \theta^2 + (1 + 2k_3)\theta^3 \wedge \theta^1 + \theta^2 \wedge \theta^3, \\ d\theta^2 & = & k_1\theta^1 \wedge \theta^2 - k_3\theta^2 \wedge \theta^3, \\ d\theta^3 & = & -k_1\theta^3 \wedge \theta^1 + k_2\theta^2 \wedge \theta^3. \end{array} \tag{75}$$

3.4. The Interpretation of the invariants

If $I(x)$ is an scalar invariant, that is, $\Phi^*\bar{I} = I$, then differentials must also agree: $\Phi^* d\bar{I} = dI$. If an invariant coframe, say $\theta = \{\theta^1, \theta^2, \theta^3\}$, is obtained on U, then the differential of a scalar function is computed by the coframe (covariant) derivatives:

$$dI = \frac{\partial I}{\partial \theta^i}\theta^i. \tag{76}$$

Functionally independent structure functions of an invariant coframe are called fundamental invariants. k_1, k_2, and k_3 are the fundamental invariants of the problem. Invariants that are obtained by coframe derivative are called derived invariants. Note that not all derived invariants are functionally independent. Differential relations on invariants are found by taking the exterior derivative of the structure equations. Now let us investigate the previously given cases.

Subcase ii. From the integrability conditions $d^2\theta^i = 0$, we obtain

$$\begin{aligned} 0 &= 2\tfrac{\partial k_3}{\partial \theta^2} - 2\tfrac{\partial k_2}{\partial \theta^3} + 2k_1 + k_3, \\ 0 &= \tfrac{\partial k_1}{\partial \theta^3} - \tfrac{\partial k_3}{\partial \theta^1} + k_1 k_3, \\ 0 &= \tfrac{\partial k_2}{\partial \theta^1} - \tfrac{\partial k_1}{\partial \theta^2} + k_1(1-k_2). \end{aligned} \tag{77}$$

On the other hand, in terms of the connection $\hat{\beta} = \sum_i k_i\theta^i$, structure equations (74) can be written as

$$\begin{aligned} d\theta^1 &= 2\hat{\beta}\wedge\theta^1 + \theta^1\wedge\theta^2 + \theta^2\wedge\theta^3, \\ d\theta^2 &= \hat{\beta}\wedge\theta^2, \\ d\theta^3 &= \hat{\beta}\wedge\theta^3. \end{aligned} \tag{78}$$

If we compute $d\hat{\beta}$ and use (77), we obtain $d\hat{\beta} = -2k_3\theta^2\wedge\theta^3$. It follows from here that connection $\hat{\beta}$ is flat iff $k_3 = 0$. Besides, if $k_1 = 0$ then we have $2\frac{\partial k_3}{\partial \theta^2} - 2\frac{\partial k_2}{\partial \theta^3} + k_3 = 0$, $\partial_{\theta^1}k_3 = \partial_{\theta^1}k_2 = 0$. That is, k_2 and k_3 depend only on the Hamiltonians, i.e. $k_2 = k_2(H_1, H_2)$ and $k_3 = k_3(H_1, H_2)$.

Subcase iii. Now, since $d\theta^1 \equiv (\tilde{p} - 2k_2)\theta^1\wedge\theta^2 \pmod{\theta^3}$, we see by taking the exterior derivative of this identity that we get $(d\tilde{p} - 2dk)\wedge\theta^1\wedge\theta^2\wedge\theta^3 = 0$ and so $\tilde{p}$ and k_2 are functionally dependent. First-order differential relations are found from $d^2\theta^i = 0$ as

$$\begin{aligned} 0 &= 2\tfrac{\partial k_3}{\partial \theta^2} - 2\tfrac{\partial k_2}{\partial \theta^3} + \tfrac{\partial \tilde{p}}{\partial \theta^3} + 2k_1 + k_2 + \tilde{p}k_3, \\ 0 &= \tfrac{\partial k_1}{\partial \theta^3} - \tfrac{\partial k_3}{\partial \theta^1} + k_1 k_3, \\ 0 &= \tfrac{\partial k_2}{\partial \theta^1} - \tfrac{\partial k_1}{\partial \theta^2} + k_1(\tilde{p} - k_2). \end{aligned} \tag{79}$$

On the other hand, in terms of the connection $\hat{\beta} = \sum_i k_i\theta^i$, structure equations (74) can be written as

$$\begin{aligned} d\theta^1 &= 2\hat{\beta}\wedge\theta^1 + \tilde{p}\theta^1\wedge\theta^2 + \theta^3\wedge\theta^1 + \theta^2\wedge\theta^3, \\ d\theta^2 &= \hat{\beta}\wedge\theta^2, \\ d\theta^3 &= \hat{\beta}\wedge\theta^3. \end{aligned} \tag{80}$$

If we compute $d\hat{\beta}$ and compare with (79), we see that connection $\hat{\beta}$ is flat iff $\tilde{p} = 1$ and $k_1 + k_2 + k_3 = 0$. Again, if $k_1 = 0$ then k_2 and k_3 depend only on the Hamiltonians, i.e. $k_2 = k_2(H_1, H_2)$ and $k_3 = k_3(H_1, H_2)$. Notice in both cases that if k_1 vanishes then the other invariants can be given in terms of Hamiltonians.

4. Conclusion

In this work, we have considered the equivalence problem for compatible bi-Hamiltonian structures of nonvanishing vector fields by means of Cartan's method of equivalence. The problem under consideration is interesting in possessing examples of both transitive and intransitive structure equations in the context of equivalence problems. These cases are determined by the integrability of the 1-form θ^1 in the direction of the vector field $\dot{x}(t) = v(x(t))$ and for the integrable case we have faced the intransitive involutive structure equations. In this case, we have single fundamental invariant independent of the group parameter whose successive coframe derivatives can be expressed as a function of this structure invariant. The invariant parametrizes the classifying curve of the regular, analytic lifted coframe and accordingly, any such two coframes so the bi-Hamiltonian structures are equivalent whenever their classifying curves overlap. The nonintegrable case was separated into subcases depending on the possible normalizations of the torsion terms, which have given rise to the $\{e\}$-structure on both $U \times G$ and U. The identity structure on $U \times G$ is obtained for rank zero coframe and hence, structure equations for such a coframe are the Maurer-Cartan equations for a Lie group. For the latter case, we have seen that the vanishing of one of the structure invariants of the problem may be identified with the functionally independent Hamiltonians.

References

[1] Cartan E. Sur les variétés à connexion projective. B Soc Math Fr 1924; 52: 205-241 (in French).

[2] Cartan E. Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes. Ann Math Pura Appl 1932; 4: 17-90 (in French).

[3] Czyzycki T, Hrivnák J. Equivalence problem and integrability of the Riccati equations. NoDEA 2010; 17: 371-388.

[4] Ehlers K. Geometric equivalence on nonholonomic three-manifolds. In: Proceedings of the Fourth International Conference on Dynamical Systems and Differential Equations; 24–27 May 2002, Wilmington, NC, USA. Berlin, Germany: Springer, 2002, pp. 246-255.

[5] Fels ME. The equivalence problem for systems of second-order ordinary differential equations. P Lond Math Soc 1995; 71: 221-240.

[6] Gardner RB. The Method of Equivalence and Its Applications. CBMS-NSF Regional Conference Series in Applied Mathematics, 58. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 1989.

[7] Gardner RB, Shadwick WF. Feedback equivalence of control systems. Syst Control Lett 1987; 8: 463-465.

[8] Gardner RB, Shadwick WF. An equivalence problem for a two-form and a vector field on $\mathbb{R}^3$. In: Nicas A, Shadwick WF, editors. Differential Geometry, Global Analysis, and Topology. Providence, RI, USA: AMS, 1991, pp. 41-50.

[9] Gümral H, Nutku Y. Poisson structure of dynamical systems with three degrees of freedom. J Math Phys 1993; 34: 5691-5723.

[10] Işim Efe M, Abadoğlu E. Global existence of bi-Hamiltonian structures on orientable three dimensional manifolds. SIGMA 2017; 13: 055.

[11] Kamran N, Lamb KG, Shadwick WF. The local equivalence problem for $d^2y/dx^2 = F(x, y, dy/dx)$ and the Painlevé transcendents. J Differ Geom 1985; 22: 139-150.

[12] Kamran N, Olver PJ. Equivalence problems for first order Lagrangians on the line. J Differ Equations 1989; 80: 32-78.

[13] Kamran N, Olver PJ. Equivalence of higher-order Lagrangians. III. New invariant differential equations. Nonlinearity 1992; 5: 601.

[14] Kartak VV. Solution of the equivalence problem for the Painlevé IV equation. Teoret Mat Fiz 2012; 173: 245-267.

[15] Montgomery R. A Tour of Subriemannian Geometries, Their Geodesics and Applications. Providence, RI, USA: AMS, 2002.

[16] Morozov OI. Contact-equivalence problem for linear hyperbolic equations. J Math Sci 2006; 135: 2680-2694.

[17] Noda T. Equivalence problem of second order PDE for scale transformations. Hokkaido Math J 2011; 40: 313-335.

[18] Olver PJ. Canonical forms and integrability of bi-Hamiltonian systems. Phys Lett A 1990; 148: 177-187.

[19] Olver PJ. Equivalence, Invariants, and Symmetry. Cambridge, UK: Cambridge University Press, 1995.

[20] Poincaré H. Les functions analytique de deux variables et la représentation conforme. Rend Cire Math Palermo 1907; 23: 185-220 (in French).

[21] Sato H, Yoshikawa AY. Third order ordinary differential equations and Legendre connections. J Math Soc Japan 1998; 50: 993-1013.