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Difference uniqueness theorems on meromorphic functions in several variables

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Abstract: In this paper, we mainly investigate the uniqueness problem on meromorphic functions in $\mathbb{C}^m$ sharing small functions with their difference operators or shifts, and we obtain some interesting results that act as some extensions of previous results from one complex variable to several complex variables.

Key words: Uniqueness, meromorphic functions, difference operators, several variables

1. Introduction and main results

Let f be a meromorphic function in the complex domain. In this paper, we assume that the reader is familiar with standard notations such as characteristic function $T(r, f)$, counting function $N(r, f)$, and fundamental results of the Nevanlinna theory of meromorphic functions (see [8, 12, 17]). We say that $\alpha(z)$ is a small function with respect to f if $T(r, \alpha) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Usually, $S(f)$ is used to denote the family of all small functions with respect to f. For two meromorphic functions f, g, if $f - \alpha$ and $g - \alpha$ have the same zeros, counting multiplicity (ignoring multiplicity), then f and g share the small function α CM (IM).

In the last decades, uniqueness problems on meromorphic functions have been studied deeply due to their important value in Nevanlinna theory, and many interesting results have been established (see [9, 13, 16, 18]). As a very active subject, the problems on uniqueness of the entire function sharing values with its derivatives were initiated by Rubel and Yang[15]. In 1986, Jank et al.[10] obtained the following result:

Theorem 1.1 [10] *Let f be a nonconstant entire function, and let $a(\neq 0)$ be a finite constant. If f and f' share the value a IM, and $f''(z) = a$ whenever $f(z) = a$, then $f = f'$.*

After that, variations and generations for Theorem 1.1 have been extensively studied throughout the last decades. In [19], Zhong gave an example to show that f'' can not be replaced by $f^{(k)}(k \geq 3)$ in Theorem 1.1. In addition, Zhong obtained the following result:

Theorem 1.2 [19] *Let f be a nonconstant entire function, and let n be a positive integer. If f and f' share a finite, nonzero value a CM, and if $f^{(n)}(z) = f^{(n+1)}(z) = a$ whenever $f(z) = a$, then $f = f^{(n)}$.*

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In 2001, Li and Yang[9] considered the cases of the higher order derivative and proved the following two theorems for all entire functions:

Theorem 1.3 [9, Theorem 2.104] *Let f be an entire function, let a be nonzero finite value, and let*

$$L(f) = a_1 f' + a_2 f'' + \cdots + a_n f^{(n)} \tag{1.1}$$

with $a_1, a_2, \cdots, a_n$ being constants and $a_n \neq 0$. If $f, L(f)$, and $L'(f)$ share the value a CM, then $\sum_{k=1}^{n} a_k \neq 0$ and

$$f(z) = \frac{ce^z}{\sum_{k=1}^{n} a_k} + a - \frac{a}{\sum_{k=1}^{n} a_k},$$

or

$$f(z) = -a\left(\sum_{k=1}^{n} a_k\right) e^{2z} - ae^z + a, \quad \sum_{k=1}^{n} 2^k a_k = 0,$$

where c is a nonzero constant.

Theorem 1.4 [9, Theorem 2.105] *Let f be a nonconstant entire function, let a be nonzero finite value, let $n(\geq 2)$ be a positive integer, and let $L(f)$ be the function defined as in Theorem 1.3. If f, f', and $L(f)$ share the value a CM, then f must assume the following form:*

$$f(z) = be^{cz} - \frac{a(1-c)}{c},$$

where b, c are nonzero constants with $\sum_{k=1}^{n} a_k c^{k-1} = 1$.

Corresponding to the uniqueness problems on meromorphic functions sharing values with their derivatives, many authors considered the case of uniqueness of meromorphic functions sharing values or small functions with their shifts or difference operators, and some significant contributions have been made (see, e.g., [3–5, 14]).

Recently, many authors have paid attention to the uniqueness problems in the case of higher dimension (see, e.g., [2, 11]). For example, in 2014, Cao[2] obtained difference analogues of the second main theorem for meromorphic functions in several complex variables, and difference analogues of Picard-type theorems were also obtained as follows.

Theorem 1.5 [2, Theorem 1.10] *Let f be a meromorphic function with hyperorder $\rho_2(f) < 1$ on $\mathbb{C}^m$, and let $\tau : \mathbb{C}^m \to \mathbb{C}^m, \tau(z) = z + c$ and $c \in \mathbb{C}^m \backslash \{\boldsymbol{0}\}$ satisfy that for any $\xi \in S_m(1)$ there exists one $\widetilde{c} \in \mathbb{C}^1 \backslash \{0\}$ such that $c = \widetilde{c}\xi$. If three distinct values of f have forward invariant preimages with respect to τ, then f is a periodic function with period c.*

As we mentioned above, a large number of research works on the uniqueness problem have been done in a complex plane (see, e.g., [3–5, 9, 10, 13, 14, 19]). One may ask whether there exist some corresponding uniqueness results for meromorphic functions sharing values with their shifts or difference operators in the case of higher dimension.

The purpose of this paper is to study some uniqueness problems on meromorphic functions in several complex variables, and some difference uniqueness results can be verified as shown in Theorem 1.6, Theorem 1.7, and Theorem 1.8.

For a given meromorphic function $f:\mathbb{C}^m\to\mathbb{P}^1$ and nonzero vector $c=(c^1,c^2,\cdots,c^m)\in\mathbb{C}^m\backslash\{\mathbf{0}\}$, we define the shift by $f(z+c)$ and the difference operators by

$$\begin{aligned}\Delta_c f(z) &= f(z^1+c^1,\cdots,z^m+c^m)-f(z^1,\cdots,z^m),\\ \Delta_c^n f(z) &= \Delta_c\circ\Delta_c^{n-1}f(z), n\in\mathbb{N}, n\geq 2,\end{aligned}$$

where $z=(z^1,z^2,\cdots z^m)\in\mathbb{C}^m$.

Furthermore, we define a difference polynomial in $f(z)$ as follows:

$$P(f)=a_0f(z)+a_1f(z+c)+\cdots+a_nf(z+nc),(n\in\mathbb{N}^+),$$

where $z\in\mathbb{C}^m, c\in\mathbb{C}^m\backslash\{\mathbf{0}\}$, and $a_k(0\leq k\leq n)\in\mathbb{C}$ are not all zero complex numbers. Obviously, $P(f)$ can be regarded as the more general difference polynomial in f. In particular, if $a_k=C_n^k(-1)^{n-k}(0\leq k\leq n)$, then $P(f)=\Delta_c^n f$. Noting that for $\Delta_c^n f$, $\sum_{k=0}^n a_k=\sum_{k=0}^n C_n^k(-1)^{n-k}=0$, we assume that $\sum_{k=0}^n a_k=0$ for some a_k of $P(f)$ in this paper.

In this paper, we use short notations in some necessary cases for brevity as follows:

$$f(z):=\overline{f}^0, f(z+c):=\overline{f}^1,\cdots,f(z+kc):=\overline{f}^k.$$

First, a different analogue of Theorem 1.3 for meromorphic functions from one complex variable to several complex variables can be showed as follows.

Theorem 1.6 *Let $f:\mathbb{C}^m\to\mathbb{P}^1$ be a nonconstant meromorphic function of finite order and let $a(z)$, $b(z)(\not\equiv 0)\in S(f)$ be two periodic meromorphic functions with period c, where $z,c\in\mathbb{C}^m$. If $f(z)-a(z), P(f)-b(z)$, and $\Delta_c\circ P(f)-b(z)$ share $0,\infty$ CM, then $P(f)=\Delta_c\circ P(f)$.*

From Theorem 1.6, the following corollary, which is almost an accurate extension of previous uniqueness results from one complex variable to several complex variables, is immediately obtained.

Corollary 1.1 *Let $f:\mathbb{C}^m\to\mathbb{P}^1$ be a nonconstant meromorphic function of finite order, $n\in\mathbb{N}^+$, and let $a(z)$, $b(z)(\not\equiv 0)\in S(f)$ be two periodic meromorphic functions with period c, where $z,c\in\mathbb{C}^m$. If $f(z)-a(z),\Delta_c^n f-b(z)$, and $\Delta_c^{n+1}f-b(z)$ share $0,\infty$ CM, then $\Delta_c^n f=\Delta_c^{n+1}f$.*

Furthermore, if $P(f)\not\equiv 0$, then by the same conditions in Theorem 1.6, we obtain the following theorem, which can be seen as an improvement of Theorem 1.6.

Theorem 1.7 *Let $f:\mathbb{C}^m\to\mathbb{P}^1$ be a nonconstant meromorphic function of finite order and let $a(z)$, $b(z)(\not\equiv 0)\in S(f)$ be two periodic meromorphic functions with period c, where $z,c\in\mathbb{C}^m$. Assume that $P(f)\not\equiv 0$. If $f(z)-a(z),P(f)-b(z)$, and $\Delta_c\circ P(f)-b(z)$ share $0,\infty$ CM, then*

$$\Delta_c f(z)=f(z)-a(z)+\frac{b(z)}{A}\quad or\quad \sum_{k=0}^n a_k4^k=0,$$

where $A=\sum_{k=0}^n a_k2^k$ is a nonzero constant.

In particular, if $a_k = C_n^k(-1)^{n-k}(0 \le k \le n)$, $a(z) \equiv b(z)(\not\equiv 0)$, then $A = 1$, and Theorem 1.7 can be rewritten as follows

Corollary 1.2 *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function of finite order and let $b(z)(\not\equiv 0) \in S(f)$ be a periodic meromorphic function with period c, where $z, c \in \mathbb{C}^m$. Assume that $\Delta_c^n f \not\equiv 0$. If $f(z), \Delta_c^n(f)$, and $\Delta_c^{n+1}(f)$ share $b(z), \infty$ CM, then $\Delta_c f(z) = f(z)$.*

Example 1.1 below shows that the conditions and conclusions in Theorem 1.6 can be satisfied, and it also implies that the condition $P(f) \not\equiv 0$ in Theorem 1.7 is necessary.

Example 1.1 *Let $m = 2, c = (1,1), z = (z^1, z^2), a(z) \equiv 1, b(z) \equiv 2$, and $f(z) = e^{\pi i(z^1+z^2)} + a(z)$. Let $P(f) = \sum_{k=0}^n a_k f(z+kc)$. Obviously, $f(z+kc) = f(z)(k = 0,1,2)$, $\Delta_c f = 0$, and $P(f) = 0$. Thus, $f(z) - a(z) = e^{\pi i(z^1+z^2)}, P(f) - b(z) = -2$, and $\Delta_c \circ P(f) - b(z) = -2$ share 0 CM. However, $0 = \Delta_c f \not\equiv f(z) - a(z) + \frac{b(z)}{A}$.*

Example 1.2 *Let $m = 2, c = (1,0), z = (z^1, z^2)$, and $f(z) = e^{(z^1+z^2)\ln 2}$. Let $n = 2$ and $P(f) = a_0 f(z) + a_1 f(z+c) + a_2 f(z+2c)$. Obviously, $f(z+kc) = 2^k e^{(z^1+z^2)\ln 2}(k = 0,1,2)$ and $P(f) = e^{(z^1+z^2)\ln 2}(a_0 + 2a_1 + 2^2 a_2)$. Hence,*

Case 1: *$a_0 = 4, a_1 = -5, a_2 = 1$, i.e. $P(f) \not\equiv \Delta_c^2 f$. Let $a(z) \equiv 1, b(z) \equiv -2$. Obviously, $A = -2$. $f(z) - a(z) = e^{(z^1+z^2)\ln 2} - 1$, $P(f) - b(z) = -2(e^{(z^1+z^2)\ln 2} - 1)$, and $\Delta_c \circ P(f) - b(z) = -2(e^{(z^1+z^2)\ln 2} - 1)$ share 0 CM. Thus, $P(f) = \Delta_c \circ P(f)$ and $\Delta_c f = f$.*

Case 2: *$a_k = C_n^k(-1)^{n-k}(k = 0,1,2)$, i.e. $P(f) \equiv \Delta_c^2 f$. Let $a(z) \equiv b(z)$. Obviously, $f, P(f), \Delta_c \circ P(f)$ share any small function $a(z)$ CM, and $P(f) = \Delta_c f(z) = f(z) = e^{(z^1+z^2)\ln 2}$.*

The discussion above implies that the examples satisfy all the conditions and the conclusions of Theorem 1.7. From Case 1 and Case 2 in Example 1.2, similarities and differences between $P(f)$ and $\Delta_c^n f$ can be found. Therefore, the difference operator $P(f)$ may be seen as the more generalized form of $\Delta_c^n f$.

Corresponding to Theorem 1.4, there is also a uniqueness result in several complex variables, as shown in Theorem 1.8.

Theorem 1.8 *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function of finite order and let $a(z)$, $b(z) \in S(f)$ be two periodic meromorphic functions with period c, where $z, c \in \mathbb{C}^m$. If $f(z) - a(z), f(z+c) - a(z)$, and $P(f) - b(z)$ share $0, \infty$ CM, then $\frac{f(z+c)-a(z)}{f(z)-a(z)} = A$, where A is a nonzero constant. In particular, if $b(z) \not\equiv 0$, then $f(z) = f(z+c)$ or $\sum_{k=0}^n a_k A^k = 0$.*

The discussion below implies that the conditions and the conclusions of Theorem 1.8 are reasonable.

Remark 1.1 *For the case of $b(z) \not\equiv 0$ in Theorem 1.8, if $P(f) \equiv \Delta_c^n f$, then $A = 1$ and $f(z) = f(z+c)$ hold. On the other hand, if $P(f) \not\equiv \Delta_c^n f$, then $f(z) = f(z+c)$ may not hold. For example, let $m = 2$, $c = (1,0)$, $z = (z^1, z^2)$, $a(z) = 0$, and $f(z) = -b(z)e^{-(z^1+z^2)\ln 3}$, and $b(z)(\not\equiv 0)$ is a small periodic function with period c. Then $f(z+c) = \frac{1}{3}f(z)$ and $f(z+2c) = \frac{1}{9}f(z)$. Let $n = 2$ and $P(f) = f(z) - 4f(z+c) + 3f(z+2c)$.*

Obviously, $P(f) \equiv 0$. *Then* $f(z)-a(z) = -b(z)e^{-(z^1+z^2)\ln 3}, f(z+c)-a(z) = -\frac{1}{3}b(z)e^{-(z^1+z^2)\ln 3}$, *and* $P(f)-b(z) = -b(z)$ *share* 0 *CM. Note here that* $A=\frac{1}{3}$ *and* $\sum_{k=0}^{2} a_k A^k = 0$, *however* $f(z) \not\equiv f(z+c)$.

Example 1.3 *Let* $m=2$, $c=(c^1,c^2), z=(z^1,z^2)$, *and* $f(z) = e^{z^1+z^2}+a(z)$, *and* $a(z)$ *is a small periodic function with period* c. *Let* $n=2$ *and* $P(f) = a_0 f(z) + a_1 f(z+c) + a_2 f(z+2c)$. *Obviously,* $f(z+kc) = e^{k(c^1+c^2)}e^{z^1+z^2}+a(z)(k=1,2)$ *and* $P(f) = e^{z^1+z^2}(a_0 + a_1 e^{(c^1+c^2)} + a_2 e^{2(c^1+c^2)})$. *Hence,*

Case 1: $b=0, a_0=1, a_1=-4, a_2=3$, *i.e.* $P(f)\not\equiv \Delta_c^2 f$. *Let* $a=2, c=(1,0)$. *Then* $f(z)-a(z) = e^{z^1+z^2}, f(z+c)-a(z) = ee^{z^1+z^2}$, *and* $P(f)-b(z) = e^{z^1+z^2}(3e-1)(e-1)$ *share* 0 *CM, and* $\frac{f(z+c)-a(z)}{f(z)-a(z)} = e$.

Case 2: $b\neq 0, a_0=1, a_1=-4, a_2=3$, *i.e.* $P(f)\not\equiv \Delta_c^2 f$. *Let* $b=1, c=(\pi i,\pi i)$. *Then* $f(z)-a(z) = e^{z^1+z^2}, f(z+c)-a(z) = e^{z^1+z^2}$, *and* $P(f)-b(z) = -1$ *sharing* 0 *CM holds for any given small periodic function with period* c. *Obviously,* $f(z) = f(z+c)$.

Case 3: $a_k = C_n^k(-1)^{n-k}(k=0,1,2)$, *i.e.* $P(f) = \Delta_c^2 f$. *Similarly, as shown in Case 1 and Case 2, Theorem 1.8 holds for* $b=0, a=2, c=(1,0)$ *and* $b=1, a=1, c=(\pi i,\pi i)$, *respectively.*

The remainder of this paper is organized as follows. In Section 2, basic notions are shown, as well as some necessary results including some further instructions for Nevanlinna theory in $\mathbb{C}^m$, which play important roles in the later proofs. In Section 3, we give the proof of Theorem 1.6. In addition, Theorem 1.7 and Theorem 1.8 are proved in Section 4 and the last section, respectively.

2. Basic notions and auxiliary lemmas from Nevanlinna theory

Set $\|z\| = (|z^1|^2+|z^2|^2+\cdots+|z^m|^2)^{\frac{1}{2}}$ for $z=(z^1,z^2,\cdots,z^m)\in\mathbb{C}^m$. For $r>0$, define

$$B_m(r) := \{z\in\mathbb{C}^m \,|\, \|z\|<r\}, \quad S_m(r) := \{z\in\mathbb{C}^m \,|\, \|z\|=r\}.$$

Let $d=\partial+\overline{\partial}$, $d^c = (4\pi\sqrt{-1})^{-1}(\partial-\overline{\partial})$. Thus, $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}$. Write

$$\upsilon_m(z) := (dd^c\|z\|^2)^{m-1}, \quad \sigma_m(z) := d^c\log\|z\|^2 \wedge (dd^c\log\|z\|^2)^{m-1},$$

for $z\in\mathbb{C}^m\backslash\{\mathbf{0}\}$.

For a divisor ν on $\mathbb{C}^m$, define the following counting function of ν by

$$N(r,\nu) = \int_1^r \frac{n(t)}{t^{2m-1}}dt, \quad (1<r<\infty),$$

where

$$n(t) = \begin{cases} \int_{|\nu|\cap B(t)} \nu(z)\upsilon_m(z), & \text{if} \quad m\geq 2, \\ \sum_{|z|\leq t}\nu(z), & \text{if} \quad m=1. \end{cases}$$

Let $f(z)$ be a nonzero entire function on $\mathbb{C}^m$. For a point $z_0\in\mathbb{C}^m$, we write $f(z) = \sum_{i=0}^{\infty}P_i(z-z_0)$, where the term $P_i(z)$ is a homogeneous polynomial of degree i. Denote the zero-multiplicity of f at z_0 by $\nu_f(z_0) = \min\{i|P_i\neq 0\}$ in the sense of [6, Definition 2.1]. Set $|\nu_f| := \mathrm{Supp}\nu_f$, which is a pure $(n-1)$-dimensional analytic subset or empty set.

Let $f(z)$ be a nonzero meromorphic function on $\mathbb{C}^m$ in the sense that f can be written as a quotient of two relatively prime holomorphic functions. For each $z \in \mathbb{C}^m$, write $f(z) = (f_1(z), f_2(z))$ where $f_1(\not\equiv 0), f_2$ are two relatively prime holomorphic functions such that $\dim\{z \in \mathbb{C}^m | f_1(z) = f_2(z) = 0\} \leq m-2$. Thus, f may be regarded as a meromorphic mapping $f : \mathbb{C}^m \to \mathbb{P}^1$ such that $f^{-1}(\infty) \neq \mathbb{C}^m$. Define $\nu_f^0 := \nu_{f_2}, \nu_f^\infty := \nu_{f_1}$, which are independent of the choice of f_1, f_2.

For a meromorphic function f on $\mathbb{C}^m$, we usually write $N(r, f) := N(r, \nu_f^\infty)$ and $N(r, \frac{1}{f}) := N(r, \nu_f^0)$. Thus, Jensen's formula is given as

$$N(r, \frac{1}{f}) - N(r, f) = \int_{S_m(r)} \log|f|\sigma_m(z) - \log|f(0)|,$$

for all $r > 0$, provided that $f(0) \neq 0, \infty$. The proximity function of f is defined by

$$m(r, f) = \int_{S_m(r)} \log^+ |f|\sigma_m(z),$$

where $\log^+ x = \max\{\log x, 0\}$ for any $x > 0$. $T(r, f)$ denotes the Nevanlinna characteristic function of f such that $T(r, f) = m(r, f) + N(r, f)$.

In order to prove the main theorems in this paper, the following auxiliary lemmas from Nevanlinna theory in $\mathbb{C}^m$ are needed.

Lemma 2.1 [11, Theorem 3.1] *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function such that $f(0) \neq 0, \infty$, and let $c \in \mathbb{C}^m, \epsilon > 0$. If the hyperorder $\varsigma(f) = \varsigma < 2/3$, then*

$$\int_{\partial B_m(r)} \log^+ \left| \frac{f(z+c)}{f(z)} \right| \sigma_m(z) = o\left(\frac{T(r, f)}{r^{1-\frac{3}{2}\varsigma-\epsilon}} \right)$$

where $r \to \infty$ outside of a possible exceptional set $E \subset [1, +\infty)$ of finite logarithmic measure $\int_E 1/dt < \infty$.

Lemma 2.2 [11, Theorem 4.3] *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a meromorphic function and let $c \in \mathbb{C}^m, \epsilon > 0$. If hyperorder $\varsigma(f) = \varsigma < 2/3$, then*

$$T(r, f(z+c)) = T(r, f) + o\left(\frac{T(r, f)}{r^{1-\frac{3}{2}\varsigma-\epsilon}} \right)$$

where $r \to \infty$ outside of an exceptional set of finite logarithmic measure.

Lemma 2.3 [9, Theorem 1.26] *Let $f(z)$ be a nonconstant meromorphic function in the parabolic manifold M. Assume that*

$$R(z, w) = \frac{A(z, w)}{B(z, w)}.$$

Then

$$T(r, R_f) = \max\{p, q\}T(r, f) + O\left(\sum_{j=0}^{p} T(r, a_j) + \sum_{j=0}^{q} T(r, b_j) \right),$$

where $R_f(z) = R(z, f(z))$ *and two relatively prime polynomials* $A(z,w), B(z,w)$ *are given respectively as follows:*

$$A(z,w) = \sum_{j=0}^{p} a_j(z)w^j, B(z,w) = \sum_{j=0}^{q} b_j(z)w^j.$$

Lemma 2.4 [1, Corollary 4.5] *Let* $a_1(z), a_2(z), \cdots, a_n(z)$ *be* n *meromorphic functions in* $\mathbb{C}^m$ *and* $g_1(z), g_2(z), \cdots,$ $g_n(z)$ *be* n *entire functions in* $\mathbb{C}^m$ *satisfying*

$$\sum_{i=1}^{n} a_i(z)e^{g_i(z)} \equiv 0.$$

If for all $1 \le i \le n$

$$T(r, a_i) = o(T(r, e^{g_j - g_k})), j \ne k,$$

then $a_i(z) \equiv 0$ *for* $1 \le i \le n$.

Lemma 2.5 [9, Theorem 1.101] *Suppose that* $f_1(z), f_2(z), \cdots, f_n(z)$ *are linearly independent meromorphic functions in* $\mathbb{C}^m$ *such that*

$$f_1 + f_2 + \cdots + f_n \equiv 1.$$

Then for $1 \le j \le n$, $R > \rho > r > r_0$,

$$\begin{aligned} T(r, f_j) &\le N(r, f_j) + \sum_{k=1}^{n} \left\{ N(r, \frac{1}{f_k}) - N(r, f_k) \right\} + N(r, W) \\ &- N(r, \frac{1}{W}) + l \log \left\{ (\frac{\rho}{r})^{2m-1} \frac{T(R)}{\rho - r} \right\} + O(1), \end{aligned}$$

where $W = W_{\nu_1 \cdots \nu_{n-1}}(f_1, f_2, \cdots, f_n) \not\equiv 0$ *is a Wronskian determinant,*

$$n - 1 \le l = |\nu_1| + \cdots + |\nu_{n-1}| \le \frac{n(n-1)}{2},$$

and where

$$T(r) = \max_{1 \le k \le n} \{T(r, f_k)\}.$$

The following lemma extends the result due to Halburd and Korhonen[7, Theorem 3.2] on difference equations from one variable to several variables, which will be used in the later proofs of main results in this paper frequently.

Lemma 2.6 *Let* $f : \mathbb{C}^m \to \mathbb{P}^1$ *be a nonconstant meromorphic function of finite order satisfying the following equation:*

$$Q(z, f(z)) = 0, \tag{2.1}$$

where $Q(z, f(z))$ *is difference polynomial in* $f(z)$, $z \in \mathbb{C}^m$. *If* $Q(z, a(z)) \not\equiv 0$ *for a slowly moving target* $a(z)$, *then*

$$m\left(r, \frac{1}{f-a}\right) = S(r, f)$$

where $r \to \infty$ *outside of a possible exceptional set* $E \subset [1, +\infty)$ *of finite logarithmic measure* $\int_E 1/t dt < \infty$.

Proof Without loss of generality, assume that the difference polynomial $Q(z,f(z))$ of degree n can be written in the following form:

$$Q(z,f(z)) = \sum_{|I|=0}^{n} a_I f(z)^{i_0} f(z+c_1)^{i_1}\cdots f(z+c_l)^{i_l}, \tag{2.2}$$

where $I=(i_0,i_1,\cdots,i_l)\in\mathbb{N}^{l+1}$ denotes a multiindex with $|I|=i_0+i_1+\cdots+i_l$, $a_I=a_I(z)$ being small functions with respect to f in $z\in\mathbb{C}^m$, and $c_j\in\mathbb{C}^m(1\le j\le l)$ being some nonzero complex vectors. Taking $g=f-a$, then $Q(z,f(z))=Q_1(z,g(z))+Q_2(z)$, where $Q_1(z,g(z))$ denotes a difference polynomial in g such that all of its terms are at least of degree one. Therefore, $Q(z,g(z))$ can be shown as follows:

$$Q_1(z,g(z)) = \sum_{|I|=1}^{n} b_I g(z)^{i_0} g(z+c_1)^{i_1}\cdots g(z+c_l)^{i_l}, \tag{2.3}$$

where $b_I=b_I(z)$ are small functions with respect to f. Note that $Q_2(z)$ is a difference polynomial in $a(z)$, $a_I(z)$, and $Q_1(z,g(z))=-Q_2(z)$. Obviously, $T(r,Q_2)=S(r,f)$. On the other hand, when $|g(z)|\le 1$, we have

$$\begin{aligned}\left|\frac{Q_1(z,g(z))}{g(z)}\right| &= \frac{1}{|g(z)|}\left|\sum_{|I|=1}^{n} b_I g(z)^{i_0} g(z+c_1)^{i_1}\cdots g(z+c_l)^{i_l}\right| \\ &\le \sum_{|I|=1}^{n} |b_I| \left|\frac{g(z)}{g(z)}\right|^{i_0}\left|\frac{g(z+c_1)}{g(z)}\right|^{i_1}\cdots\left|\frac{g(z+c_l)}{g(z)}\right|^{i_l},\end{aligned}$$

where $|I|=i_0+i_1+\cdots+i_l\ge 1$. In view of the definition of $m(r,1/g)$, it can be seen that $m(r,1/g)$ vanishes on the part of $|z|=r$ where $|g(z)|>1$. Thus, from Lemma 2.1 and $T(r,Q_2)=S(r,f)$, we have

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &= m\left(r,\frac{1}{g}\right) \le m\left(r,\frac{Q_1(z,g(z))}{g}\right) + m\left(r,\frac{1}{-Q_2(z)}\right) \\ &\le \sum_{|I|=1}^{n}\left(m(r,b_I)+\sum_{j=1}^{l} i_j m\left(r,\frac{g(z+c_j)}{g(z)}\right)\right) + S(r,f) \\ &= S(r,f), \end{aligned}$$

where $r\to\infty$ outside of a possible exceptional set $E\subset[1,+\infty)$ of finite logarithmic measure $\int_E 1/dt<\infty$.

□

Lemma 2.7 *Let $\alpha(z)$ be a polynomial in z, $z=(z^1,z^2,\cdots,z^m)\in\mathbb{C}^m$. If $\alpha(z)$ is of degree $n(\ge 1)$, then $\deg(\alpha(z+c)-\alpha(z))<n$ holds for any given $c=(c^1,c^2,\cdots,c^m)\in\mathbb{C}^m$.*

Proof Without loss of generality, assume that

$$\alpha(z) = \sum_{|I|=n} a_I (z^1)^{i_1}(z^2)^{i_2}\cdots(z^m)^{i_m} + \sum_{|I|=0}^{n-1} b_I (z^1)^{i_1}(z^2)^{i_2}\cdots(z^m)^{i_m}$$

where a_I (not all zero), b_I are complex numbers, and $I=(i_1,\cdots,i_m)\in\mathbb{N}^m$ denotes a multiindex with $|I|=i_1+\cdots+i_m$. Next, consider the polynomial $(z^1)^{i_1}(z^2)^{i_2}\cdots(z^m)^{i_m}, i_1+i_2+\cdots+i_m=n$. Given

$c = (c^1, c^2, \cdots, c^m) \in \mathbb{C}^m$, assume $i_j \geq 1$ for all $1 \leq j \leq m$ (if there exists $i_k = 0$, only consider the rest of the term). Furthermore, we have

$$\begin{aligned} & (z^1+c^1)^{i_1}(z^2+c^2)^{i_2}\cdots(z^m+c^m)^{i_m} \\ = \; & (z^1)^{i_1}(z^2+c^2)^{i_2}\cdots(z^m+c^m)^{i_m} + P_{1,n-1}(z) \\ = \; & (z^1)^{i_1}(z^2)^{i_2}(z^3+c^3)^{i_3}\cdots(z^m+c^m)^{i_m} + P_{2,n-1}(z) + P_{1,n-1}(z) \\ = \; & \cdots \\ = \; & (z^1)^{i_1}(z^2)^{i_2}\cdots(z^m)^{i_m} + P_{m,n-1}(z) + \cdots + P_{2,n-1}(z) + P_{1,n-1}(z), \end{aligned}$$

where $P_{j,n-1}(z)$ $(1 \leq j \leq m)$ are some polynomials of degree at most $n-1$. Therefore,

$$\deg((z^1+c^1)^{i_1}(z^2+c^2)^{i_2}\cdots(z^m+c^m)^{i_m} - (z^1)^{i_1}(z^2)^{i_2}\cdots(z^m)^{i_m}) \leq n-1.$$

Similarly to the above discussion, it can be computed that

$$\begin{aligned} \alpha(z+c)-\alpha(z) &= \sum_{|I|=n} a_I((z^1+c^1)^{i_1}\cdots(z^m+c^m)^{i_m} - (z^1)^{i_1}\cdots(z^m)^{i_m}) \\ &+ \sum_{|I|=0}^{n-1} b_I((z^1+c^1)^{i_1}\cdots(z^m+c^m)^{i_m} - (z^1)^{i_1}\cdots(z^m)^{i_m}). \end{aligned} \tag{2.4}$$

From the polynomial in the right side of (2.4), it can be seen that the degree of $\alpha(z+c)-\alpha(z)$ is at most $n-1$. Thus, we complete the proof of Lemma 2.7. □

3. The proof of Theorem 1.6

Assume $P(f) \not\equiv \Delta_c \circ P(f)$. Obviously, $P(f) \not\equiv 0$. By Lemma 2.2, we have

$$T(r, P(f)) \leq \sum_{k=0}^{n} T(r, f(z+kc)) + O(1) \leq O(T(r,f)). \tag{3.1}$$

Note that $f(z)-a(z), P(f)-b(z)$, and $\Delta_c \circ P(f) - b(z)$ share $0, \infty$ CM. Hence, there exist two polynomials $\alpha(z), \beta(z)$, $z \in \mathbb{C}^m$ such that

$$\frac{P(f)-b(z)}{f(z)-a(z)} = e^{\alpha(z)}, \frac{\Delta_c \circ P(f) - b(z)}{f(z)-a(z)} = e^{\beta(z)}. \tag{3.2}$$

Letting $\phi(z) = \frac{P(f)-\Delta_c \circ P(f)}{f(z)-a(z)}$, then $\phi(z) = e^{\alpha(z)} - e^{\beta(z)} \not\equiv 0$. By $\sum_{k=0}^{n} a_k = 0$ and Lemma 2.1, we have

$$\begin{aligned} T(r,\phi) &= m(r,\phi) \leq m\left(r, \frac{P(f)}{f(z)-a(z)}\right) + m\left(r, \frac{\Delta_c \circ P(f)}{f(z)-a(z)}\right) \\ &\leq m\left(r, \sum_{k=0}^{n} a_k \frac{f(z+kc)-a(z)}{f(z)-a(z)}\right) + m\left(r, \frac{\Delta_c \circ P(f)}{P(f)} \cdot \frac{P(f)}{f(z)-a(z)}\right) \\ &\leq S(r,f) + S(r, P(f)). \end{aligned}$$

Note here that $P(f) \not\equiv 0$. Together with (3.1), we know that $T(r,\phi) = S(r,f)$.

Furthermore, by the second main theorem, we have

$$\begin{aligned} T(r, \frac{e^{\alpha}}{\phi}) &\leq \overline{N}(r, \frac{e^{\alpha}}{\phi}) + \overline{N}(r, \frac{\phi}{e^{\alpha}}) + \overline{N}(r, \frac{1}{e^{\alpha}/\phi - 1}) + S(r, \frac{e^{\alpha}}{\phi}) \\ &\leq \overline{N}(r, \frac{\phi}{e^{\beta}}) + S(r, f) + S(r, e^{\alpha}) \\ &\leq S(r, f) + S(r, e^{\alpha}). \end{aligned}$$

From the discussion above, it can be seen that $T(r, e^{\alpha}) = S(r, f)$. Similarly, we conclude that $T(r, e^{\beta}) = S(r, f)$.

Next, the fact that $m(r, \frac{1}{f-a}) = S(r, f)$ will be proved.

If $a(z) \not\equiv b(z)$, by the first equation in (3.2), we have

$$\begin{aligned} m(r, \frac{1}{f-a}) &\leq m(r, \frac{b}{f-a}) + m(r, \frac{1}{b}) \\ &\leq m\left(r, \frac{P(f)}{f-a}\right) + m(r, e^{\alpha}) + +m(r, \frac{1}{b}) \\ &= S(r, f). \end{aligned}$$

If $a(z) \equiv b(z) (\not\equiv 0)$, then the difference polynomial $Q(z, g(z))$ in $g(z)$ is considered as follows:

$$Q(z, g(z)) = P(g) - b(z) - e^{\alpha(z)}(g(z) - a(z)).$$

It follows from (3.2) that $Q(z, f(z)) \equiv 0$ and $Q(z, a(z)) = -b(z) \not\equiv 0$. By Lemma 2.6, $m(r, \frac{1}{f-a}) = S(r, f)$ holds.

On the other hand, (3.2) can be rewritten as follows:

$$P(f(z)) = b(z) + e^{\alpha(z)}(f(z) - a(z)), \quad \Delta_c \circ P(f(z)) = b(z) + e^{\beta(z)}(f(z) - a(z)).$$

Then it can be verified that

$$\begin{aligned} P(f(z+c)) &= b(z) + e^{\alpha(z+c)}(f(z+c) - a(z)), \\ P(f(z+c)) &= \Delta_c \circ P(f(z)) + P(f(z)) \\ &= 2b(z) + (e^{\alpha(z)} + e^{\beta(z)})(f(z) - a(z)). \end{aligned}$$

Note here that $a(z)$ and $b(z) (\not\equiv 0) \in S(f)$ are two periodic meromorphic functions with period c. Therefore, we can conclude that

$$f(z+c) - a(z) = (e^{\alpha(z) - \alpha(z+c)} + e^{\beta(z) - \alpha(z+c)})(f(z) - a(z)) + b(z)e^{-\alpha(z+c)}. \tag{3.3}$$

To simplify the above equality, we set

$$g(z) = e^{\alpha(z) - \alpha(z+c)} + e^{\beta(z) - \alpha(z+c)}, \quad h(z) = b(z)e^{-\alpha(z+c)}. \tag{3.4}$$

Thus, $f(z+c) - a(z) = g(z)(f(z) - a(z)) + h(z)$.

For $c \in \mathbb{C}^m$, $f : \mathbb{C}^m \to \mathbb{P}^1$, we use the short notations for brevity:

$$f(z) := \overline{f}^0, f(z+c) := \overline{f}^1, \cdots, f(z+kc) := \overline{f}^k.$$

From (3.3), we have

$$\begin{aligned}
\overline{f}^1 - a &= \overline{g}^0(\overline{f}^0 - a) + \overline{h}^0, \\
\overline{f}^2 - a &= \overline{g}^1(\overline{f}^1 - a) + \overline{h}^1 = \overline{g}^0\overline{g}^1(\overline{f}^0 - a) + \overline{h}^0\overline{g}^1 + \overline{h}^1, \\
\overline{f}^3 - a &= \overline{g}^1\overline{g}^2(\overline{f}^1 - a) + \overline{h}^1\overline{g}^2 + \overline{h}^2 = \overline{g}^0\overline{g}^1\overline{g}^2(\overline{f}^0 - a) + \overline{h}^0\overline{g}^1\overline{g}^2 + \overline{h}^1\overline{g}^2 + \overline{h}^2.
\end{aligned}$$

Now apply induction for the positive integer k to prove

$$\begin{aligned}
\overline{f}^k - a &= \overline{g}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-1}(\overline{f}^0 - a) + \overline{h}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-1} \\
&+ \overline{h}^1\overline{g}^2\overline{g}^3\cdots\overline{g}^{k-1} + \overline{h}^2\overline{g}^3\overline{g}^4\cdots\overline{g}^{k-1} + \cdots \\
&+ \overline{h}^{k-2}\overline{g}^{k-1} + \overline{h}^{k-1}.
\end{aligned} \tag{3.5}$$

For $j = 1$, $\overline{f}^1 - a = \overline{g}^0(\overline{f}^0 - a) + \overline{h}^0$ holds.
For $j = k-1 (k \geq 2)$, assume that

$$\begin{aligned}
\overline{f}^{k-1} - a &= \overline{g}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-2}(\overline{f}^0 - a) + \overline{h}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-2} \\
&+ \overline{h}^1\overline{g}^2\overline{g}^3\cdots\overline{g}^{k-2} + \overline{h}^2\overline{g}^3\overline{g}^4\cdots\overline{g}^{k-2} + \cdots \\
&+ \overline{h}^{k-3}\overline{g}^{k-2} + \overline{h}^{k-2}.
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
\overline{f}^k - a &= \overline{g}^1\overline{g}^2\overline{g}^3\cdots\overline{g}^{k-1}(\overline{f}^1 - a) + \overline{h}^1\overline{g}^2\overline{g}^3\cdots\overline{g}^{k-1} \\
&+ \overline{h}^2\overline{g}^3\overline{g}^4\cdots\overline{g}^{k-1} + \overline{h}^3\overline{g}^4\overline{g}^5\cdots\overline{g}^{k-1} + \cdots \\
&+ \overline{h}^{k-2}\overline{g}^{k-1} + \overline{h}^{k-1}.
\end{aligned}$$

Substituting $\overline{f}^1 - a = \overline{g}^0(\overline{f}^0 - a) + \overline{h}^0$ into the above equality,

$$\begin{aligned}
\overline{f}^k - a &= \overline{g}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-1}(\overline{f}^0 - a) + \overline{h}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-1} \\
&+ \overline{h}^1\overline{g}^2\overline{g}^3\cdots\overline{g}^{k-1} + \overline{h}^2\overline{g}^3\overline{g}^4\cdots\overline{g}^{k-1} + \cdots \\
&+ \overline{h}^{k-2}\overline{g}^{k-1} + \overline{h}^{k-1}.
\end{aligned}$$

Thus, (3.5) is proved.

For the sake of simplicity of computation, we set

$$\gamma_k = \overline{g}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-1}, \tag{3.6}$$

$$\zeta_k = \overline{h}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-1} + \overline{h}^1\overline{g}^2\overline{g}^3\cdots\overline{g}^{k-1} + \cdots + \overline{h}^{k-2}\overline{g}^{k-1} + \overline{h}^{k-1}. \tag{3.7}$$

Thus, $\overline{f}^k - a = \gamma_k(\overline{f}^0 - a) + \zeta_k$. In particular, $\gamma_0 = 1, \zeta_0 = 0$.

By Lemma 2.2, and $T(r, e^{\alpha}) = S(r, f)$, $T(r, e^{\beta}) = S(r, f)$, we have

$$T(r, g) = S(r, f), T(r, h) = S(r, f).$$

Furthermore, we obtain that for all $k \geq 0$,

$$T(r, \gamma_k) = S(r, f), T(r, \zeta_k) = S(r, f). \tag{3.8}$$

By the definition of $P(f)$, we have

$$P(f) - b = \sum_{k=0}^{n} a_k(\overline{f}^k - a) - b = (\overline{f}^0 - a)\sum_{k=0}^{n} a_k\gamma_k + \sum_{k=0}^{n} a_k\zeta_k - b. \tag{3.9}$$

Set $P_0(f) = P(f) - b - (f - a)\sum_{k=0}^{n} a_k\gamma_k$. If $\sum_{k=0}^{n} a_k\zeta_k - b \not\equiv 0$, then $P_0(f) \not\equiv 0$. Owing to $P(f) - b$ and $f - a$ sharing 0 CM, $N(r, \frac{1}{f-a}) \leq N(r, \frac{1}{P_0(f)})$ can be verified. Hence, by (3.8) and $m(r, \frac{1}{f-a}) = S(r, f)$, we have

$$\begin{aligned} T(r, f) + S(r, f) &= N(r, \frac{1}{f-a}) \leq N(r, \frac{1}{P_0(f)}) \\ &= N\left(r, \frac{1}{\sum_{k=0}^{n} a_k\zeta_k - b}\right) \\ &\leq S(r, f), \end{aligned}$$

which yields a contradiction. Therefore, we have

$$\sum_{k=0}^{n} a_k\zeta_k - b \equiv 0. \tag{3.10}$$

Together with (3.2) and (3.9), we have

$$\sum_{k=0}^{n} a_k\gamma_k = e^{\alpha}. \tag{3.11}$$

Noting that $\gamma_k = \overline{g}^0\overline{g}^1\overline{g}^2\cdots\overline{g}^{k-1}$, (3.11) can be rewritten as follows:

$$a_0 + a_1\overline{g}^0 + a_2\overline{g}^0\overline{g}^1 + a_3\overline{g}^0\overline{g}^1\overline{g}^2 + \cdots + a_n\overline{g}^0\overline{g}^1\cdots\overline{g}^{n-1} = e^{\alpha}, \tag{3.12}$$

where $\overline{g}^0 = e^{\overline{\alpha}^0 - \overline{\alpha}^1}(1 + e^{\overline{\beta}^0 - \overline{\alpha}^0})$.

We consider it in two cases.

Case 1: $\deg(\beta - \alpha) \geq \deg(\alpha)$. Letting $\omega = \beta - \alpha$, it follows from Lemma 2.7 that for any $i \neq j$,

$$\begin{aligned} \deg(\overline{\alpha}^i - \overline{\alpha}^j) &< \deg(\alpha) \leq \deg(\omega), \\ \deg(\overline{\omega}^i - \overline{\omega}^j) &< \deg(\omega). \end{aligned} \tag{3.13}$$

Thus, $\overline{g}^k(0 \le k \le n)$ can be represented as follows:

$$\begin{aligned}
\overline{g}^0 &= e^{\overline{\alpha}^0-\overline{\alpha}^1}(1+e^{\overline{\omega}^0}) := \eta_0 + \theta_0 e^{\overline{\omega}^0},\\
\overline{g}^1 &= e^{\overline{\alpha}^1-\overline{\alpha}^2}(1+e^{\overline{\omega}^1}) = e^{\overline{\alpha}^1-\overline{\alpha}^2}(1+e^{\overline{\omega}^0}\cdot e^{\overline{\omega}^1-\overline{\omega}^0}) := \eta_1 + \theta_1 e^{\overline{\omega}^0},\\
\overline{g}^2 &= e^{\overline{\alpha}^2-\overline{\alpha}^3}(1+e^{\overline{\omega}^2}) = e^{\overline{\alpha}^2-\overline{\alpha}^3}(1+e^{\overline{\omega}^0}\cdot e^{\overline{\omega}^2-\overline{\omega}^0}) := \eta_2 + \theta_2 e^{\overline{\omega}^0},\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\overline{g}^{n-2} &= e^{\overline{\alpha}^{n-2}-\overline{\alpha}^{n-1}}(1+e^{\overline{\omega}^0}\cdot e^{\overline{\omega}^{n-2}-\overline{\omega}^0}) := \eta_{n-2} + \theta_{n-2} e^{\overline{\omega}^0},\\
\overline{g}^{n-1} &= e^{\overline{\alpha}^{n-1}-\overline{\alpha}^{n}}(1+e^{\overline{\omega}^0}\cdot e^{\overline{\omega}^{n-1}-\overline{\omega}^0}) := \eta_{n-1} + \theta_{n-1} e^{\overline{\omega}^0},
\end{aligned} \tag{3.14}$$

where $\eta_k = e^{\overline{\alpha}^k-\overline{\alpha}^{k+1}}, \theta_k = e^{\overline{\alpha}^k-\overline{\alpha}^{k+1}}\cdot e^{\overline{\omega}^k-\overline{\omega}^0}$ for all $0 \le k \le n-1$.

Subcase 1.1: $\deg(\omega) > \deg(\alpha) \ge 0$. Obviously, $\deg(\omega) \ge 1, T(r, e^\alpha) = o(T(r, e^\omega))$. In view of the definitions of $\eta_k, \theta_k(0 \le k \le n-1)$ in (3.14), we conclude that $T(r, \eta_k) = o(T(r, e^\omega))$ and $T(r, \theta_k) = o(T(r, e^\omega))$ for all $0 \le k \le n-1$.

Substituting $\overline{g}^0, \overline{g}^1, \cdots, \overline{g}^{n-1}$ into (3.12), we have

$$\begin{aligned}
e^\alpha &= a_0 + a_1(\eta_0+\theta_0 e^\omega) + a_2(\eta_0+\theta_0 e^\omega)(\eta_1+\theta_1 e^\omega) + \cdots\\
&+ a_n(\eta_0+\theta_0 e^\omega)(\eta_1+\theta_1 e^\omega)\cdots(\eta_{n-1}+\theta_{n-1} e^\omega)\\
&= b_0 + b_1 e^\omega + b_2 e^{2\omega} + b_3 e^{3\omega} + \cdots + b_n e^{n\omega},
\end{aligned} \tag{3.15}$$

where $T(r, b_k) = o(T(r, e^\omega))$ for all $0 \le k \le n-1$. In particular,

$$\begin{aligned}
b_0 &= a_0 + a_1\eta_0 + a_2\eta_0\eta_1 + \cdots + a_n\eta_0\eta_1\cdots\eta_{n-1}\\
&= a_0 + a_1 e^{\overline{\alpha}^0-\overline{\alpha}^1} + a_2 e^{\overline{\alpha}^0-\overline{\alpha}^1}e^{\overline{\alpha}^1-\overline{\alpha}^2} + \cdots + a_n e^{\overline{\alpha}^0-\overline{\alpha}^1}\cdots e^{\overline{\alpha}^{n-1}-\overline{\alpha}^n}\\
&= a_0 + a_1 e^{\overline{\alpha}^0-\overline{\alpha}^1} + a_2 e^{\overline{\alpha}^0-\overline{\alpha}^2} + \cdots + a_n e^{\overline{\alpha}^0-\overline{\alpha}^n}.
\end{aligned} \tag{3.16}$$

Applying Lemma 2.4 to (3.15), it can be seen that $b_0 - e^\alpha \equiv 0, b_k \equiv 0(1 \le k \le n)$.

If $e^\alpha(\ne 0)$ is a constant, then for all $0 \le k \le n-1$, $\eta_k = e^{\overline{\alpha}^k-\overline{\alpha}^{k+1}} \equiv 1$ and $b_0 = \sum_{k=0}^n a_k \equiv 0$. Thus, $e^\alpha = b_0 \equiv 0$, which is a contradiction.

If $e^\alpha(\ne 0)$ is not a constant, i.e. α is a nonconstant polynomial, from (3.16),

$$a_0 + a_1 e^{\overline{\alpha}^0-\overline{\alpha}^1} + a_2 e^{\overline{\alpha}^0-\overline{\alpha}^2} + \cdots + a_n e^{\overline{\alpha}^0-\overline{\alpha}^n} = e^\alpha.$$

Hence, by (3.13)

$$\begin{aligned}
T(r, e^\alpha) &= T(r, a_0 + a_1 e^{\overline{\alpha}^0-\overline{\alpha}^1} + a_2 e^{\overline{\alpha}^0-\overline{\alpha}^2} + \cdots + a_n e^{\overline{\alpha}^0-\overline{\alpha}^n})\\
&\le o(T(r, e^\alpha)),
\end{aligned}$$

which is impossible.

Subcase 1.2: $\deg(\omega) = \deg(\alpha) = 0$.

Obviously, e^{β} is also a constant. Hence, we can assume that $e^{\alpha} = T_1, e^{\beta} = T_2$, where T_1, T_2 are two distinct complex numbers for $e^{\alpha} \not\equiv e^{\beta}$. In view of (3.4), we deduce that $\overline{g}^k = \overline{g}^0 = g \equiv 1 + \frac{T_2}{T_1}$ and $\overline{h}^k = \overline{h}^0 = h \equiv \frac{b}{T_1}$ for all $0 \leq k \leq n-1$. Together with (3.6),

$$\begin{aligned} \gamma_k &= g^k = (1 + \frac{T_1}{T_2})^k \\ \zeta_k &= h(1 + g + g^2 + \cdots + g^{k-1}) = \frac{h}{1-g}(1 - g^k). \end{aligned}$$

Noting that $b(z) \not\equiv 0$ and $\sum_{k=0}^{n} a_k = 0$, by (3.11) and (3.10),

$$\begin{aligned} T_1 &= \sum_{k=0}^{n} a_k g^k, \\ T_2 &= \frac{T_2}{b} \cdot b = \frac{T_2}{b} \cdot \sum_{k=0}^{n} a_k \frac{h(1-g^k)}{(1-g)} = \sum_{k=0}^{n} a_k g^k, \end{aligned}$$

which yields a contradiction for $T_1 \neq T_2$.

Subcase 1.3: $\deg(\omega) = \deg(\alpha) \geq 1$

For this case, a contradiction can be obtained in a similar way as shown in Subcase 1.1. Here, we prove it in a different way.

Obviously, e^{α} is not a constant. From (3.12) and (3.14), we have

$$e^{\alpha} = b_0 + b_1 e^{\omega} + b_2 e^{2\omega} + b_3 e^{3\omega} + \cdots + b_n e^{n\omega}, \tag{3.17}$$

where $T(r, b_k) = o(T(r, e^{\omega}))$ for all $0 \leq k \leq n-1$. In particular,

$$\begin{aligned} b_0 &= a_0 + a_1 e^{\overline{\alpha}^0 - \overline{\alpha}^1} + a_2 e^{\overline{\alpha}^0 - \overline{\alpha}^2}, \\ b_n &= a_n e^{\overline{\alpha}^0 - \overline{\alpha}^n} \cdot e^{\overline{\omega}^1 + \cdots + \overline{\omega}^{n-1} - (n-1)\overline{\omega}^0}. \end{aligned} \tag{3.18}$$

Next, assume that $\deg(j\omega - \alpha) = \deg(\omega) \geq 1$ for all $0 \leq j \leq n$. Let $\varphi_0 = b_0, \varphi_1 = b_1 e^{\omega}, \varphi_2 = b_2 e^{2\omega}, \cdots, \varphi_n = b_n e^{n\omega}, \varphi_{n+1} = -e^{\alpha}$; thus, $\sum_{k=0}^{n+1} \varphi_k = 0$. From basic linear algebra, we deduce that there exist a set $\kappa \subset \{0, 1, 2, \cdots, n\}$ and some nonzero complex numbers $\lambda_k (k \in \kappa)$ such that

$$\varphi_{n+1} = \sum_{k \in \kappa} \lambda_k \varphi_k, \tag{3.19}$$

and $\{\varphi_k | k \in \kappa\}$ is linearly independent. Dividing both of the two sides of (3.19) by φ_{n+1}, we have

$$1 = \sum_{k \in \kappa} \lambda_k \frac{\varphi_k}{\varphi_{n+1}} = \sum_{k \in \kappa} \lambda_k b_k e^{k\omega - \alpha}.$$

Note that $\{\lambda_k b_k e^{k\omega-\alpha}\}_{k\in\kappa}$ is linearly independent. In addition, it is not difficult to verify that zeros and poles of $\{\lambda_k b_k e^{k\omega-\alpha}\}(k\in\kappa)$ and their Wronskian determinant come only from the zeros and poles of $b_k(k\in\kappa)$. Then by Lemma 2.5 and $T(r,b_k)=o(T(r,e^\omega))$ for all $0\le k\le n-1$, we have

$$T(r,\lambda_k b_k e^{k\omega-\alpha})\le O\left(\sum_{k\in\kappa}T(r,b_k)\right)=o(T(r,e^w)).$$

This is a desired contradiction for $\deg(k\omega-\alpha)=\deg(\omega)(k\in\kappa)$. Hence, there exists $k_0\in\{1,2,\cdots,n\}$ such that $\deg(k_0\omega-\alpha)<\deg(\omega)$. Then $e^\alpha=e^{k_0\omega}\cdot e^{\alpha-k_0\omega}:=b^*_{k_0}e^{k_0\omega}$, where $T(r,b_{k_0})=o(T(r,e^\omega))$. Thus, (3.17) can be rewritten as follows:

$$b_0+b_1e^\omega+b_2e^{2\omega}+\cdots+(b_{k_0}-b^*_{k_0})e^{k_0\omega}+\cdots+b_ne^{n\omega}=0,$$

where $T(r,b_k)=o(T(r,e^\omega))$ for all $0\le k\le n-1$.

By Lemma 2.4, $b_0\equiv 0, b_1\equiv 0,\cdots,b_{k_0}-b^*_{k_0}\equiv 0,\cdots,b_n\equiv 0$.

If $k_0\neq n$, then by (3.18), $b_n=a_ne^{\overline{\alpha}^0-\overline{\alpha}^n}\cdot e^{\overline{\omega}^1+\cdots+\overline{\omega}^{n-1}-(n-1)\overline{\omega}^0}\neq 0$, which yields a contradiction for $b_n\equiv 0$.

If $k_0=n$, i.e. $\deg(n\omega-\alpha)<\deg(\omega)$, then by (3.7) and (3.14), we have

$$\begin{aligned}\zeta_k &= \overline{h}^0(\eta_1+\theta_1e^\omega)(\eta_2+\theta_2e^\omega)\cdots(\eta_{k-1}+\theta_{k-1}e^\omega)\\ &+ \overline{h}^1(\eta_2+\theta_2e^\omega)(\eta_3+\theta_3e^\omega)\cdots(\eta_{k-1}+\theta_{k-1}e^\omega)\\ &+ \cdots+\overline{h}^{k-2}(\eta_{k-1}+\theta_{k-1}e^\omega)+\overline{h}^{k-1},\end{aligned}$$

where $\eta_k,\theta_k(0\le k\le n-1)$ are small functions with respect to e^ω. By (3.4) and some calculation, $\zeta_k(1\le k\le n)$ can be rewritten in a new form:

$$\zeta_k=be^{-\alpha}(\tau_0+\tau_1e^\omega+\tau_2e^{2\omega}+\cdots+\tau_{k-1}e^{(k-1)\omega}),$$

where $\tau_j(0\le j\le k-1)$ are polynomials of $\eta_j,\theta_j,e^{\alpha-\overline{\alpha}^j}(0\le j\le k-1)$. Obviously, $T(r,\tau_j)=o(T(r,e^\omega))(0\le j\le k-1)$. Substituting ζ_k into (3.10), we get

$$be^{-\alpha}\sum_{k=0}^{n}a_k(\tau_0+\tau_1e^\omega+\tau_2e^{2\omega}+\cdots+\tau_{k-1}e^{(k-1)\omega})=b. \tag{3.20}$$

Noting here $b(z)\not\equiv 0$, a routine computation yields

$$d_0+d_1e^\omega+d_2e^{2\omega}+\cdots+d_{n-1}e^{(n-1)\omega}=e^\alpha, \tag{3.21}$$

where $T(r,d_j)=o(T(r,e^\omega))(0\le j\le n-1)$.

On the other hand, $e^\alpha=e^{\alpha-n\omega}\cdot e^{n\omega}:=-d_ne^{n\omega}(d_n\neq 0)$. It follows from $\deg(n\omega-\alpha)<\deg(\omega)$ that $T(r,d_n)=o(T(r,e^\omega))$. Thus, (3.21) turns into $\sum_{k=0}^n d_ke^{k\omega}=0$. Applying Lemma 2.4, we have $d_k\equiv 0(0\le k\le n)$, which is impossible for $d_n\not\equiv 0$.

Case 2: $\deg(\beta-\alpha)<\deg(\alpha)$. In view of $\overline{g}^0=e^{\overline{\alpha}^0-\overline{\alpha}^1}(1+e^{\overline{\beta}^0-\overline{\alpha}^0})$, it can be seen that $T(r,\overline{g}^0)=o(T(r,e^\alpha))$.

Thus, $T(r,\overline{g}^k)=o(T(r,e^{\alpha}))$ and $T(r,\overline{g}^0\overline{g}^1\cdots\overline{g}^k)=o(T(r,e^{\alpha}))$ for all $0\leq k\leq n-1$. Together with (3.12), we have

$$\begin{aligned} T(r,e^{\alpha}) &= T(r,a_0+a_1\overline{g}^0+a_2\overline{g}^0\overline{g}^1+a_3\overline{g}^0\overline{g}^1\overline{g}^2+\cdots+a_n\overline{g}^0\overline{g}^1\cdots\overline{g}^{n-1}) \\ &\leq o(T(r,e^{\alpha})), \end{aligned}$$

which is impossible. That completes the proof of Theorem 1.6.

4. The proof of Theorem 1.7

Set $g(z)=f(z)+b(z)-a(z)$. It follows from the definition of $P(f)$ and $\Delta_c\circ P(f)$ that

$$\begin{aligned} g(z)-b(z) &= f(z)-a(z), \\ P(g(z))-b(z) &= P(f(z))-b(z), \\ \Delta_c\circ P(g(z))-b(z) &= \Delta_c\circ P(f(z))-b(z). \end{aligned}$$

Note that $f(z)-a(z), P(f)-b(z)$, and $\Delta_c\circ P(f)-b(z)$ share $0,\infty$ CM. This means that g, $P(g)$ and $\Delta_c\circ P(g)$ share $b(z),\infty$ CM. From Theorem 1.6, there exists a polynomial $\alpha(z)$, $z\in\mathbb{C}^m$ such that

$$\frac{P(g)-b(z)}{g(z)-b(z)}=e^{\alpha(z)},\ \frac{\Delta_c\circ P(g)-b(z)}{g(z)-b(z)}=e^{\alpha(z)}.$$

The above equality can be rewritten as follows:

$$P(g)=e^{\alpha}(g-b)+b,\quad \Delta_c\circ P(g)=e^{\alpha}(g-b)+b. \tag{4.1}$$

On the other hand,

$$\Delta_c\circ P(g)=P(g(z+c))-P(g(z))=e^{\overline{\alpha}^1}(\overline{g}^1-b)-e^{\alpha}(g-b).$$

Owing to the second equality of (4.1), we can conclude that

$$\overline{g}^1=g(z+c)=2(g-b)e^{\alpha-\overline{\alpha}^1}+b(1+e^{-\overline{\alpha}^1}). \tag{4.2}$$

Applying the induction, we have for $0\leq k\leq n$

$$\overline{g}^k=g(z+kc)=2^k(g-b)e^{\alpha-\overline{\alpha}^k}+b(2^k-1)e^{-\overline{\alpha}^k}+b.$$

One can complete the proof similarly to the proof of Theorem 1.6, so we omit the details.

Substituting $\overline{g}^k$ into $P(g)$ and $\Delta_c\circ P(g)$, we have the following:

$$\begin{aligned} P(g) &= \sum_{k=0}^{n}a_k\overline{g}^k=\sum_{k=0}^{n}a_k(2^k(g-b)e^{\alpha-\overline{\alpha}^k}+b(2^k-1)e^{-\overline{\alpha}^k}+b) \\ &= (g-b)\left(\sum_{k=0}^{n}a_k2^ke^{\alpha-\overline{\alpha}^k}\right)+b\left(\sum_{k=0}^{n}a_k(2^k-1)e^{-\overline{\alpha}^k}\right), \end{aligned}$$

$$\begin{aligned}
\Delta_c \circ P(g) &= P(g(z+c)) - P(g(z)) = \sum_{k=0}^{n} a_k \overline{g}^{k+1} - \sum_{k=0}^{n} a_k \overline{g}^{k} \\
&= -a_0 g + \sum_{k=0}^{n-1} (a_k - a_{k+1}) \overline{g}^{k+1} + a_n \overline{g}^{n+1} \\
&= (g-b)\left(\sum_{k=0}^{n-1} (a_k - a_{k+1}) 2^{k+1} e^{\alpha - \overline{\alpha}^{k+1}} + a_n 2^{n+1} e^{\alpha - \overline{\alpha}^{n+1}} - a_0 \right) \\
&+ b\left(\sum_{k=0}^{n-1} (a_k - a_{k+1})(2^{k+1}-1) e^{-\overline{\alpha}^{k+1}} + a_n (2^{n+1}-1) e^{-\overline{\alpha}^{n+1}} \right) \\
&= (g-b)\left(\sum_{k=0}^{n} a_k 2^k e^{\alpha - \overline{\alpha}^{k}} (2e^{\overline{\alpha}^{k} - \overline{\alpha}^{k+1}} - 1) \right) \\
&+ b\left(\sum_{k=0}^{n} a_k ((2^{k+1}-1) e^{-\overline{\alpha}^{k+1}} - (2^k - 1) e^{-\overline{\alpha}^{k}}) \right).
\end{aligned}$$

For brevity, set

$$\begin{aligned}
A_n &= \sum_{k=0}^{n} a_k 2^k e^{\alpha - \overline{\alpha}^{k}}, \quad B_n = \sum_{k=0}^{n} a_k (2^k - 1) e^{-\overline{\alpha}^{k}} \\
A_n^* &= \sum_{k=0}^{n} a_k 2^k e^{\alpha - \overline{\alpha}^{k}} (2e^{\overline{\alpha}^{k} - \overline{\alpha}^{k+1}} - 1) \qquad (4.3) \\
B_n^* &= \sum_{k=0}^{n} a_k ((2^{k+1}-1) e^{-\overline{\alpha}^{k+1}} - (2^k - 1) e^{-\overline{\alpha}^{k}}).
\end{aligned}$$

Thus, $P(g)$ and $\Delta_c \circ P(g)$ can be rewritten as follows:

$$\begin{aligned}
P(g) &= (g-b)A_n + bB_n, \\
\Delta_c \circ P(g) &= (g-b)A_n^* + bB_n^*.
\end{aligned}$$

Together with (4.1), we have

$$\begin{aligned}
(A_n - e^{\alpha})g - b(1 - B_n) &= 0, \qquad (4.4) \\
(A_n^* - e^{\alpha})g - b(1 - B_n^*) &= 0,
\end{aligned}$$

which yields

$$(A_n - e^{\alpha})(1 - B_n^*) = (1 - B_n)(A_n^* - e^{\alpha}). \qquad (4.5)$$

On the other hand, it follows from (4.3) that

$$\begin{aligned} e^{\alpha}B_n &= \sum_{k=0}^{n} a_k(2^k-1)e^{\alpha-\overline{\alpha}^k} = A_n - \sum_{k=0}^{n} a_k e^{\alpha-\overline{\alpha}^k} = A_n - e^{\alpha}P(e^{-\alpha}) \\ e^{\alpha}B_n^* &= \sum_{k=0}^{n} a_k((2^{k+1}-1)e^{\alpha-\overline{\alpha}^{k+1}} - (2^k-1)e^{\alpha-\overline{\alpha}^k}) \\ &= A_n^* - \sum_{k=0}^{n} a_k e^{\alpha-\overline{\alpha}^{k+1}} + \sum_{k=0}^{n} a_k e^{\alpha-\overline{\alpha}^k} \\ &= A_n^* - e^{\alpha}\Delta_c \circ P(e^{-\alpha}). \end{aligned}$$

Furthermore, substituting B_n, B_n^* into (4.5), we conclude that

$$(A_n - e^{\alpha})\Delta_c \circ P(e^{-\alpha}) = (A_n^* - e^{\alpha})P(e^{-\alpha}). \tag{4.6}$$

In addition, we can deduce that

$$A_n^* = e^{\alpha}\sum_{k=0}^{n} a_k 2^{k+1} e^{-\overline{\alpha}^{k+1}} - A_n.$$

Thus, (4.6) can be rewritten as follows:

$$e^{-\alpha}A_nP(e^{-\overline{\alpha}^1}) + P(2e^{-\alpha} - e^{-\overline{\alpha}^1}) = \sum_{k=0}^{n} a_k 2^{k+1} e^{-\overline{\alpha}^{k+1}} P(e^{-\alpha}).$$

Substituting A_n into the above equality, we have

$$P(2e^{-\alpha} - e^{-\overline{\alpha}^1}) = \sum_{k=0}^{n} a_k 2^{k+1} e^{-\overline{\alpha}^{k+1}} P(e^{-\alpha}) - \sum_{k=0}^{n} a_k 2^k e^{-\overline{\alpha}^k} P(e^{-\overline{\alpha}^1}).$$

Setting $d_k = 2^k e^{\overline{\alpha}^{n+1}-\overline{\alpha}^k} (0 \le k \le n+1)$, the above equality can be shown as follows:

$$e^{\overline{\alpha}^{n+1}} P(2e^{-\alpha} - e^{-\overline{\alpha}^1}) = \sum_{k=0}^{n} a_k d_{k+1} P(e^{-\alpha}) - \sum_{k=0}^{n} a_k d_k P(e^{-\overline{\alpha}^1}). \tag{4.7}$$

Case 1: α is a constant.

Obviously, $e^{\alpha} = A(\neq 0)$. From (4.2), we have

$$\begin{aligned} \Delta_c g &= \overline{g}^1 - g = 2(g-b) + b(1+\frac{1}{A}) - g \\ &= g - b + \frac{b}{A}, \end{aligned}$$

which implies $\Delta_c f(z) = f(z) - a(z) + \frac{b(z)}{A}$. Furthermore, by (4.3) and (4.4),

$$A_n = \sum_{k=0}^{n} a_k 2^k, B_n = \frac{1}{A}\sum_{k=0}^{n} a_k 2^k,$$

$$(A_n - A)g - b\left(1 - \frac{A_n}{A}\right) = 0.$$

If $A_n \neq A$, then $g = -\frac{b}{A}$. Therefore, $T(r,g) \leq O(T(r,b)) = S(r,g)$, which is impossible. Hence, $A = A_n = \sum_{k=0}^{n} a_k 2^k$ is a nonzero constant. Thus, the first conclusion of Theorem 1.7 holds.

Case 2: α is nonconstant polynomial.

In this case, for any $0 \leq i < j \leq n+1$, we have $T(r, e^{\overline{\alpha}^i - \overline{\alpha}^j}) = S(r, e^{\alpha})$. Hence, $T(r, d_k) = S(r, e^{\alpha})(0 \leq k \leq n+1)$. Therefore, (4.7) can be rewritten as follows:

$$t_{n+2} = t_0 e^{-\alpha} + t_1 e^{-\overline{\alpha}^1} + \cdots + t_n e^{-\overline{\alpha}^n} + t_{n+1} e^{-\overline{\alpha}^{n+1}}, \tag{4.8}$$

where $T(r, t_k) = S(r, e^{\alpha})(0 \leq k \leq n+2)$ and $t_{n+2} = e^{\overline{\alpha}^{n+1}} P(2e^{-\alpha} - e^{-\overline{\alpha}^1})$.

Subcase 2.1: $t_{n+2} \not\equiv 0$. By (4.8), we have

$$\begin{aligned}
T(r, e^{\alpha}) &= m(r, e^{\alpha}) \\
&= m\left(r, \frac{t_0 + t_1 e^{\alpha - \overline{\alpha}^1} + \cdots + t_n e^{\alpha - \overline{\alpha}^n} + t_{n+1} e^{\alpha - \overline{\alpha}^{n+1}}}{t_{n+2}}\right) \\
&\leq m(r, \frac{1}{t_{n+2}}) + \sum_{k=0}^{n+1} m(r, t_k) + \sum_{k=1}^{n+1} m(r, e^{\alpha - \overline{\alpha}^k}) + S(r, e^{\alpha}) \\
&\leq S(r, e^{\alpha}),
\end{aligned}$$

which yields a desired contradiction.

Subcase 2.2: $t_{n+2} \equiv 0$. In this case, we have

$$0 = P(2e^{-\alpha} - e^{-\overline{\alpha}^1}) = 2a_0 e^{-\overline{\alpha}^0} + \sum_{k=0}^{n-1} (2a_{k+1} - a_k) e^{-\overline{\alpha}^{k+1}} - a_n e^{-\overline{\alpha}^{n+1}}.$$

If $\deg(\overline{\alpha}^i - \overline{\alpha}^j) > 0$ holds for any $0 \leq i < j \leq n+1$, then by Lemma 2.4, we have

$$\begin{aligned}
2a_0 &= 0, \quad 2a_1 - a_0 = 0 \\
&\cdots \qquad \cdots \\
2a_n - a_{n-1} &= 0, \quad -a_n = 0,
\end{aligned}$$

which implies $a_k = 0(0 \leq k \leq n)$, a contradiction.

Thus, there exist i_0, j_0 such that $\deg(\overline{\alpha}^{i_0} - \overline{\alpha}^{j_0}) = 0$. We may assume that $e^{\overline{\alpha}^0 - \overline{\alpha}^1} = B$, and $e^{\overline{\alpha}^i - \overline{\alpha}^j} = B^{j-i}$ for any $0 \leq i, j \leq n$. Furthermore, by some calculation, we have the following:

$$P(2e^{-\alpha} - e^{-\overline{\alpha}^1}) = (2 - B) e^{-\alpha} \sum_{k=0}^{n} a_k B^k.$$

Since $P(2e^{-\alpha} - e^{-\overline{\alpha}^1}) = 0$, we have $\sum_{k=0}^{n} a_k B^k = 0$ or $B = 2$.

If $\sum_{k=0}^{n} a_k B^k = 0$, by (4.3) and (4.4), $B_n = e^{-\alpha} A_n$ and $g = \frac{b(1-B_n)}{A_n - e^{\alpha}} = -be^{-\alpha}$. Hence, $\overline{g}^k = -be^{-\overline{\alpha}^k}$

and

$$\begin{aligned} P(f) &= P(g) = \sum_{k=0}^{n} a_k \overline{g}^k = -b\sum_{k=0}^{n} a_k e^{-\overline{\alpha}^k} \\ &= -be^{-\alpha}\sum_{k=0}^{n} a_k e^{\alpha-\overline{\alpha}^k} = -be^{-\alpha}\sum_{k=0}^{n} a_k B^k \\ &= 0, \end{aligned}$$

which is impossible for the hypothesis.

If $B=2$, then from the discussion above, it can be seen that $\sum_{k=0}^{n} a_k 2^k \neq 0$. On the other hand,

$$\begin{aligned} P(e^{-\alpha}) &= \sum_{k=0}^{n} a_k e^{-\overline{\alpha}^k} = e^{-\alpha}\sum_{k=0}^{n} a_k e^{\alpha-\overline{\alpha}^k} = e^{-\alpha}\sum_{k=0}^{n} a_k 2^k, \\ P(e^{-\overline{\alpha}^1}) &= \sum_{k=0}^{n} a_k e^{-\overline{\alpha}^{k+1}} = e^{-\alpha}\sum_{k=0}^{n} a_k e^{\alpha-\overline{\alpha}^{k+1}} = 2e^{-\alpha}\sum_{k=0}^{n} a_k 2^k. \end{aligned}$$

Obviously, $\Delta_c \circ P(e^{-\alpha}) = P(e^{-\alpha}) (\neq 0)$. Furthermore, owing to (4.6), we have $A_n = A_n^*$. From (4.3), we get $A_n = \sum_{k=0}^{n} a_k 4^k$ and $A_n^* = 3\sum_{k=0}^{n} a_k 4^k$. It can be concluded that $A_n = \sum_{k=0}^{n} a_k 4^k = 0$. Thus, we complete the proof of Theorem 1.7.

5. The proof of Theorem 1.8

Since $f(z)-a(z), f(z+c)-a(z)$, and $P(f)-b(z)$ share $0,\infty$ CM, there exist two polynomials $\alpha(z), \beta(z)$ such that

$$\frac{f(z+c)-a(z)}{f(z)-a(z)} = e^{\alpha(z)}, \frac{P(f)-b(z)}{f(z)-a(z)} = e^{\beta(z)}. \tag{5.1}$$

By the above equalities and Lemma 2.1, we deduce that

$$T(r, e^{\alpha}) = m(r, e^{\alpha}) = S(r, f).$$

By the first equality of (5.1), we can deduce that for any $k (\geq 1)$

$$\frac{\overline{f}^k - a}{f-a} = \frac{\overline{f}^k - a}{\overline{f}^{k-1} - a} \cdots \frac{\overline{f}^1 - a}{f-a} = e^{\sum_{j=0}^{k-1} \overline{\alpha}^j}.$$

Here, we use the short notations mentioned above for brevity. Thus, $P(f)$ can be rewritten as follows:

$$P(f) = \left(a_0 + \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1} \overline{\alpha}^j} \right)(f-a). \tag{5.2}$$

Together with the second equality of (5.1),

$$\left(e^{\beta} - a_0 - \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1} \overline{\alpha}^j} \right)(f-a) + b \equiv 0. \tag{5.3}$$

Case 1: $b(z) \equiv 0$. By the fact that $f - a \not\equiv 0$ and (5.3), we have

$$a_0 + a_1 e^{\alpha} + a_2 e^{\alpha + \overline{\alpha}^1} + \cdots + a_n e^{\sum_{k=0}^{n-1} \overline{\alpha}^k} = e^{\beta}. \tag{5.4}$$

If α is a constant, then Theorem 1.8 holds. Hence, we assume that α is not a constant, i.e. $\deg(\alpha) \geq 1$. (5.4) can be rewritten as follows:

$$b_0 + b_1 e^{\alpha} + b_2 e^{2\alpha} + \cdots + b_n e^{n\alpha} = e^{\beta}, \tag{5.5}$$

where $b_0 = a_0, b_k = e^{\sum_{j=0}^{k-1} (\overline{\alpha}^j - \alpha)} (\neq 0)$ for $1 \leq k \leq n$. It follows from Lemma 2.7 that $T(r, b_k) = S(r, e^{\alpha})$ for $0 \leq k \leq n$.

Subcase 1.1: $b_0 \neq 0$.

From (5.5), we conclude that $T(r, e^{\beta}) = nT(r, e^{\alpha}) + S(r, e^{\alpha})$. Applying the second main theorem, we have

$$\begin{aligned}
nT(r, e^{\alpha}) &= T(r, b_0 + b_1 e^{\alpha} + b_2 e^{2\alpha} + \cdots + b_n e^{n\alpha}) \\
&\leq N(r, e^{\beta}) + N(r, \frac{1}{e^{\beta}}) + N(r, \frac{1}{e^{\beta} - b_0}) + S(r, e^{\beta}) \\
&\leq N\left(r, \frac{1}{e^{\alpha}(b_1 + b_2 e^{\alpha} + \cdots + b_n e^{(n-1)e^{\alpha}})}\right) + S(r, e^{\alpha}) \\
&\leq T(r, b_1 + b_2 e^{\alpha} + \cdots + b_n e^{(n-1)e^{\alpha}}) + S(r, e^{\alpha}) \\
&\leq (n-1)T(r, e^{\alpha}) + S(r, e^{\alpha}),
\end{aligned}$$

which yields a contradiction for α is not a constant.

Subcase 1.2: $b_0 = 0$.

Obviously, (5.5) can be written as follows:

$$b_1 + b_2 e^{\alpha} + \cdots + b_n e^{(n-1)\alpha} = e^{\beta - \alpha}. \tag{5.6}$$

It is not difficult to see that $\deg(\beta - \alpha) \leq \deg(\alpha)$. Therefore, if $\deg(\beta - \alpha) = \deg(\alpha)$, then $n \geq 2$. By Lemma 2.3,

$$\begin{aligned}
(n-1)T(r, e^{\alpha}) &= T(r, b_1 + b_2 e^{\alpha} + \cdots + b_n e^{(n-1)\alpha}) \\
&\leq N(r, e^{\beta - \alpha}) + N(r, \frac{1}{e^{\beta - \alpha}}) + N(r, \frac{1}{e^{\beta - \alpha} - b_1}) + S(r, e^{\alpha}) \\
&\leq N\left(r, \frac{1}{e^{\alpha}(b_2 + b_3 e^{\alpha} + \cdots + b_n e^{(n-2)e^{\alpha}})}\right) + S(r, e^{\alpha}) \\
&\leq T(r, b_2 + b_3 e^{\alpha} + \cdots + b_n e^{(n-2)e^{\alpha}}) + S(r, e^{\alpha}) \\
&\leq (n-2)T(r, e^{\alpha}) + S(r, e^{\alpha}),
\end{aligned}$$

which is impossible. Hence, $\deg(\beta - \alpha) < \deg(\alpha)$. Thus, (5.6) can be rewritten as follows:

$$b_1 - e^{\beta - \alpha} + b_2 e^{\alpha} + \cdots + b_n e^{(n-1)e^{\alpha}} = 0. \tag{5.7}$$

Applying Lemma 2.4, it can be seen that $b_1 = e^{\beta-\alpha}$, $b_k \equiv 0 (2 \le k \le n)$, which contradicts the fact that $b_k = e^{\sum_{j=0}^{k-1}(\overline{\alpha}^j - \alpha)} \not\equiv 0 (1 \le k \le n)$.

Case 2: $b(z) \not\equiv 0$.

In this case, all we need is to prove that $e^{\alpha} \equiv A$, and $A = 1$ or $\sum_{k=0}^{n} a_k A^k = 0$. From (5.1), we have

$$\begin{aligned} T(r, e^{\beta}) &= m(r, e^{\beta}) \le m\left(r, \frac{P(f)}{f-a}\right) + m\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &\le \sum_{k=0}^{n} m\left(r, \frac{\overline{f}^k - a}{f-a}\right) + T(r, f) + S(r, f) \\ &\le T(r, f) + S(r, f). \end{aligned}$$

Subcase 2.1: $T(r, e^{\beta}) = S(r, f)$.

If $e^{\beta} - a_0 - \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1}\overline{\alpha}^j} \not\equiv 0$, then from (5.3),

$$f = \frac{-b}{e^{\beta} - a_0 - \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1}\overline{\alpha}^j}} + a,$$

which implies $T(r, f) = S(r, f)$, a contradiction. Thus,

$$e^{\beta} = a_0 + \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1}\overline{\alpha}^j}.$$

It follows from (5.3) that $b(z) \equiv 0$, a contradiction.

Subcase 2.2: $T(r, e^{\beta}) \ne S(r, f)$.

If α is a constant, then we may assume that $e^{\alpha} = A (\ne 0)$. From (5.2), we have

$$\begin{aligned} P(f) &= (f-a) \sum_{k=0}^{n} a_k A^k \\ P(f(z+c)) &= (\overline{f}^1 - a) \sum_{k=0}^{n} a_k A^k. \end{aligned}$$

Together with the second equality of (5.1), we deduce

$$\begin{aligned} b &= (f-a)\left(\sum_{k=0}^{n} a_k A^k - e^{\beta}\right), \\ b &= (f-a)A\left(\sum_{k=0}^{n} a_k A^k - e^{\overline{\beta}^1}\right), \end{aligned} \tag{5.8}$$

which implies $(1-A)\sum_{k=0}^{n} a_k A^k = e^{\beta}(1 - Ae^{\overline{\beta}^1 - \beta})$. By Lemma 2.4, we have $(1-A)\sum_{k=0}^{n} a_k A^k = 0$. Hence, $A = 1$ or $\sum_{k=0}^{n} a_k A^k = 0$.

If α is not a constant, then it follows from $T(r, e^{\beta}) \neq S(r, f)$ that

$$e^{\beta} \not\equiv a_0 + \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1} \overline{\alpha}^j}.$$

Therefore, by (5.3), we have

$$f - a = \frac{-b}{e^{\beta} - a_0 - \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1} \overline{\alpha}^j}}.$$

Furthermore, by the first equality of (5.1), we have

$$e^{\alpha} = \frac{\overline{f}^1 - a}{f - a} = \frac{\frac{-b}{e^{\overline{\beta}^1} - a_0 - \sum_{k=1}^{n} a_k e^{\sum_{j=1}^{k} \overline{\alpha}^j}}}{\frac{-b}{e^{\beta} - a_0 - \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1} \overline{\alpha}^j}}}.$$

The above equality can be rewritten as follows:

$$e^{\alpha + \overline{\beta}^1} - e^{\beta} = a_0(e^{\alpha} - 1) + \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1} \overline{\alpha}^j}(e^{\overline{\alpha}^k} - 1).$$

From the right side of the above equality and $T(r, e^{\alpha}) = S(r, f)$, it can be seen that $e^{\beta}(e^{\alpha + \overline{\beta}^1 - \beta} - 1) = e^{\alpha + \overline{\beta}^1} - e^{\beta}$ is small function with respect to f. By Lemma 2.7 and $T(r, e^{\beta}) \neq S(r, f)$, we deduce that $e^{\alpha + \overline{\beta}^1 - \beta} - 1 = S(r, e^{\beta})$. If $e^{\alpha + \overline{\beta}^1 - \beta} - 1 \not\equiv 0$ holds, then

$$e^{\beta} = \frac{e^{\alpha + \overline{\beta}^1} - e^{\beta}}{e^{\alpha + \overline{\beta}^1 - \beta} - 1},$$

which implies that e^{β} is small function with respect to f, a contradiction. Thus, $e^{\alpha + \overline{\beta}^1 - \beta} - 1 \equiv 0$. Owing to (5.1), we can conclude that

$$\frac{P(f(z+c)) - b(z)}{P(f(z)) - b(z)} \equiv 1,$$

which implies $P(f(z)) = P(f(z+c))$. From (5.2) and the first equality of (5.1), we have

$$\begin{aligned} P(f(z+c)) &= \left(a_0 + \sum_{k=1}^{n} a_k e^{\sum_{j=0}^{k-1} \overline{\alpha}^{j+1}}\right)(\overline{f}^1 - a) \\ &= \left(\sum_{k=0}^{n} a_k e^{\sum_{j=0}^{k} \overline{\alpha}^j}\right)(f - a). \end{aligned}$$

Owing to $f - a \neq 0$ and $P(f(z)) = P(f(z+c))$, we have

$$\begin{aligned} a_0 &+ (a_1 - a_0)e^{\alpha} + (a_2 - a_1)e^{\alpha + \overline{\alpha}^1} + (a_3 - a_2)e^{\sum_{j=0}^{2} \overline{\alpha}^j} + \cdots \\ &+ (a_n - a_{n-1})e^{\sum_{j=0}^{n-1} \overline{\alpha}^j} - a_n e^{\sum_{j=0}^{n} \overline{\alpha}^j} = 0. \end{aligned}$$

Furthermore, the above equality can be rewritten as follows:

$$d_0 + d_1e^{\alpha} + d_2e^{2\alpha} + \cdots + d_ne^{n\alpha} + d_{n+1}e^{(n+1)\alpha} = 0, \tag{5.9}$$

where

$$\begin{aligned}
d_0 &= a_0, \\
d_1 &= a_1 - a_0, \\
d_2 &= (a_2 - a_1)e^{\overline{\alpha}^1 - \alpha}, \\
d_3 &= (a_3 - a_2)e^{\sum_{j=1}^{2}(\overline{\alpha}^j - \alpha)}, \\
&\cdots \quad \cdots \\
d_n &= (a_n - a_{n-1})e^{\sum_{j=1}^{n-1}(\overline{\alpha}^j - \alpha)}, \\
d_{n+1} &= -a_ne^{\sum_{j=1}^{n}(\overline{\alpha}^j - \alpha)}.
\end{aligned}$$

Note here that $T(r, d_k) = S(r, e^{\alpha})(0 \le k \le n+1)$. Applying Lemma 2.4 for (5.9), we have $d_k \equiv 0(0 \le k \le n+1)$. Hence, $a_0 = a_1 = \cdots = a_n = 0$, which contradicts the fact that $a_k(0 \le k \le n)$ are not all zero. That completes the proof of Theorem 1.8.

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