

## Coefficients inequalities for classes of meromorphic functions

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**Abstract:** A typical problem in the theory of analytic functions is to study a functional made up of combinations of coefficients of the original function. Usually, there is a parameter over which the extremal value of the functional is needed. One of the important functionals of this type is the Fekete–Szegő functional defined on the class of analytic functions. In this paper we transfer the Fekete–Szegő problem to some classes of meromorphic functions.

**Key words:** Meromorphic functions, Fekete–Szegő problem, subordination, Hadamard product

### 1. Introduction

Let  $\mathcal{A}$  denote the class of functions that are analytic in  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  and let  $\Omega := \{\omega \in \mathcal{A} : |\omega(z)| \leq |z| \quad (z \in \mathbb{D})\}$ . We denote by  $\Sigma$  the class of normalized meromorphic functions  $f$  of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \quad (1)$$

defined on the punctured unit disk  $\Delta := \mathbb{D} \setminus \{0\}$ .

Let  $\alpha$  be a real number,  $0 \leq \alpha \leq 1$ , and let  $\phi, \varphi, \psi, \chi \in \Sigma$ ,  $p, q \in \mathcal{A}$  be of the form

$$\begin{aligned} \varphi(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} \varphi_n z^n, & \phi(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} \phi_n z^n \quad (z \in \Delta), \\ \chi(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} \chi_n z^n, & \psi(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} \psi_n z^n \quad (z \in \Delta), \\ p(z) &= 1 + \sum_{n=1}^{\infty} p_n z^n, & q(z) &= 1 + \sum_{n=1}^{\infty} q_n z^n \quad (z \in \mathbb{D}), \end{aligned}$$

with

$$\varphi_n - \phi_n > 0, \quad \chi_n - \psi_n > 0, \quad p_1, q_1 > 0 \quad (n = 0, 1, \dots). \quad (2)$$

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By  $\Sigma_\alpha(\phi, \varphi; \psi, \chi; p)$  we denote the class of functions  $f \in \Sigma$  such that  $(\varphi * f)(z)(\chi * f)(z) \neq 0$  ( $z \in \Delta$ ) and

$$(1 - \alpha) \frac{\phi * f}{\varphi * f} + \alpha \frac{\psi * f}{\chi * f} \prec p,$$

where  $*$  denotes the Hadamard product or convolution and  $\prec$  is the symbol of subordination. Moreover, let us define

$$\Sigma^*(p) := \Sigma_0(-z\varphi'(z), \varphi; \varphi, \varphi; p), \quad \varphi(z) = \frac{1}{z} + \sum_{n=0}^{\infty} z^n.$$

The class  $\Sigma_\alpha(\phi, \varphi; \psi, \chi; p)$  is related to the classes of meromorphic Mocanu functions, meromorphic starlike functions, and meromorphic convex functions.

We denote by  $\mathcal{C}(\phi, \varphi; p, q)$  the class of functions  $f \in \Sigma$  for which there exist a function  $g \in \Sigma^*(q)$  such that  $(\varphi * g)(z) \neq 0$  ( $z \in \Delta$ ) and

$$\frac{\phi * f}{\varphi * g} \prec p.$$

The class is related to the class of meromorphic close-to-convex functions.

Motivated by Fekete and Szegő [3] (see also [1,2,4,5,6,7]), we consider the coefficient functional  $\Lambda(f) = a_1 - \mu a_0^2$ . The object of this paper is to estimate the absolute value of the functional  $\Lambda$  in the defined classes of functions. Some consequences of the main results are also considered.

## 2. The main results

The following lemma will be required in our present investigation.

**Lemma 1** [4] *Let  $\omega \in \Omega$  be a function of the form*

$$\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n \quad (z \in \mathbb{D}). \tag{3}$$

Then,

$$|\omega_n| \leq 1 \quad (n = 1, 2), \quad |\omega_2| \leq 1 - |\omega_1|^2, \quad |\omega_2 - \mu\omega_1^2| \leq \max\{1, |\mu|\} \quad (\mu \in \mathbb{C}).$$

The results are sharp with the extremal functions of the form

$$\omega(z) = z, \quad \omega_a(z) = z \frac{z+a}{1+\bar{a}z} \quad (z \in \mathbb{D}, |a| < 1).$$

**Theorem 1** *If  $f \in \Sigma_\alpha(\phi, \varphi; \psi, \chi; p)$ , then*

$$|a_0| \leq \frac{p_1}{(1 - \alpha)(\varphi_0 - \phi_0) + \alpha(\chi_0 - \psi_0)}, \tag{4}$$

$$|a_1| \leq \frac{p_1}{(1 - \alpha)(\varphi_0 - \phi_0) + \alpha(\chi_0 - \psi_0)} \max\{1, |\beta|\}, \tag{5}$$

$$|a_1 - \mu a_0^2| \leq \frac{p_1}{(1 - \alpha)(\varphi_1 - \phi_1) + \alpha(\chi_1 - \psi_1)} \max\{1, |\gamma|\} \quad (\mu \in \mathbb{C}), \tag{6}$$

where

$$\beta = \frac{p_2}{p_1} + \frac{(1 - \alpha) \varphi_0 (\phi_0 - \varphi_0) + \alpha \chi_0 (\psi_0 - \chi_0)}{[(1 - \alpha) (\phi_0 - \varphi_0) + \alpha (\psi_0 - \chi_0)]^2} p_1 \tag{7}$$

$$\gamma = \frac{(1 - \alpha) (\phi_1 - \varphi_1) + \alpha (\psi_1 - \chi_1)}{[(1 - \alpha) (\phi_0 - \varphi_0) + \alpha (\psi_0 - \chi_0)]^2} p_1 \mu - \beta. \tag{8}$$

The results are sharp.

**Proof** Let  $f \in \Sigma_\alpha (\phi, \varphi; \psi, \chi; p)$ . Then there exists a function  $\omega \in \Omega$  of the form (3) such that

$$(1 - \alpha) \frac{\phi * f}{\varphi * f} + \alpha \frac{\psi * f}{\chi * f} = p \circ \omega. \tag{9}$$

It follows easily that

$$(p \circ \omega)(z) = 1 + (p_1 \omega_1) z + (p_1 \omega_2 + p_2 \omega_1^2) z^2 + \dots \quad (z \in \mathbb{D}). \tag{10}$$

Moreover, if we put

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} = 1 + \sum_{n=1}^{\infty} x_n z^n, \quad \frac{(\psi * f)(z)}{(\chi * f)(z)} = 1 + \sum_{n=1}^{\infty} y_n z^n \quad (z \in \Delta),$$

and

$$(1 - \alpha) \frac{(\phi * f)(z)}{(\varphi * f)(z)} + \alpha \frac{(\psi * f)(z)}{(\chi * f)(z)} = 1 + w_1 z + w_2 z^2 + \dots \quad (z \in \Delta), \tag{11}$$

then

$$x_1 = (\phi_0 - \varphi_0) a_0, \quad x_n = (\phi_{n-1} - \varphi_{n-1}) a_{n-1} - \sum_{k=1}^{n-1} x_{n-k} \varphi_k a_k \quad (n = 2, 3, \dots),$$

$$y_1 = (\psi_0 - \chi_0) a_0, \quad y_n = (\psi_{n-1} - \chi_{n-1}) a_{n-1} - \sum_{k=1}^{n-1} y_{n-k} \chi_k a_k \quad (n = 2, 3, \dots),$$

and

$$w_1 = [(1 - \alpha) (\phi_0 - \varphi_0) + \alpha (\psi_0 - \chi_0)] a_0,$$

$$w_n = [(1 - \alpha) (\phi_{n-1} - \varphi_{n-1}) + \alpha (\psi_{n-1} - \chi_{n-1})] a_{n-1}$$

$$- \sum_{k=1}^{n-1} [(1 - \alpha) x_{n-k} \varphi_k + \alpha y_{n-k} \chi_k] a_k \quad (n = 2, 3, \dots). \tag{12}$$

Therefore, by (9)–(12) we obtain

$$a_0 = \frac{p_1 \omega_1}{(1 - \alpha) (\phi_0 - \varphi_0) + \alpha (\psi_0 - \chi_0)}, \tag{13}$$

$$a_1 = \frac{p_1 \omega_2 + p_2 \omega_1^2 + [(1 - \alpha) \varphi_0 (\phi_0 - \varphi_0) + \alpha \chi_2 (\psi_0 - \chi_0)] a_0}{(1 - \alpha) (\phi_1 - \varphi_1) + \alpha (\psi_1 - \chi_1)}. \tag{14}$$

Thus, by Lemma 1 we have sharp estimation (4). Moreover, by (13) and (14) we get

$$a_1 - \mu a_0^2 = \frac{p_1}{(1 - \alpha)(\phi_1 - \varphi_1) + \alpha(\psi_1 - \chi_1)} \{\omega_2 - \gamma\omega_1^2\},$$

where  $\gamma$  is given by (8). Hence, by Lemma 1 we obtain (6). In particular, taking  $\mu = 0$  in (6) we obtain the estimation (5). To prove that the results are sharp let us define the functions  $f_1, f_2 \in \Sigma$  such that

$$(1 - \alpha) \frac{(\phi * f_1)(z)}{(\varphi * f_1)(z)} + \alpha \frac{(\psi * f_1)(z)}{(\chi * f_1)(z)} = p(z) \quad (z \in \Delta),$$

$$(1 - \alpha) \frac{(\phi * f_2)(z)}{(\varphi * f_2)(z)} + \alpha \frac{(\psi * f_2)(z)}{(\chi * f_2)(z)} = p(z^2) \quad (z \in \Delta).$$

The functions exist since we can obtain their coefficients from (12). In particular, the coefficients  $a_0$  and  $a_1$  are given by (13) and (14). It is clear that  $f_1, f_2 \in \Sigma_\alpha(\phi, \varphi; \psi, \chi; p)$  and the functions  $f_1, f_2$  realize the equality in the estimation (6). Therefore, the obtained results are sharp.  $\square$

**Theorem 2** Assume that  $\phi_0\phi_1 \neq 0$ . If a function  $f \in \mathcal{C}(\phi, \varphi; p, q)$  is of the form (1), then

$$|a_1 - \mu a_0^2| \leq \frac{1}{2|\phi_1|} [Z + \max\{0, U_\mu\} + \max\{0, V_\mu\}], \tag{15}$$

$$|a_1| \leq \frac{1}{2|\phi_1|} [Z + \max\{0, U_0\} + \max\{0, V_0\}], \tag{16}$$

$$|a_0| = \frac{p_1 + q_1|\varphi_0|}{|\phi_0|}, \tag{17}$$

where

$$U_\mu = \left| \varphi_1(q_2 + q_1^2) + 2\mu \frac{\varphi_0^2 \phi_1 q_1^2}{\phi_0} \right| + W_\mu - |\varphi_1||q_1|, \quad Z = 2|p_1| + |\varphi_1||q_1|, \tag{18}$$

$$V_\mu = 2 \left| p_2 - \mu \frac{\phi_1 p_1^2}{\phi_0^2} \right| + W_\mu - 2|p_1|, \quad W_\mu = |\varphi_0||p_1||q_1| \left| 1 - 2\mu \frac{\phi_1}{\phi_0^2} \right|. \tag{19}$$

The results (15) and (16) are sharp for  $U_\mu V_\mu \geq 0$  and  $U_0 V_0 \geq 0$ , respectively.

**Proof** Let  $f \in \mathcal{C}(\phi, \varphi; p, q)$ . Then there exists a function  $g \in \Sigma^*(q)$  and functions  $\omega, \eta \in \Omega$ ,

$$\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n, \quad \eta(z) = \sum_{n=1}^{\infty} \eta_n z^n \quad (z \in \mathbb{D}),$$

such that

$$\frac{\phi * f}{\varphi * g} = p \circ \omega, \quad \frac{zg'(z)}{g(z)} = -(q \circ \eta)(z) \quad (z \in \Delta). \tag{20}$$

Thus, by (10) we have

$$b_0 = -q_1\eta_1, \quad 2b_1 = q_1\eta_2 - (q_2 - q_1^2)\eta_1^2, \quad a_0 = \frac{\omega_1 p_1 - \varphi_0 \eta_1 q_1}{\phi_0}, \tag{21}$$

$$2\varphi_1 a_1 = -\varphi_1 \{q_1\eta_2 + (q_2 - q_1^2)\eta_1^2\} - 2\varphi_0 p_1 q_1 \omega_1 \eta_1 + 2p_1 \omega_2 + 2p_2 \omega_1^2; \tag{22}$$

by Lemma 1 we obtain the sharp estimation (17). Let  $\mu$  be a complex number. Then, by (21) and (22), we have

$$2\varphi_1 |a_1 - \mu a_0^2| \leq |\varphi_1 q_1| |\eta_2| + 2|p_1| |\omega_2| + \left| \varphi_1 (q_2 - q_1^2) + 2\mu \frac{\varphi_0^2 \phi_1 q_1^2}{\phi_0^2} \right| |\eta_1|^2 + 2 \left| p_2 - \mu \frac{\phi_1 p_1^2}{\phi_0^2} \right| |\omega_1|^2 + 2|\varphi_0 p_1 q_1| \left| 1 - 2\mu \frac{\phi_1}{\phi_0^2} \right| |\omega_1| |\eta_1|.$$

Therefore, by Lemma 1 we get

$$2|\phi_1| |a_1 - \mu a_0^2| \leq (U - W) |\eta_1|^2 + (V - W) |\omega_1|^2 + 2W |\eta_1| |\omega_1| + Z \tag{23}$$

or equivalently

$$2|\phi_1| |a_1 - \mu a_0^2| \leq U |\eta_1|^2 + V |\omega_1|^2 - W (|\eta_1| - |\omega_1|)^2 + Z, \tag{24}$$

where  $U = U_\mu, V = V_\mu, W = W_\mu, Z$  are defined by (18) and (19). Thus, we obtain

$$2|\phi_1| |a_1 - \mu a_0^2| \leq U |\eta_1|^2 + V |\omega_1|^2 + Z, \tag{25}$$

and, in consequence, by Lemma 1 we have (15). It is easy to verify that the equality in (25) is attained by choosing  $\omega_1 = \eta_1 = 1, \omega_2 = \eta_2 = 0$  if  $U \geq 0, V \geq 0$  or  $\omega_1 = \eta_1 = 0, \omega_2 = \eta_2 = 1$  if  $U \leq 0, V \leq 0$ . Therefore, we consider functions  $f_1, f_2 \in \Sigma$  such that

$$\frac{(\phi * f_1)(z)}{(\varphi * g)(z)} = p(z), \quad \frac{(\phi * f_2)(z)}{(\varphi * g)(z)} = p(z^2) \quad (z \in \Delta).$$

Then the functions belong to the class  $\mathcal{C}(\phi, \varphi; p)$  and they realize the equality in the estimation (15) for  $UV \geq 0$ . Putting  $\mu = 0$  in (15) we get the sharp estimation (16).  $\square$

The following theorem gives the complete sharp estimation of the Fekete–Szegő functional in the class  $\mathcal{C}(\phi, \varphi; p, q)$ .

**Theorem 3** Assume that  $\phi_0 \phi_1 \neq 0$ . If a function  $f \in \mathcal{C}(\phi, \varphi; p, q)$  is of the form (1), then

$$|a_1 - \mu a_0^2| \leq \begin{cases} \frac{1}{2|\phi_1|} (U + V + Z) & \text{if } \begin{matrix} (-U \leq V \leq W, U \leq W) \\ \text{or } (U \geq 0, V \geq 0), \end{matrix} \\ \frac{Z}{2|\phi_1|} & \text{if } U \leq W, V \leq W, U + V \leq 0, \\ \frac{1}{2|\phi_1|} \left( Z + V - W + \frac{W^2}{W-U} \right) & \text{if } U < 0, V \geq W, \\ \frac{1}{2|\phi_1|} \left( Z + U - W + \frac{W^2}{W-V} \right) & \text{if } V < 0, U \geq W, \end{cases} \tag{26}$$

where  $U = U_\mu, V = V_\mu, W = W_\mu, Z$  are given by (18) and (19). The result is sharp.

**Proof** If  $UV \geq 0$ , then the result follows from Theorem 2. Let  $U \geq W, V < 0$  and let

$$h(x) := -(W - V)x^2 + 2Wx + (U - W) + Z.$$

Then, by (23) and Lemma 1, we obtain

$$2|\phi_1| |a_1 - \mu a_0^2| \leq h(|\omega_1|).$$

An easy computation shows that the function  $h$  has a maximum in  $[0, 1]$  for  $x = \frac{W}{W-V} \leq 1$ . Thus, we get (26) for  $U \geq W$  and  $V < 0$ . If we define the functions  $\eta(z) = z$ ,  $\omega(z) = z \frac{z+a}{1+az}$ , for  $a = \frac{W}{W-V}$ , then  $\omega_1 = \frac{W}{W-V}$ ,  $\eta_1 = 1$ , and  $\omega_2 = 1 - |a|^2$ ,  $\eta_2 = 0$ . Therefore, we have the equality in (23) and, in consequence, the result is sharp.

Now, let  $U < 0$ ,  $W \leq V$  and  $\tilde{h}(x) := -(W - U)x^2 + 2Wx + (V - W) + Z$ . Then, by (23) and Lemma 1, we obtain

$$2|\phi_1| |a_1 - \mu a_0^2| \leq \tilde{h}(|\eta_1|).$$

An easy computation shows that the function  $\tilde{h}$  has a maximum in  $[0, 1]$  for  $x = \frac{W}{W-U} \leq 1$ . Thus, we get (26) for  $U < 0$  and  $W \leq V$ . If we define the functions  $\omega(z) = z$ ,  $\eta(z) = z \frac{z+a}{1+az}$ , for  $a = \frac{W}{W-U}$ , i.e.  $\omega_1 = 1$ ,  $\eta_1 = \frac{W}{W-U}$ , and  $\omega_2 = 0$ ,  $\eta_2 = 1 - |a|^2$ , then we obtain the equality in (23) and the result is sharp.

In the last step, let us assume  $(0 \leq U \leq W, V \leq 0)$  or  $(0 \leq V \leq W, U \leq 0)$  and let us define

$$G(x, y) = -(W - U)x^2 - (W - V)y^2 + 2Wxy + Z.$$

Then, by (23), we have

$$2|\phi_1| |a_1 - \mu a_0^2| \leq G(|\omega_1|, |\eta_1|). \tag{27}$$

It is clear that  $G$  is the continuous function on  $B := [0, 1] \times [0, 1]$ . Therefore, if we denote by  $P$  the set of critical points of the function  $G$  in  $B$ , then by (27) we get

$$2|\phi_1| |a_1 - \mu a_0^2| \leq \max G(B) = \max G(\partial B \cup P). \tag{28}$$

An easy computation shows that

$$P \setminus \partial B = \begin{cases} \emptyset & \text{if } W^2 \neq (W - U)(W - V) \\ & \text{or } U = W, \\ \{(x, y) \in \text{int}B : x = \frac{W}{W-U}y\} & \text{if } W^2 = (W - U)(W - V) \\ & \text{and } U \neq W. \end{cases}$$

If we assume  $W^2 = (U - W)(V - W) \neq 0$ , then we have

$$G\left(\frac{W}{W-U}y, y\right) = \frac{W^2 - (W - U)(W - V)}{U - W}y^2 + Z = Z \quad (y \in [0, 1]).$$

Moreover, we obtain

$$G(x, 0) = -(W - U)x^2 + Z \leq Z \quad (x, y \in [0, 1]),$$

$$G(0, y) = -(W - V)y^2 + Z \leq Z \quad (x, y \in [0, 1]),$$

$$G(x, 1) = -(W - U)x^2 + 2Cx + V - W + Z \leq G(1, 1) = U + V + Z \quad (x \in [0, 1]),$$

$$G(1, y) = -(W - V)y^2 + 2Cy + U - W + Z \leq G(1, 1) = U + V + Z \quad (y \in [0, 1]).$$

Hence, we get

$$\max G(\partial B \cup P) = \begin{cases} U + V + Z & \text{if } U + V \geq 0 \\ Z & \text{if } U + V \leq 0 \end{cases},$$

and by (28) we obtain (26) for  $(0 \leq U \leq W, V \leq 0)$  or  $(0 \leq V \leq W, U \leq 0)$ . Moreover, if we put  $\omega_1 = \eta_1 = 1$ ,  $\omega_2 = \eta_2 = 0$  for  $U + V \geq 0$  or  $\omega_1 = \eta_1 = 0$ ,  $\omega_2 = \eta_2 = 1$  for  $U + V \leq 0$ , then we get the equality in (27). Therefore, the result is sharp and the proof is completed.  $\square$

**Remark 1** *By choosing parameters in the defined classes of functions we can obtain several additional results. Some of these results were obtained in earlier works; see, for example, [8, 9].*

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