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Formal residue and computer-assisted proofs of combinatorial identities

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Abstract: The coefficient of x^{-1} of a formal Laurent series $f(x)$ is called the formal residue of $f(x)$. Many combinatorial numbers can be represented by the formal residues of hypergeometric terms. With these representations and the extended Zeilberger algorithm, we generate recurrence relations for summations involving combinatorial sequences such as Stirling numbers and their q-analog. As examples, we give computer proofs of several known identities and derive some new identities. The applicability of this method is also studied.

Key words: Formal residue, extended Zeilberger algorithm, Stirling number

1. Introduction

Finding recurrence relations for summations is a key step in computer proofs of combinatorial identities. In the 1990s, Wilf and Zeilberger [20, 21] developed the method of creative telescoping to generate recurrence relations for hypergeometric summations. Since then, many extensions and new algorithms have been discovered and designed for various kinds of summations. See, for example, [3, 5] for holonomic sequences, [1, 19] for multivariable hypergeometric terms, [17] for nested sums and products, [11, 12] for Stirling-like numbers, and [4, 13, 14] for nonholonomic sequences.

Our approach is motivated by the work of Chen and Sun [2]. By using the Cauchy contour integral representations, they transformed sums involving Bernoulli numbers into hypergeometric summations. Then the recurrence relations for the sums can be derived by the extended Zeilberger algorithm [1].

In the present paper, we combine the formal residue operator and the extended Zeilberger algorithm to generate recurrence relations for combinatorial sums. With this residue method, we give computer proofs of some known identities and derive two new identities. By considering the annihilator of our representations for Stirling numbers of both kinds, we show that our method is a good choice when dealing with summations on Stirling numbers. Moreover, we study the applicability of this method. We show that in the case of one variable, it is similar to the Sister Celine method.

We note that Egorychev [7] provided integral representations for many combinatorial numbers and used them to prove combinatorial identities. Fürst [8] reformulated Egorychev's method in terms of formal residue operators. Egorychev transformed sums into geometric series and then evaluated them by some manipulation rules. We transform sums into hypergeometric sums and find the recurrence relations that they satisfy.

The paper is organized as follows. In Section 2, we describe the residue method. Then, in Section 3, we

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2010 *AMS Mathematics Subject Classification:* 33F10, 11B73, 68W30, 05A19 give several examples involving Stirling numbers of both kinds. Section 4 is devoted to deriving two new Stirling number identities. In Section 5, we consider the q-Stirling numbers as well as other combinatorial sequences that also fall in the scope of our method. Finally, in Section 6, we study the applicability of the residue method.

2. The method of residue

Let $\mathbb{K}$ be a field and $\mathbb{K}((z))$ be the set of formal Laurent series in the indeterminate z over $\mathbb{K}$. For any element

$$f(z) = \sum_{n=n_0}^{\infty} a_n z^n \in \mathbb{K}((z)), \tag{1}$$

the *formal residue operator* $\operatorname*{res}_z$ (or res if no confusion) is defined by

$$\operatorname{res} f(z) = \operatorname*{res}_z f(z) = a_{-1}.$$

Clearly, the kth coefficient of $f(z)$ can be represented by the formal residue as follows:

$$a_k = \operatorname{res} \frac{f(z)}{z^{k+1}}.$$

We see that this representation is equivalent to the Cauchy integral representation of a_k,

$$a_k = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{f(z)}{z^{k+1}} dz.$$

Based on the formal residue, we give a computer-assisted method to derive recurrence relations for sums involving nonhypergeometric sequences. Consider a definite sum with the form of

$$f(\boldsymbol{n}) = \sum_{k=-\infty}^{\infty} F(\boldsymbol{n}, k),$$

where $\boldsymbol{n} = (n_1, \ldots, n_r)$ is the vector of parameters. The residue method consists of the following three steps.

1. Rewrite the summand $F(\boldsymbol{n}, k)$ as $\operatorname*{res}_z \tilde{F}(\boldsymbol{n}, k, z)$, where $\tilde{F}(\boldsymbol{n}, k, z)$ is a hypergeometric term.

2. Take a finite subset $S \subset \mathbb{N}^r$ and apply the extended Zeilberger algorithm to the similar terms $\{\tilde{F}(\boldsymbol{n} + \boldsymbol{\alpha}, k, z)\}_{\boldsymbol{\alpha} \in S}$, where $\mathbb{N}$ denotes the set of nonnegative integers. Here two hypergeometric terms are said to be similar if their ratio is a rational function. Now, if the algorithm succeeds, we obtain a relation of the form
$$\sum_{\boldsymbol{\alpha} \in S} p_{\boldsymbol{\alpha}}(\boldsymbol{n}) \tilde{F}(\boldsymbol{n} + \boldsymbol{\alpha}, k, z) = \Delta_k G(\boldsymbol{n}, k, z), \tag{2}$$
where $p_{\boldsymbol{\alpha}}(\boldsymbol{n})$ are polynomial coefficients independent of k and z and $G(\boldsymbol{n}, k, z)$ is a hypergeometric term similar to $\tilde{F}(\boldsymbol{n}, k, z)$.

3. Summing over k and applying the operator res_z, we are led to a recurrence relation for the sum $f(\boldsymbol{n})$,
$$\sum_{\boldsymbol{\alpha} \in S} p_{\boldsymbol{\alpha}}(\boldsymbol{n}) f(\boldsymbol{n} + \boldsymbol{\alpha}) = \operatorname*{res}_z G(\boldsymbol{n}, +\infty, z) - \operatorname*{res}_z G(\boldsymbol{n}, -\infty, z).$$

Remark. In most cases, $G(\boldsymbol{n}, k, z)$ is finitely supported and hence we do not need to calculate $\operatorname{res}_z G(\boldsymbol{n}, +\infty, z)$ and $\operatorname{res}_z G(\boldsymbol{n}, -\infty, z)$.

To conclude this section, we give an example to illustrate the method of residue. More examples can be found in Sections 3–5.

Example 1 *We have [9, identity (6.15)]*

$$\sum_k \binom{n}{k} S_2(k, m) = S_2(n+1, m+1), \tag{3}$$

where $S_2(n, m)$ is the Stirling number of the second kind.

Proof It is known that

$$\sum_{n=k}^{\infty} S_2(n, k) z^n = \frac{z^k}{(1-z)(1-2z)\cdots(1-kz)}.$$

Therefore,

$$S_2(n, k) = \operatorname*{res}_z \frac{z^k}{z^{n+1}(1-z)(1-2z)\cdots(1-kz)}. \tag{4}$$

Denote the left-hand side of (3) by $L(n, m)$. We thus have

$$L(n, m) = \operatorname*{res}_z \sum_k \binom{n}{k} \frac{z^m}{z^{k+1}\prod_{i=1}^m (1-iz)}.$$

Now consider the inner summand

$$C(n, m, k, z) = \binom{n}{k} \frac{z^m}{z^{k+1}\prod_{i=1}^m (1-iz)}.$$

Applying the extended Zeilberger algorithm to the four similar terms

$$C(n+i, m+j, k, z), \quad 0 \le i, j \le 1,$$

we find that

$$C(n+1, m+1, k, z) - (m+2)C(n, m+1, k, z) - C(n, m, k, z) = \Delta_k \frac{-kzC(n, m, k, z)}{(n+1-k)(1-(m+1)z)}.$$

Summing over k and applying the formal residue operator, we derive that

$$-L(n, m) - (m+2)L(n, m+1) + L(n+1, m+1) = 0.$$

This agrees with the recurrence relation satisfied by $S_2(n+1, m+1)$. Finally, the identity follows by checking the initial values:

$$L(0, m) = S_2(0, m) = S_2(1, m+1), \quad L(n, 0) = 1 = S_2(n+1, 1).$$

□

We remark that most of the sums appearing in this paper can also be treated by Koutschan's implementation of the creative telescoping algorithm on a nonholonomic sequence (for more detail, see [13]). The only exception is Example 2. We also remark that the extended Zeilberger algorithm has been implemented by Hou. The corresponding package is available from the URL `http://www.combinatorics.net.cn/homepage/hou/apci.html`.

3. Stirling number identities

In this section we shall provide several examples involving Stirling numbers of both kinds to illustrate the residue method. Recall that

$$\sum_{k=0}^{n} S_1(n,k) z^k = (z)_{\underline{n}} = z(z-1)\cdots(z-n+1),$$

and

$$\sum_{n=k}^{\infty} S_2(n,k) z^n = \frac{z^k}{(1-z)(1-2z)\cdots(1-kz)},$$

where $S_1(n,k)$ and $S_2(n,k)$ are Stirling numbers of the first kind and of the second kind, respectively. We thus have

$$S_1(n,k) = \operatorname*{res}_z \frac{(z)_{\underline{n}}}{z^{k+1}}$$

and

$$S_2(n,k) = \operatorname*{res}_z \frac{z^k}{z^{n+1}(1-z)(1-2z)\cdots(1-kz)}.$$

It is worth mentioning that we use the ordinary generating functions of Stirling numbers instead of their exponential generating functions, which were extensively used in [8]. Let

$$F_1(n,k,z) = \frac{(z)_{\underline{n}}}{z^{k+1}} \quad \text{and} \quad F_2(n,k,z) = \frac{z^k}{z^{n+1}(1-z)(1-2z)\cdots(1-kz)}.$$

We see that both $F_1(n,k,z)$ and $F_2(n,k,z)$ are hypergeometric terms of n and k. Let N and K be the shift operators with respect to n and k, respectively. Denote the ring of linear difference operators with rational coefficients by

$$\mathbb{K}(n,k)\langle N,K\rangle = \left\{ \sum_{i=0}^{I} \sum_{j=0}^{J} r_{i,j}(n,k) N^i K^j \colon I,J \in \mathbb{N},\ r_{i,j}(n,k) \in \mathbb{K}(n,k) \right\}.$$

Let $\operatorname{Ann} f(n,k)$ be the annihilator of $f(n,k)$, namely,

$$\operatorname{Ann} f(n,k) = \{L \in \mathbb{K}(n,k)\langle N,K\rangle \colon Lf(n,k) = 0\}.$$

We are now ready to give the following theorem, which is the underlying mechanism of our method.

Theorem 2 *We have the following inclusion relations:*

$$\operatorname{Ann} S_1(n,k) \subset \operatorname{Ann} F_1(n,k,z), \qquad \operatorname{Ann} S_2(n,k) \subset \operatorname{Ann} F_2(n,k,z).$$

Proof For $F_1(n,k,z)$, it is readily seen that

$$\langle N-(z-n), zK-1\rangle \subset \operatorname{Ann} F_1(n,k,z).$$

Eliminating the parameter z by the left Euclidean division algorithm gives

$$(NK - 1 + nK) F_1(n,k,x) = 0.$$

Taking note of $\operatorname{Ann}(S_1(n,k)) = \langle NK - 1 + nK\rangle$ (see [11]), we are led to

$$\operatorname{Ann}(S_1(n,k)) \subset \operatorname{Ann} F_1(n,k,z).$$

The second inclusion relation for $S_2(n,k)$ can be proved similarly. □

Given a function $F(\boldsymbol{n},k)$, we denote by $\tilde{F}(\boldsymbol{n},k,z)$ the function obtained from $F(\boldsymbol{n},k)$ by replacing $S_1(n,k)$ and $S_2(n,k)$ with $F_1(n,k,z)$ and $F_2(n,k,z)$, respectively. Suppose that there exist $Q \in \mathbb{K}(\mathbf{n},k)\langle \mathbf{N},K\rangle$ and $L \in \mathbb{K}(\mathbf{n})\langle \mathbf{N}\rangle$ such that

$$L - (K-1)Q \in \operatorname{Ann} F(\mathbf{n},k).$$

Then we also have

$$L - (K-1)Q \in \operatorname{Ann} \tilde{F}(\mathbf{n},k,z),$$

which leads to an equation of the form of (2). The extended Zeilberger algorithm will succeed in finding such L and Q. This fact indicates that the residue method always works as long as the existence of such L and Q is guaranteed.

With the residue method, we can prove all identities on Stirling numbers that appeared in [11]. Moreover, we can deal with sums involving products of Stirling numbers, which typically are identities (6.24), (6.25), (6.28), and (6.29) in [9]. Here we only give two examples.

Example 3 *We have [9, identity (6.24)]*

$$\sum_k S_1(k,m)S_2(n+1,k+1) = \binom{n}{m}. \tag{5}$$

Proof Denote the left-hand side by $L(n,m)$. We have

$$L(n,m) = \operatorname*{res}_x \operatorname*{res}_y \sum_k \frac{x^{k+1}}{x^{n+2}\prod_{i=1}^{k+1}(1-ix)} \frac{(y)_{\underline{k}}}{y^{m+1}}.$$

For the inner summand $F(n,m,k)$, Gosper's algorithm gives

$$F(n,m,k) = G(n,m,k+1) - G(n,m,k),$$

where

$$G(n,m,k) = \frac{x^k}{x^{n+1}\prod_{i=1}^{k}(1-ix)} \frac{(y)_{\underline{k}}}{y^{m+1}} \frac{1}{1-x(1+y)}.$$

Since the denominator contains $1-x(1+y)$ as a factor, we are unable to deduce a closed form of $\operatorname{res}_x \operatorname{res}_y G$. However, summing over k from 0 to n, we get

$$L(n,m) = \operatorname*{res}_x \operatorname*{res}_y \frac{1}{(1-x(1+y))}\left(-\frac{1}{\prod_{i=1}^{n+1}(1-ix)}\frac{(y)_{\underline{n+1}}}{y^m} + \frac{1}{x^{n+1}y^{m+1}}\right).$$

Notice that

$$\operatorname*{res}_y \operatorname*{res}_x \frac{1}{(1-x(1+y))}\frac{1}{\prod_{i=1}^{n+1}(1-ix)}\frac{(y)_{\underline{n+1}}}{y^m} = \operatorname*{res}_y 0 = 0,$$

and

$$\operatorname*{res}_y \operatorname*{res}_x \frac{1}{(1-x(1+y))} \frac{1}{x^{n+1}y^{m+1}} = \operatorname*{res}_y \frac{(1+y)^n}{y^{m+1}} = \binom{n}{m}.$$

This completes the proof. □

Example 4 *We have [9, identity (6.28)]*

$$\sum_k \binom{n}{k} S_2(k,l) S_2(n-k,m) = \binom{l+m}{l} S_2(n,l+m). \tag{6}$$

Proof Denote the left-hand side by $L(n,m,l)$. We have

$$L(n,m,l) = \operatorname*{res}_x \operatorname*{res}_y \sum_k \binom{n}{k} \frac{x^l y^m}{x^{k+1}y^{n-k+1}\prod_{i=1}^{l}(1-ix)\prod_{j=1}^{m}(1-jy)}.$$

For the inner summand $F(n,m,l,k)$, the extended Zeilberger algorithm gives

$$-F(n,m+1,l) - F(n,m,l+1) - (m+2+l)F(n,m+1,l+1) + F(n+1,m+1,l+1) = \Delta_k G(n,m,l,k),$$

where

$$G(n,m,l,k) = \binom{n}{k} \frac{-kx^l y^m}{(n+1-k)x^{k+1}y^{n-k+1}\prod_{i=1}^{l+1}(1-ix)\prod_{j=1}^{m+1}(1-jy)}.$$

Summing over k and applying the operators res_x and res_y, we get a recurrence relation:

$$-L(n,m+1,l) - L(n,m,l+1) - (m+2+l)L(n,m+1,l+1) + L(n+1,m+1,l+1) = 0.$$

It is easy to check that the right-hand side of (6) satisfies the same recurrence relation. Finally, the identity holds by checking the initial values

$$L(0,m,l) = \delta_{0,l}\delta_{0,m}, \quad L(n,0,l) = S_2(n,l), \quad L(n,m,0) = S_2(n,m).$$

□

4. New identities

In this section, we use two examples to illustrate how to discover new identities by the residue method. In the first example, we generate new identities by introducing a new parameter in the original summand, while in the second example, we use Zeilberger's algorithm to construct new identities, as done by Chen and Sun [2].

We first consider the identity

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} S_1(n+k,k) = (2n-1)!!, \tag{7}$$

which was proposed by Kauers and Sheng-Lang Ko as the American Mathematical Monthly Problem 11545. It was proved by Fürst [8] by the residue representation

$$S_1(n,k) = \frac{n!}{k!} \operatorname*{res}_z \frac{\ln^k(1+z)}{z^{n+1}}.$$

In fact, this identity can be generalized as follows.

Theorem 5 *Let n and m be nonnegative integers. Then we have*

$$\sum_{k=-m}^{n}(-1)^k\binom{2n}{n+k}S_1(n+k,k+m)=\begin{cases}(2n-1)!!, & \text{if } m=0,\\ 0, & \text{if } m\geq 1.\end{cases} \tag{8}$$

Proof Denote the left-hand side of (8) by $L(n,m)$. By the residue representation, we have

$$L(n,m)=\operatorname*{res}_z \sum_{k=-m}^{n}(-1)^k\binom{2n}{n+k}\frac{(z)_{\underline{n+k}}}{z^{m+k+1}}.$$

The extended Zeilberger algorithm gives the recurrence relation

$$2(n+1)(2n+3)L(n,m)-\frac{(2n+3)(4n+3)}{2n+1}L(n+1,m)+L(n+2,m) \\ +2(n+1)(2n+3)L(n+1,m+1)=0. \tag{9}$$

We now prove that $L(n,n-r+1)=0$ for $n\geq r$ by induction on the nonnegative integer r. Since $S_1(n+k,n+1+k)=0$ for any integer k, we have $L(n,n+1)=0$, i.e. the assertion holds for $r=0$. For $r=1$, we have

$$L(n,n)=\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}=0, \quad n\geq 1.$$

Now suppose that the assertion holds for $1\leq r\leq r_0$ where $r_0\geq 1$. The recurrence relation (9) implies that

$$2(n-1)(2n-1)L(n-2,n-r_0)-\frac{(2n-1)(4n-5)}{2n-3}L(n-1,n-r_0)+L(n,n-r_0) \\ +2(n-1)(2n-1)L(n-1,n-r_0+1)=0.$$

By induction, we have

$$L(n-2,n-r_0)=L(n-1,n-r_0)=L(n-1,n-r_0+1)=0.$$

Therefore, $L(n,n-r_0)=0$, which completes the induction. Notice that the assertion is equivalent to the statement $L(n,m)=0$ for any nonnegative integers n and $m\geq 1$.

For $m=0$, the recurrence relation (9) becomes

$$2(n+1)(2n+3)L(n,0)-\frac{(2n+3)(4n+3)}{2n+1}L(n+1,0)+L(n+2,0)=0.$$

It is easy to check that $(2n-1)!!$ satisfies this recurrence relation and coincides with the initial values $L(0,0)=L(1,0)=1$. $\square$

Sitgreaves [18] found the following identity (see also [7]):

$$\sum_{k=0}^{n}\binom{n+m}{k}(-1)^kS_2(n+m-k,n-k)=0, \quad m\geq 0, \quad n\geq m+1. \tag{10}$$

From this result, we can establish the following theorem.

Theorem 6 *For nonnegative integers* $n \geq m \geq 0$, *we have*

$$\sum_{k=0}^{n} \binom{n+m+1}{k} (-1)^k S_2(n+m-k, n-k) = (-1)^{n+m} m!. \tag{11}$$

Proof Denoting the left-hand side of (10) by $L(n,m)$, we have

$$L(n,m) = \operatorname*{res}_z \sum_{k=0}^{n} \binom{n+m}{k} (-1)^k \frac{z^{n-k}}{z^{n+m-k+1} \prod_{i=1}^{n-k}(1-iz)}.$$

For the inner summand $F(n,m,k)$, the original Zeilberger algorithm gives

$$(n+m+1)F(n,m,k) - z(m+1)F(n,m+1,k) - zF(n,m+2,k) = \Delta_k \frac{(n+m+1)kF(n,m,k)}{(n+m+1-k)(n+m+2-k)z}.$$

Summing over k and applying the residue operator, we obtain

$$(n+m+1)L(n,m) - (m+1)\sum_{k=0}^{n} \binom{n+m+1}{k} (-1)^k S_2(n+m-k, n-k)$$
$$- \sum_{k=0}^{n} \binom{n+m+2}{k} (-1)^k S_2(n+m+1-k, n-k) = 0.$$

Denote the left-hand side of (11) by $S(n,m)$. Substituting $L(n,m) = 0$ in the above identity, we deduce that

$$(m+1)S(n,m) + S(n,m+1) = 0.$$

Thus, we have

$$S(n,m+1) = (-1)^{m+1}(m+1)!S(n,0).$$

Note that

$$S(n,0) = \sum_{k=0}^{n} \binom{n+1}{k} (-1)^k S_2(n-k, n-k) = \sum_{k=0}^{n} \binom{n+1}{k} (-1)^k = (-1)^n.$$

We finally derive that

$$S(n,m+1) = (-1)^{n+m+1}(m+1)!,$$

as desired. □

5. More combinatorial sequences

It is readily seen that our approach is also applicable to many other combinatorial sequences as long as the corresponding generating function is hypergeometric. More generally, the residue operator can be replaced by any linear operator L. For example, a classical treatment for identities involving harmonic numbers $H_n = \sum_{k=1}^{n} 1/k$ (see [15]) is to use the fact

$$H_n = \delta \binom{n+x}{x},$$

where $\delta f(x) = \frac{df(x)}{dx}|_{x=0}$.

Here we list several sequences which could be treated by this method.

5.1. q-Stirling numbers

A kind of q-analog of Stirling numbers is given by [6, 10]

$$S_1^q(n,k) = S_1^q(n-1,k-1) - [n-1]S_1^q(n-1,k), \qquad S_1^q(0,k) = \delta_{0,k},$$
$$S_2^q(n,k) = S_2^q(n-1,k-1) + [k]S_2^q(n-1,k), \qquad S_2^q(0,k) = \delta_{0,k},$$

where

$$[n] = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}.$$

Their generating functions are

$$\sum_{k=0}^{n} S_1^q(n,k)z^k = \prod_{k=0}^{n-1}(z-[k]), \quad \sum_{n\geq k} S_2^q(n,k)z^n = \frac{z^k}{\prod_{i=1}^{k}(1-[i]z)}.$$

Thus, we have

$$S_1^q(n,k) = \operatorname*{res}_z \frac{\prod_{i=0}^{n-1}(z-[i])}{z^{k+1}}, \quad S_2^q(n,k) = \operatorname*{res}_z \frac{z^{k-n-1}}{\prod_{i=1}^{k}(1-[i]z)}.$$

Note that we also have

$$\operatorname{Ann} S_1^q(n,k) \subset \operatorname{Ann} \frac{\prod_{i=0}^{n-1}(z-[i])}{z^{k+1}} \quad \text{and} \quad \operatorname{Ann} S_2^q(n,k) \subset \operatorname{Ann} \frac{z^{k-n-1}}{\prod_{i=1}^{k}(1-[i]z)}.$$

Using these representations and the q-analog of the extended Zeilberger algorithm, we can derive recurrence relations for sums involving q-Stirling numbers. For instance, let us consider the sum (see [12])

$$L(n,m) = \sum_{k=m}^{n}(-1)^{n-k}\begin{bmatrix}k\\m\end{bmatrix}_q S_1^q(n,k)q^{-k},$$

where

$$\begin{bmatrix}k\\m\end{bmatrix}_q = \frac{[k][k-1]\cdots[k-m+1]}{[m][m-1]\cdots[1]}$$

represents the q-binomial coefficients. Our approach gives the recurrence relation

$$L(n,m) + \frac{q(q^{m+1}-q^m+q^n-1)}{q-1}L(n,m+1) - qL(n+1,m+1) = 0.$$

Similarly, for the sum

$$L(n,m) = \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q S_2^q(k,m)q^{-k},$$

we have

$$(1-q^{n+1})L(n,m)+L(n+1,m)+\frac{(1-q^{m+1})(1-q^{n+1})}{1-q}L(n,m+1)$$
$$+\frac{(q^{m+1}-q^2+q-1)}{q-1}L(n+1,m+1)-qL(n+2,m+1)=0.$$

5.2. Exponential functions

Noting that

$$k^n=\operatorname*{res}_x\frac{1}{(1-kx)x^{n+1}},$$

we can use the residue method to deal with sums involving k^n. For example, consider the sum

$$L(n,m)=\sum_k\binom{m}{k}k^n(-1)^{m-k}.$$

Applying the extended Zeilberger algorithm to the summand

$$F(n,m,k)=\binom{m}{k}\frac{(-1)^{m-k}}{(1-kx)x^{n+1}},$$

we find that

$$-(m+1)L(n,m)-(m+1)L(n,m+1)+L(n+1,m+1)=0.$$

Since $m!S_2(n,m)$ satisfies the same recurrence relation and has the same initial values, we finally derive that (see [9, identity (6.19)])

$$\sum_k\binom{m}{k}k^n(-1)^{m-k}=m!S_2(n,m).$$

5.3. Bernoulli polynomials

Identities involving Bernoulli and Euler numbers were verified in [2]. Here we only point out that we may also use the extended Zeilberger algorithm to derive differential equations satisfied by the sum. We take the Bernoulli polynomial $B_n(x)$ as an example. Recall the generating function

$$\sum_{n=0}^{\infty}B_n(x)\frac{z^k}{k!}=\frac{ze^{xz}}{e^z-1}.$$

We have

$$L(n,x,y)=\sum_k\binom{n}{k}y^{n-k}B_k(x)=\operatorname*{res}_z\frac{1}{e^z-1}\sum_k\binom{n}{k}y^{n-k}\frac{e^{xz}k!}{z^k}.$$

The extended Zeilberger algorithm generates

$$\frac{\partial}{\partial x}F(n+1,k,x,y)-(n+1)F(n,k,x,y)=\Delta_k\left(-\frac{(n+1)(n+1-k-ty)}{t(n+1-k)}F(n,k,x,y)\right).$$

We thus have

$$\frac{\partial}{\partial x} L(n+1,x,y) = (n+1)L(n,x,y).$$

This relation together with the fact $L(n,0,y) = B_n(y)$ indicates that

$$\sum_k \binom{n}{k} y^{n-k} B_k(x) = B_n(x+y).$$

6. Applicability of the residue method

We have shown in Section 3 that for sums involving Stirling numbers, the residue method succeeds if the creative telescoping algorithm works, whereas the converse is uncertain. In this section, we consider sums of the form

$$\sum_k F(\boldsymbol{n},k) a_k,$$

where $F(\boldsymbol{n},k)$ is a hypergeometric term and the generating function of a_k is independent of k. By the residue method, we aim to find a finite set S and (k,z)-free polynomial coefficients $\{p_{\boldsymbol{\alpha}}(\boldsymbol{n})\}_{\boldsymbol{\alpha}\in S}$ such that

$$\sum_{\boldsymbol{\alpha}\in S} p_{\boldsymbol{\alpha}}(\boldsymbol{n}) \frac{F(\boldsymbol{n}+\boldsymbol{\alpha},k)}{z^{k+1}} = \Delta_k G(\boldsymbol{n},k,z).$$

We will show that in most cases, the above equation holds only for $G(\boldsymbol{n},k,z)=0$. In this case, we have

$$\sum_{\boldsymbol{\alpha}\in S} p_{\boldsymbol{\alpha}}(\boldsymbol{n}) F(\boldsymbol{n}+\boldsymbol{\alpha},k) = 0,$$

which is similar to the equation that appears in the Sister Celine method.

We first give a lemma on the C-finiteness of hypergeometric terms.

Lemma 7 *Let $f(k)$ be a hypergeometric term and*

$$\frac{f(k+1)}{f(k)} = u\frac{A(k)}{B(k)}\frac{C(k+1)}{C(k)} \tag{12}$$

be the GP-representation (see [16] for the definition). If $f(k)$ is C-finite, then

$$A(k) = B(k) = 1.$$

Proof Suppose that $f(k)$ is C-finite; that is, there exist constants $a_0, a_1, \ldots, a_d$, not all zeros, such that

$$a_0 f(k) + a_1 f(k+1) + \cdots + a_d f(k+d) = 0.$$

Dividing $f(k)$ on both sides and substituting (12), we derive that

$$a_0 \;+\; a_1 uz\frac{A(k)}{B(k)}\frac{C(k+1)}{C(k)} \;+\; a_2 u^2 z\frac{A(k)A(k+1)}{B(k)B(k+1)}\frac{C(k+1)}{C(k)} \;+\; \cdots \;+\; a_d u^d \frac{\prod_{i=0}^{d-1} A(k+i)}{\prod_{i=0}^{d-1} B(k+i)}\frac{C(k+i)}{C(k)} \;=\; 0.$$

Hence,

$$\sum_{i=0}^{d} a_i u^i C(k+i) \prod_{j=0}^{i-1} A(k+j) \prod_{j=i}^{d-1} B(k+j) = 0.$$

Since $A(k)$ divides all the terms of the left hand side except the first one, it must also divide the first term. By the definition of GP-representation, $A(k)$ is coprime to $C(k)$ and $B(k+j)$. We thus deduce that $A(k) = 1$. With a similar discussion, we derive that $B(k) = 1$. □

Now we are ready to give the main theorem.

Theorem 8 *Let* $\boldsymbol{n} = (n_1, \dots, n_r)$ *and* $F(\boldsymbol{n}, k)$ *be a hypergeometric term. Suppose that there are a finite set* $S \subseteq \mathbb{N}^r$ *and* (k, z)*-free polynomial coefficients* $\{p_{\boldsymbol{\alpha}}(\boldsymbol{n})\}_{\boldsymbol{\alpha} \in S}$ *such that*

$$\sum_{\boldsymbol{\alpha} \in S} p_{\boldsymbol{\alpha}}(\boldsymbol{n}) \frac{F(\boldsymbol{n}+\boldsymbol{\alpha}, k)}{z^{k+1}} = \Delta_k G(\boldsymbol{n}, k, z). \tag{13}$$

Let

$$g(k) = \sum_{\boldsymbol{\alpha} \in S} p_{\boldsymbol{\alpha}}(\boldsymbol{n}) F(\boldsymbol{n}+\boldsymbol{\alpha}, k),$$

and let

$$\frac{g(k+1)}{g(k)} = u \frac{A(k)C(k+1)}{B(k)C(k)}$$

be the GP-representation. Then we have $A(k) = B(k) = 1$*.*

Proof Since $F(\boldsymbol{n}, k)$ is hypergeometric, there exists a rational function $R(k, z)$ (since $\boldsymbol{n}$ is irrelevant, we omit these variables) such that

$$G(\boldsymbol{n}, k, z) = R(k, z) F(\boldsymbol{n}, k, z).$$

Multiplying both sides of (13) by $z^{k+1}/F(\boldsymbol{n}, k)$, we see that

$$h(k) = r(k) \frac{R(k+1, z)}{z} - R(k, z)$$

is independent of z, where

$$r(k) = \frac{F(\boldsymbol{n}, k+1)}{F(\boldsymbol{n}, k)}.$$

Suppose that $R(k, z) = P(k, z)/Q(k, z)$, where $P(k, z)$ and $Q(k, z)$ are relatively prime polynomials in k, z. Then

$$r(k)P(k+1, z)Q(k, z) - zP(k, z)Q(k+1, z) = h(k) z Q(k+1, z) Q(k, z). \tag{14}$$

Noting that $r(k)$ and $h(k)$ are independent of z, by comparing the degrees in z of both sides, we obtain $\deg_z P(k, z) = \deg_z Q(k, z)$.

We first prove that $z \nmid Q(k, z)$. Suppose on the contrary that there is a positive integer m such that $z^m \mid Q(k, z)$ but $z^{m+1} \nmid Q(k, z)$. By (14), we see that $z^{m+1} \mid P(k+1, z)Q(k, z)$. Therefore, $z \mid P(k+1, z)$ and hence $z \mid P(k, z)$, but this contradicts the condition that $P(k, z)$ and $Q(k, z)$ are relatively prime.

Then we show that $Q(k,z)$ is independent of k. For any irreducible factor $p(k,z)$ of $Q(k,z)$, we deduce from (14) that $p(k,z) \mid zP(k,z)Q(k+1,z)$. Since $z \nmid Q(k,z)$ and $P(k,z), Q(k,z)$ are relatively prime, we have $p(k,z) \mid Q(k+1,z)$, which implies $p(k-1,z) \mid Q(k,z)$. By iterating the above discussion, we get $p(k-i,z) \mid Q(k,z)$ for any nonnegative integer i. Therefore, $p(k,z)$ must be independent of k. Since $p(k,z)$ is an arbitrary factor of $Q(k,z)$, we obtain that $Q(k,z)$ is independent of k.

From (14), we see that $z \mid P(k+1,z)$, so we assume that

$$P(k,z) = z\sum_{i=0}^{d} p_i(k)z^i, \quad Q(k,z) = \sum_{i=0}^{d+1} q_i z^i,$$

where all q_i are independent of k. Substituting these expressions into (14) and comparing the coefficient of each power of z, we find that

$$r(k)p_0(k+1) = q_0 h(k), \tag{15}$$

$$r(k)p_1(k+1) - p_0(k) = q_1 h(k), \tag{16}$$

$$\vdots$$

$$-p_d(k) = q_{d+1}h(k).$$

By (15), we have $p_0(k+1) = q_0 h(k)/r(k)$. Substituting it into (16), we get

$$p_1(k+2) = q_0\frac{h(k+1)}{r(k)r(k+1)} + q_1\frac{h(k+1)}{r(k)}.$$

Continuing this discussion, we finally derive that

$$q_0h(k) + q_1r(k)h(k+1) + q_2r(k)r(k+1)h(k+2) + \cdots + q_{d+1}r(k)r(k+1)\cdots r(k+d)h(k+d+1) = 0.$$

Thus, the hypergeometric term

$$v(k) = h(k)\prod_{i=0}^{k-1} r(i)$$

is C-finite. Clearly,

$$\frac{v(k+1)}{v(k)} = \frac{h(k+1)}{h(k)}r(k) = \frac{g(k+1)}{g(k)}.$$

By Lemma 7, we deduce that $A(k) = B(k) = 1$. $\square$

For example, we consider the sum

$$L(n,m) = \sum_{k=0}^{n}\binom{n}{k}B_{m+k},$$

where B_n is the Bernoulli number defined by

$$\sum_{n=0}^{\infty} B_n\frac{z^n}{n!} = \frac{z}{e^z-1}.$$

Rewrite the sum as

$$L(n,m) = \sum_{k=m}^{n+m} \binom{n}{k-m} B_k.$$

Denote the inner summand by $F(n,m,k)$. Take

$$S = \{(i,j) \colon 0 \leq i,j \leq 1\},$$

and denote

$$g(k) = \sum_{(i,j)\in S} p_{i,j}(n,m) F(n+i, m+j, k).$$

We find that

$$\frac{g(k+1)}{g(k)} = -\frac{k-m-n-2}{k+1-m}\frac{P(k+1)}{P(k)},$$

where $P(k)$ is a certain polynomial in k. By Theorem (13), we must have $g(k)=0$. There is a nontrivial solution

$$p_{0,0} = p_{0,1} = -p_{1,0},$$

which implies that

$$L(n,m) + L(n,m+1) - L(n+1,m) = 0.$$

This coincides with the recurrence relation given by Chen and Sun [2], wherein all identities involving only one Bernoulli number are of this case.

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References

[1] Chen WYC, Hou QH, Mu YP. The extended Zeilberger's algorithm with parameters. J Symb Comput 2012; 47: 643-654.

[2] Chen WYC, Sun LH. Extended Zeilberger's algorithm for identities on Bernoulli and Euler polynomials. J Number Theory 2009; 129: 2111-2132.

[3] Chyzak F. An extension of Zeilberger's fast algorithm to general holonomic functions. Discrete Math 2000; 217: 115-134.

[4] Chyzak F, Kauers M, Salvy B. A non-holonomic systems approach to special function identities. In: Proceedings of the 2009 International Symposium on Symbolic and Algebraic Computation. New York, NY, USA: ACM, 2009, pp. 111-118.

[5] Chyzak F, Salvy B. Non-commutative elimination in Ore algebras proves multivariate holonomic identities. J Symb Comput 1998; 26: 187-227.

[6] Cigler J. Operatormethoden für q-Identitäten. Monatsh Math 1979; 88: 87-105 (in German).

[7] Egorychev GP. Integral Representation and the Computation of Combinatorial Sums. Providence, RI, USA: American Mathematical Society, 1984.

[8] Fürst C. Combinatorial sums: Egorychev's method of coefficients and Riordan arrays. MSc, Johannes Kepler University, Linz, Austria, 2011.

[9] Graham RL, Knuth DE, Patashnik O. Concrete Mathematics: A Foundation for Computer Science. 2nd ed. Reading, MA, USA: Addison-Wesley Professional, 1994.

[10] Johnson WP. Some applications of the q-exponential formula. Discrete Math 1996; 157: 207-225.

[11] Kauers M. Summation algorithms for Stirling number identities. J Symb Comput 2007; 42: 948-970.

[12] Kauers M, Schneider C. Automated proofs for some Stirling number identities. Electron J Comb 2008; 15: R2.

[13] Koutschan C. Advanced applications of the holonomic systems approach. PhD, Johannes Kepler University, Linz, Austria, 2009.

[14] Mu YP. Linear recurrence relations for sums of products of two terms. Electron J Comb 2011; 18: R170.

[15] Paule P, Schneider C. Computer proofs of a new family of harmonic number identities. Adv Appl Math 2003; 31: 359-378.

[16] Petkovšek M, Wilf HS, Zeilberger D. A=B. Wellesley, MA, USA: A.K. Peters, 1996.

[17] Schneider C. Symbolic summation in difference fields. PhD, Johannes Kepler University, Linz, Austria, 2001.

[18] Sitgreaves R. Some properties of Stirling numbers of the second kind. Fibonacci Quart 1970; 8: 172-181.

[19] Wegschaider K. Computer generated proofs of binomial multi-sum identities. PhD, Johannes Kepler University, Linz, Austria, 1997.

[20] Wilf HS, Zeilberger D. An algorithmic proof theory for hypergeometric (ordinary and ¡°q¡±) multisum/integral identities. Invent Math 1992; 108: 575-633.

[21] Zeilberger D. The method of creative telescoping. J Symb Comput 1991; 11: 195-204.