Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2018) 42: 2513 – 2526
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doi:10.3906/mat-1805-50

Green's relations and regularity on some subsemigroups of transformations that preserve equivalences

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Received: 11.05.2018 • **Accepted/Published Online:** 16.07.2018 • **Final Version:** 27.09.2018

Abstract: Let $T(X)$ be the full transformation semigroup on a set X. For two equivalence relations E and F on X with $F \subseteq E$, let

$$T(X,E,F) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in E \Rightarrow (x\alpha, y\alpha) \in F\}.$$

Then $T(X,E,F)$ is a subsemigroup of $T(X)$. In this paper, we describe Green's relations and the regularity of elements for $T(X,E,F)$. Also, the relations F and E for which $T(X,E,F)$ is a regular semigroup are described.

Key words: Transformation semigroup, equivalence, Green s relations, regular.

1. Introduction

In 1951, Green defined the equivalence relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$ on a semigroup S by the rules that, for $a, b \in S$,

$(a,b) \in \mathcal{L}$ if and only if $S^1a = S^1b$,

$(a,b) \in \mathcal{R}$ if and only if $aS^1 = bS^1$, and

$(a,b) \in \mathcal{J}$ if and only if $S^1aS^1 = S^1bS^1$

where S^1 is the semigroup with identity obtained from S by adjoining an identity if necessary. Then he also defined the equivalence relations $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. These five equivalence relations are known as *Green's relations*: see the book by Howie [4].

An element x of a semigroup S is called a *regular element* if there exists $y \in S$ such that $x = xyx$, and S is called a *regular semigroup* if every element of S is regular.

Let X be a nonempty set. As usual, $T(X)$ denotes the semigroup (under composition) of all full transformations of X (that is, all mappings $\alpha : X \to X$). It is a well-known fact that $T(X)$ is a regular semigroup (see [3]) and every semigroup is isomorphic to a subsemigroup of some full transformation semigroup (see [4]). Hence, in order to study the structure of semigroups, it suffices to consider some subsemigroups of $T(X)$. Therefore, several researchers are interested in characterization of subsemigroups of the full transformation semigroup. Particularly, characterization of regularity and Green's relations on subsemigroups of $T(X)$ have been investigated. See [1, 2, 5–11].

Let E be an equivalence relation on X. Recently, Pei [6] introduced a family of subsemigroups of $T(X)$

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2010 *AMS Mathematics Subject Classification:* 20M20

defined by

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}$$

and called it the *semigroup of transformation preserving an equivalence relation* on X. It is easy to see that if $E = X \times X$ or $E = I_X = \{(x, x) : x \in X\}$, then $T_E(X)$ is equal to $T(X)$. The author studied Green's relations and regularity on $T_E(X)$.

Suppose that E and F are equivalence relations on X with $F \subseteq E$. Sun and Pei [11] studied the subsemigroup of $T(X)$ defined by

$$T_{EF}(X) = T_E(X) \cap T_F(X).$$

They described the condition under which elements of $T_{EF}(X)$ are regular and discussed Green's relations on $T_{EF}(X)$.

The semigroup $T_E(X)$ motivates us to define $T(X, E, F)$ as follows:

$$T(X, E, F) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in F\},$$

where E and F are equivalence relations on X with $F \subseteq E$. It is easy to see that $T(X, E, F)$ is a subsemigroup of $T(X)$ and that $T(X, E, F) \subseteq T_{EF}(X) \subseteq T_E(X) \subseteq T(X)$.

The purpose of this paper is to investigate the regularity of elements and Green's relations for the semigroup $T(X, E, F)$. Accordingly, in Section 2, the condition under which elements of $T(X, E, F)$ are regular is analyzed. In Section 3, Green's relations on $T(X, E, F)$ are described.

In the remainder of this paper, let E and F be equivalence relations on a set X such that $F \subseteq E$.

2. Regularity of $T(X, E, F)$

For $\alpha \in T(X)$, the symbol $\pi(\alpha)$ will denote the decomposition of X induced by the map α, namely

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\}.$$

Hence, $\pi(\alpha) = X/\ker\alpha$ where $\ker\alpha = \{(x, y) \in X \times X : x\alpha = y\alpha\}$. Denote

$$E(\alpha) = \{A\alpha^{-1} : A \in X/E, A\alpha^{-1} \neq \emptyset\},$$

where E is an equivalence relation on X. Then $E(\alpha)$ is a partition of X.

Lemma 2.1 *Let $\alpha \in T(X, E, F)$. For each $A \in X/E$, there exists $B \in X/F$ such that $A\alpha \subseteq B$.*

Proof Let $A \in X/E$ and $a \in A$. Then there exists $B \in X/F$ such that $a\alpha \in B$. Let $y \in A\alpha$. Then $x\alpha = y$ for some $x \in A$. Since $(a, x) \in E$ and $\alpha \in T(X, E, F)$, we have $(a\alpha, y) = (a\alpha, x\alpha) \in F$. This means that $y \in B$. □

Since $F \subseteq E$ and by Lemma 2.1, we certainly have the following corollary.

Corollary 2.2 *Let $\alpha \in T(X, E, F)$. Then the following statements hold.*

(i) For each $A \in X/F$, there exists $B \in X/F$ such that $A\alpha \subseteq B$.

(ii) For each $A \in X/E$, there exists $B \in X/E$ such that $A\alpha \subseteq B$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of a set X. If for every $P \in \mathcal{P}$, there exists $Q \in \mathcal{Q}$ such that $P \subseteq Q$, we write $\mathcal{P} \preceq \mathcal{Q}$. It is obvious that $\preceq$ is a partial order on the set of all partitions of X.

Proposition 2.3 *Let $\alpha, \beta, \gamma \in T(X,E,F)$ be such that $\alpha = \beta\gamma$. Then $\pi(\beta) \preceq \pi(\alpha)$, $F(\beta) \preceq F(\alpha)$, and $E(\beta) \preceq E(\alpha)$.*

Proof (i) Let $A \in \pi(\beta)$. Then $A = y\beta^{-1}$ for some $y \in X\beta$. Thus, $A\alpha = A\beta\gamma = y\gamma$ and so $A \subseteq (A\alpha)\alpha^{-1} \subseteq (y\gamma)\alpha^{-1}$. Since $(y\gamma)\alpha^{-1} \in \pi(\alpha)$, we conclude that $\pi(\beta) \preceq \pi(\alpha)$.

(ii) Let $A \in F(\beta)$. Then $A = B\beta^{-1}$ for some $B \in X/F$ with $B\beta^{-1} \neq \emptyset$ and so $A\beta \subseteq B$. By Corollary 2.2(i), we have $B\gamma \subseteq C$ for some $C \in X/F$. Therefore, $A\alpha = A\beta\gamma \subseteq B\gamma \subseteq C$, so that $A \subseteq (A\alpha)\alpha^{-1} \subseteq C\alpha^{-1}$. Since $A \neq \emptyset$ and $C \in X/F$, $C\alpha^{-1} \in F(\alpha)$. Hence, $F(\beta) \preceq F(\alpha)$.

(iii) Similar to the proof of (ii). □

Proposition 2.4 *Let $\alpha \in T(X,E,F)$. Then the following statements hold.*

(i) If $A \cap X\alpha = B\alpha$ for some $A, B \in X/F$, then $A\alpha^{-1} = \bigcup\{y\alpha^{-1} : y \in X, y\alpha^{-1} \cap B \neq \emptyset\}$.

(ii) If $A \cap X\alpha = B\alpha$ for some $A, B \in X/E$, then $A\alpha^{-1} = \bigcup\{y\alpha^{-1} : y \in X, y\alpha^{-1} \cap B \neq \emptyset\}$.

Proof (i) Suppose that $A \cap X\alpha = B\alpha$ for some $A, B \in X/F$. Let $x \in A\alpha^{-1}$. Then $x\alpha \in A$ and so $x\alpha \in B\alpha$. Thus, $x\alpha = b\alpha$ for some $b \in B$. Therefore, $b \in (x\alpha)\alpha^{-1}$, which implies that $(x\alpha)\alpha^{-1} \cap B \neq \emptyset$ and hence

$$x \in (x\alpha)\alpha^{-1} \subseteq \bigcup\{y\alpha^{-1} : y \in X, y\alpha^{-1} \cap B \neq \emptyset\}.$$

For the reverse inclusion, let $x \in \bigcup\{y\alpha^{-1} : y \in X, y\alpha^{-1} \cap B \neq \emptyset\}$. Then $x \in y\alpha^{-1}$ for some $y \in X$ with $y\alpha^{-1} \cap B \neq \emptyset$. Thus, $x\alpha = y = b\alpha$ for some $b \in y\alpha^{-1} \cap B$. Since $b\alpha \in B\alpha = A \cap X\alpha$, $x\alpha = b\alpha \in A$. Therefore, $x \in (x\alpha)\alpha^{-1} \subseteq A\alpha^{-1}$.

(ii) Similar to the proof of (i). □

Proposition 2.5 *Let $\alpha \in T(X,E,F)$. Then α is a right zero element of $T(X,E,F)$ if and only if α is constant.*

Proof Suppose that α is nonconstant. Then there exist distinct elements $a, b \in X\alpha$. Thus, $a'\alpha = a$ and $b'\alpha = b$ for some $a', b' \in X$. Thus $b' \in B$ for some $B \in X/E$. Define $\beta \in T(X)$ by

$$x\beta = \begin{cases} a' & \text{if } x \in B, \\ b' & \text{otherwise.} \end{cases}$$

It is clear that $\beta \in T(X,E,F)$. Since $b'\beta\alpha = a'\alpha = a \neq b = b'\alpha$, we conclude that $\beta\alpha \neq \alpha$. This proves that α is not a right zero element of $T(X,E,F)$. □

As a consequence of Proposition 2.5, a necessary and sufficient condition for being a right zero semigroup can be given as follows.

Corollary 2.6 *$T(X,E,F)$ is a right zero semigroup if and only if $E = X \times X$ and $F = I_X$.*

Proof We will prove the contrapositive of this statement. We can consider two cases as follows.

Case 1. $E \neq X \times X$. Then there exist $A, B \in X/E$ such that $A \neq B$. Let $a \in A$ and $b \in B$. Define $\alpha \in T(X)$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ b & \text{otherwise.} \end{cases}$$

Certainly, $\alpha \in T(X, E, F)$ and α is nonconstant. By Proposition 2.5, we obtain that α is not a right zero element of $T(X, E, F)$.

Case 2. $F \neq I_X$. Then there exist distinct elements $c, d \in X$ such that $(c, d) \in F$. Define $\alpha \in T(X)$ by

$$x\alpha = \begin{cases} c & \text{if } x = c, \\ d & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T(X, E, F)$ and α is nonconstant. It then follows from Proposition 2.5 that α is not a right zero element of $T(X, E, F)$.

From the two cases we conclude that $T(X, E, F)$ is not a right zero semigroup.

The converse is clear. □

In fact, the following example shows that $T(X, E, F)$ is not necessarily regular.

Example 2.7 *Let* $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $X/E = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$, *and* $X/F = \{\{1, 2\}, \{3\}, \{4, 5\}, \{6, 8\}, \{7\}\}$. *Let* $\alpha \in T(X, E, F)$ *be defined by*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 6 & 3 & 3 & 2 & 1 & 2 \end{pmatrix}.$$

Suppose that α *is regular. Then* $\alpha = \alpha\beta\alpha$ *for some* $\beta \in T(X, E, F)$. *Since* $1 = 7\alpha = 7\alpha\beta\alpha = 1\beta\alpha$ *and* $3 = 4\alpha = 4\alpha\beta\alpha = 3\beta\alpha$, *we obtain that* $1\beta = 7$ *and* $3\beta \in \{4, 5\}$. *Since* $(1, 3) \in E$ *and* $\beta \in T(X, E, F)$, $(1\beta, 3\beta) \in F$, *which is a contradiction. Hence,* α *is not a regular element of* $T(X, E, F)$.

Next, we give a characterization of regular elements in $T(X, E, F)$.

Theorem 2.8 *Let* $\alpha \in T(X, E, F)$. *Then* α *is regular if and only if for each* $A \in X/E$, *there exists* $B \in X/F$ *such that* $A \cap X\alpha \subseteq B\alpha$.

Proof Suppose that α is a regular element of $T(X, E, F)$. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, E, F)$. Let $A \in X/E$. By Lemma 2.1, $A\beta \subseteq B$ for some $B \in X/F$. Let $y \in A \cap X\alpha$. Then $y = x\alpha$ for some $x \in X$ and hence $y\beta \in A\beta \subseteq B$. It then follows that $y = x\alpha = x\alpha\beta\alpha = y\beta\alpha \in B\alpha$. Hence, $A \cap X\alpha \subseteq B\alpha$.

Conversely, assume that for each $A \in X/E$, there exists $B \in X/F$ such that $A \cap X\alpha \subseteq B\alpha$. Let $A \in X/E$ be such that $A \cap X\alpha \neq \emptyset$. By assumption, we choose and fix $B_A \in X/F$ with $A \cap X\alpha \subseteq B_A\alpha$. For each $y \in A \cap X\alpha$, we choose $a_y \in B_A$ such that $y = a_y\alpha$. Let $b_A \in B_A$. Define $\beta_A : A \to X$ by

$$x\beta_A = \begin{cases} a_x & \text{if } x \in X\alpha, \\ b_A & \text{otherwise.} \end{cases}$$

Let $\beta : X \to X$ be defined by

$$\beta|_A = \begin{cases} \beta_A & \text{if } A \cap X\alpha \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/E$ and C_A is a constant map from A into X. Then $\beta \in T(X)$. Let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$ and, by assumption, there is $B_A \in X/F$ such that $A \cap X\alpha \subseteq B_A\alpha$. We consider two cases as follows.

Case 1. $A \cap X\alpha = \emptyset$. Then

$$(x\beta, y\beta) = (xC_A, yC_A) \in F,$$

by reflexivity of F.

Case 2. $A \cap X\alpha \neq \emptyset$. Then there are three cases to consider.

If $x, y \in X\alpha$, then $a_x, a_y \in B_A$ and so $(x\beta, y\beta) = (a_x, a_y) \in F$.

If $x, y \notin X\alpha$, then $(x\beta, y\beta) = (b_A, b_A) \in F$.

If $x \in X\alpha$ and $y \notin X\alpha$, then $a_x, b_A \in B_A$ and so

$$(x\beta, y\beta) = (a_x, b_A) \in F.$$

From the two cases, we have $\beta \in T(X, E, F)$, and $x\alpha\beta\alpha = a_{x\alpha}\alpha = x\alpha$ for all $x \in X$. This shows that α is a regular element of $T(X, E, F)$ as desired. □

From Example 2.7, let $A = \{1, 2, 3\} \in X/E$. Then $A \cap X\alpha \nsubseteq B\alpha$ for all $B \in X/F$. By Theorem 2.8, we have that α is not a regular element of $T(X, E, F)$.

Note that $F \subseteq E$; it follows from Theorem 2.8 and we obtain a corollary as follows.

Corollary 2.9 *Let α be a regular element of $T(X, E, F)$. Then the following statements hold.*

(i) For each $A \in X/F$, there exists $B \in X/F$ such that $A \cap X\alpha \subseteq B\alpha$.

(ii) For each $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha \subseteq B\alpha$.

We also have the following theorem, which characterizes when $T(X, E, F)$ is a regular semigroup.

Theorem 2.10 *$T(X, E, F)$ is a regular semigroup if and only if $T(X, E, F) = T(X)$ or $T(X, E, F)$ is a right zero semigroup.*

Proof Assume that $T(X, E, F) \neq T(X)$ and $T(X, E, F)$ is not a right zero semigroup. Since $T(X, E, F) \neq T(X)$, $E \neq I_X$ and $F \neq X \times X$. By Corollary 2.6, we obtain $E \neq X \times X$ or $F \neq I_X$. We distinguish two cases as follows.

Case 1. $E \neq X \times X$. Since $E \neq I_X$, there exist distinct elements $a, b \in X$ such that $(a, b) \in E$. Then $a, b \in A$ for some $A \in X/E$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ b & \text{otherwise.} \end{cases}$$

Obviously, $\alpha \in T(X, E, F)$. Suppose that α is regular. By Theorem 2.8, there exists $B \in X/F$ such that $A \cap X\alpha \subseteq B\alpha$. Since $E \neq X \times X$ and $a, b \in A$, it follows that $A \cap X\alpha = \{a, b\}$. Thus, $x\alpha = a$ and $y\alpha = b$ for some $x, y \in B$. By the definition of α, we get that $x \in A$ and $y \in X \setminus A$. These imply that $B \cap A \neq \emptyset$ and $B \cap (X \setminus A) \neq \emptyset$, a contradiction. Thereby, α is not a regular element of $T(X, E, F)$.

Case 2. $F \neq I_X$. Then there exist distinct elements $c, d \in X$ such that $(c, d) \in F$. Then $c, d \in A$ for some $A \in X/F$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} c & \text{if } x \in A, \\ d & \text{otherwise.} \end{cases}$$

Since $(c, d) \in F$, $\alpha \in T(X, E, F)$. Suppose that α is regular. By Corollary 2.9(i), there exists $B \in X/F$ such that $A \cap X\alpha \subseteq B\alpha$. Since $F \neq X \times X$ and $c, d \in A$, we get that $A \cap X\alpha = \{c, d\}$. Thus, $x\alpha = c$ and $y\alpha = d$ for some $x, y \in B$. Therefore, $x \in A$ and $y \in X \setminus A$, which implies that $B \cap A \neq \emptyset$ and $B \cap (X \setminus A) \neq \emptyset$. This is a contradiction. Hence, α is not a regular element of $T(X, E, F)$.

The converse is clear. □

Next, we observe three properties for regular elements of the semigroup $T(X, E, F)$.

Proposition 2.11 *Let α be a regular element of $T(X, E, F)$. Then the following statements hold.*

(i) If $\emptyset \neq A \cap X\alpha \subseteq B\alpha$ for some $A, B \in X/F$, then $A \cap X\alpha = B\alpha$.

(ii) If $\emptyset \neq A \cap X\alpha \subseteq B\alpha$ for some $A, B \in X/E$, then $A \cap X\alpha = B\alpha$.

Proof (i) Suppose that $\emptyset \neq A \cap X\alpha \subseteq B\alpha$ for some $A, B \in X/F$. By Corollary 2.2(i), $B\alpha \subseteq C$ for some $C \in X/F$. This implies that

$$A\alpha^{-1} = A\alpha^{-1} \cap X = (A \cap X\alpha)\alpha^{-1} \subseteq (B\alpha)\alpha^{-1} \subseteq C\alpha^{-1}.$$

Since $F(\alpha)$ is a partition of X, we get that $A\alpha^{-1} = C\alpha^{-1}$ and so $A = C$. It follows that $B\alpha \subseteq A \cap X\alpha$. Hence, $A \cap X\alpha = B\alpha$.

(ii) The proof is similar to the proof of (i). □

Proposition 2.12 *Let α and β be regular elements of $T(X, E, F)$. If $\pi(\alpha) = \pi(\beta)$, then $F(\alpha) = F(\beta)$ and $E(\alpha) = E(\beta)$.*

Proof Suppose that $\pi(\alpha) = \pi(\beta)$. Let $A \in X/F$ be such that $A\alpha^{-1} \neq \emptyset$. By Corollary 2.9(i), $\emptyset \neq A \cap X\alpha \subseteq B\alpha$ for some $B \in X/F$. It follows from Propositions 2.11(i) and 2.4(i) that $A\alpha^{-1} = \bigcup\{y\alpha^{-1} : y \in X, y\alpha^{-1} \cap B \neq \emptyset\}$. By assumption, we obtain that

$$A\alpha^{-1} = \bigcup\{y\alpha^{-1} : y \in X, y\alpha^{-1} \cap B \neq \emptyset\} = \bigcup\{z\beta^{-1} : z \in X, z\beta^{-1} \cap B \neq \emptyset\}.$$

For each $x \in A\alpha^{-1}$ we have $x \in y\beta^{-1}$ for some $y \in X$ with $y\beta^{-1} \cap B \neq \emptyset$. Then there is $b \in B$ such that $x\beta = y = b\beta$. Thus, $x\beta \in B\beta$ and therefore $(A\alpha^{-1})\beta \subseteq B\beta$. Corollary 2.2(i) implies that $B\beta \subseteq D$ for some $D \in X/F$. This implies that $A\alpha^{-1} \subseteq (A\alpha^{-1})\beta\beta^{-1} \subseteq D\beta^{-1} \in F(\beta)$. Therefore, $F(\alpha) \preceq F(\beta)$. Similarly, $F(\beta) \preceq F(\alpha)$. Hence, $F(\alpha) = F(\beta)$.

Similarly, $E(\alpha) = E(\beta)$. □

Proposition 2.13 *Let α and β be regular elements of $T(X, E, F)$. If $X\alpha = X\beta$, then for each $A \in X/E$, there exist $B, C \in X/F$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.*

Proof Suppose that $X\alpha = X\beta$. Let $A \in X/E$. We can see from Lemma 2.1 that $A\alpha \subseteq B$ for some $B \in X/F$. Regularity of β and Corollary 2.9(i) yield $B \cap X\beta \subseteq B'\beta$ for some $B' \in X/F$. It is evident that

$$A\alpha \subseteq B \cap X\alpha = B \cap X\beta \subseteq B'\beta.$$

Similarly, it can be shown that $A\beta \subseteq C\alpha$ for some $C \in X/F$. □

As a consequence of Proposition 2.13, the following result follows readily.

Corollary 2.14 *Let α and β be regular elements of $T(X,E,F)$ such that $X\alpha = X\beta$. Then the following statements hold.*

(i) For each $A \in X/F$, there exist $B, C \in X/F$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(ii) For each $A \in X/E$, there exist $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

3. Green's relations on $T(X,E,F)$

In this section, we describe Green's relations on $T(X,E,F)$. Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, we only consider the Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$, and $\mathcal{D}$ in the following.

Next, we introduce the following terminology. For $\alpha \in T(X)$ and $A \subseteq X$, we denote

$$\pi_A(\alpha) = \{P \in \pi(\alpha) : P \cap A \neq \emptyset\}.$$

Theorem 3.1 [4] *Let a and b be elements of a semigroup S. Then the following statements hold.*

(i) $(a,b) \in \mathcal{R}$ if and only if there exist $x, y \in S^1$ such that $a = bx$ and $b = ay$.

(ii) $(a,b) \in \mathcal{L}$ if and only if there exist $x, y \in S^1$ such that $a = xb$ and $b = ya$.

(iii) $(a,b) \in \mathcal{J}$ if and only if there exist $w, x, y, z \in S^1$ such that $a = wbx$ and $b = yaz$.

Lemma 3.2 *Let $\alpha, \beta \in T(X,E,F)$. Then $\alpha = \beta\mu$ for some $\mu \in T(X,E,F)$ if and only if*

(i) $\ker\beta \subseteq \ker\alpha$ and

(ii) for all $x, y \in X$, $(x\beta, y\beta) \in E$ implies that $(x\alpha, y\alpha) \in F$.

Proof The necessity is clear. To prove the sufficiency, we assume that conditions (i) and (ii) hold. For each $y \in X\beta$, there exists $a_y \in X$ such that $a_y\beta = y$. Let $A \in X/E$ be such that $A \cap X\beta \neq \emptyset$. Then there exists $y \in A \cap X\beta$. Thus, $a_y\beta = y$ for some $a_y \in X$. We choose and fix $b_A \in X$ with $(b_A, a_y\alpha) \in F$. Define $\mu_A : A \to X$ by

$$x\mu_A = \begin{cases} a_x\alpha & \text{if } x \in X\beta, \\ b_A & \text{otherwise.} \end{cases}$$

Let $x, y \in A$ be such that $x = y$. If $x, y \in X\beta$, then there are $a_x, a_y \in X$ such that $a_x\beta = x$ and $a_y\beta = y$. Thus, $(a_x, a_y) \in \ker\beta$ and so $a_x\alpha = a_y\alpha$ by (i), which implies that $x\mu_A = y\mu_A$. If $x, y \notin X\beta$, then $x\mu_A = b_A = y\mu_A$.

From the above discussion, we obtain that μ_A is well defined. Define $\mu : X \to X$ by

$$\mu|_A = \begin{cases} \mu_A & \text{if } A \cap X\beta \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/E$ where C_A is a constant map from A into X. Since X/E is a partition of X, we have that μ is well defined and so $\mu \in T(X)$. To show that $\mu \in T(X, E, F)$, let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$.

Case 1. $A \cap X\beta \neq \emptyset$. Then there exists $z \in A \cap X\beta$ such that $a_z\beta = z$ and $(b_A, a_z\alpha) \in F$. We note that $(x, z) \in E$. It suffices to consider three cases as follows.

Subcase 1.1. $x, y \in X\beta$. Then $a_x\beta = x$ and $a_y\beta = y$ for some $a_x, a_y \in X$. Thus, $(a_x\beta, a_y\beta) = (x, y) \in E$ and so $(x\mu, y\mu) = (x\mu_A, y\mu_A) = (a_x\alpha, a_y\alpha) \in F$ by (ii).

Subcase 1.2. $x \in X\beta$ and $y \notin X\beta$. Then $a_x\beta = x$ for some $a_x \in X$ and so $(a_x\beta, a_z\beta) = (x, z) \in E$. By (ii), we have $(a_x\alpha, a_z\alpha) \in F$. Since $(a_z\alpha, b_A) \in F$, $(x\mu, y\mu) = (x\mu_A, y\mu_A) = (a_x\alpha, b_A) \in F$ by transitivity of F.

Subcase 1.3. $x, y \notin X\beta$. Then by reflexivity of F, we obtain that

$$(x\mu, y\mu) = (x\mu_A, y\mu_A) = (b_A, b_A) \in F.$$

Case 2. $A \cap X\beta = \emptyset$. Then by reflexivity of F, we have $(x\mu, y\mu) = (xC_A, yC_A) \in F$.

From the two cases, we deduce that $\mu \in T(X, E, F)$. Let $x \in X$. Then $x\beta \in X\beta$ and $x\beta \in A$ for some $A \in X/E$ and so $a_{x\beta}\beta = x\beta$ for some $a_{x\beta} \in X$. Thus, $(a_{x\beta}, x) \in \ker\beta$ so that $x\alpha = a_{x\beta}\alpha = (x\beta)\mu_A = x\beta\mu$ by (i). This shows that $\alpha = \beta\mu$ as required. □

As an immediate consequence of Lemma 3.2, we have the following.

Theorem 3.3 *Let $\alpha, \beta \in T(X, E, F)$. Then $(\alpha, \beta) \in \mathcal{R}$ if and only if*

(i) $\ker\beta = \ker\alpha$,

(ii) for all $x, y \in X$, $(x\beta, y\beta) \in E$ implies that $(x\alpha, y\alpha) \in F$, and

(iii) for all $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies that $(x\beta, y\beta) \in F$.

To describe the $\mathcal{R}$-relation again, the following lemma is required.

Lemma 3.4 *Let $\alpha, \beta \in T(X, E, F)$. Then $\alpha = \beta\mu$ for some $\mu \in T(X, E, F)$ if and only if there exists a mapping $\varphi : X\beta \to X\alpha$ satisfying*

(i) $\alpha = \beta\varphi$ and

(ii) for all $x, y \in X\beta$, $(x, y) \in E$ implies that $(x\varphi, y\varphi) \in F$.

Proof The necessity is clear from Lemma 3.2 by just taking $\varphi = \mu|_{X\beta}$. To prove the sufficiency, we suppose that $\varphi : X\beta \to X\alpha$ is a mapping satisfying the conditions (i) and (ii). Let $A \in X/E$ be such that $A \cap X\beta \neq \emptyset$. Then there exists a unique $B \in X/F$ such that $(A \cap X\beta)\varphi = B \cap X\alpha$ by (ii). Fix some $b_A \in B$ and define $\mu_A : A \to B$ by

$$x\mu_A = \begin{cases} x\varphi & \text{if } x \in X\beta, \\ b_A & \text{otherwise.} \end{cases}$$

Let $\mu : X \to X$ be defined by

$$\mu|_A = \begin{cases} \mu_A & \text{if } A \cap X\beta \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/E$ and C_A is a constant map from A into X. Since X/E is a partition of X, we have that μ is well defined. Let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$.

Case 1. $A \cap X\beta \neq \emptyset$. Then there exists $B \in X/F$ such that $(A \cap X\beta)\varphi = B \cap X\alpha$ by (ii) and so $b_A \in B$.

Subcase 1.1. $x, y \in X\beta$. Then $(x\mu, y\mu) = (x\mu_A, y\mu_A) = (x\varphi, y\varphi) \in F$ by (ii).

Subcase 1.2. $x \in X\beta$ and $y \notin X\beta$. Then $x\varphi \in B$ and so $(x\mu, y\mu) = (x\mu_A, y\mu_A) = (x\varphi, b_A) \in F$.

Subcase 1.3. $x, y \notin X\beta$. Then by reflexivity of F, we have

$$(x\mu, y\mu) = (x\mu_A, y\mu_A) = (b_A, b_A) \in F.$$

Case 2. $A \cap X\beta = \emptyset$. Then by reflexivity of F, we have $(x\mu, y\mu) = (xC_A, yC_A) \in F$.

From the two cases we deduce that $\mu \in T(X, E, F)$. It is routine to check that $\alpha = \beta\mu$, as required. □

The following theorem is a direct consequence of Lemma 3.4.

Theorem 3.5 *Let $\alpha, \beta \in T(X, E, F)$. Then $(\alpha, \beta) \in \mathcal{R}$ if and only if there exists a bijection $\varphi : X\beta \to X\alpha$ satisfying*

(i) $\alpha = \beta\varphi$,

(ii) for all $x, y \in X\beta$, $(x, y) \in E$ implies that $(x\varphi, y\varphi) \in F$, and

(iii) for all $x, y \in X\alpha$, $(x, y) \in E$ implies that $(x\varphi^{-1}, y\varphi^{-1}) \in F$.

For an equivalence E on a set X and $\varphi : A \to B$ where $A, B \subseteq X$, we say that φ is E^**-preserving* if $(x, y) \in E$ if and only if $(x\varphi, y\varphi) \in E$.

As a consequence, we obtain a corollary of Theorem 3.5.

Corollary 3.6 *Let $\alpha, \beta \in T(X, E, F)$. If $(\alpha, \beta) \in \mathcal{R}$, then there exists a bijection $\varphi : X\beta \to X\alpha$ is an F^*-preserving bijection and an E^*-preserving bijection such that $\alpha = \beta\varphi$.*

Let $\alpha, \beta \in T(X, E, F)$ and φ be a map from $\pi(\alpha)$ into $\pi(\beta)$. If for each $A \in X/E$, there exists $B \in X/F$ such that

$$(\pi_A(\alpha))\varphi \subseteq \pi_B(\beta),$$

then φ is said to be EF*-admissible*. Note that, if $E = F$, then φ is said to be E*-admissible*. If φ is a bijection and both φ and φ^{-1} are EF-admissible, then φ is said to be EF^**-admissible*, and if $E = F$, we say that φ is said to be E^**-admissible*. If $\gamma \in T(X, E, F)$, then denote by γ_* the map from $\pi(\gamma)$ onto $X\gamma$ induced by γ, namely $P\gamma_* = p\gamma$ for each $P \in \pi(\gamma)$ and all $p \in P$. Obviously, γ_* is a bijection.

Proposition 3.7 *Let $\alpha, \beta \in T(X, E, F)$. Then $\varphi : \pi(\alpha) \to \pi(\beta)$ is EF-admissible if and only if for each $A \in X/E$ there exists $B \in X/F$ such that $B \cap P\varphi \neq \emptyset$ for all $P \in \pi_A(\alpha)$.*

Proof Suppose that $\varphi : \pi(\alpha) \to \pi(\beta)$ is EF-admissible. Let $A \in X/E$. Then there exists $B \in X/F$ such that

$$(\pi_A(\alpha))\varphi \subseteq \pi_B(\beta).$$

Let $P \in \pi_A(\alpha)$. Then $P\varphi \in \pi_B(\beta)$. Hence, $B \cap P\varphi \neq \emptyset$.

Conversely, suppose that for each $A \in X/E$, there exists $B \in X/F$ such that $B \cap P\varphi \neq \emptyset$ for all $P \in \pi_A(\alpha)$. Let $A \in X/E$. Then there exists $B \in X/F$ such that $B \cap P\varphi \neq \emptyset$ for all $P \in \pi_A(\alpha)$. Let $P \in \pi_A(\alpha)$. Then $P\varphi \in \pi(\beta)$ and $B \cap P\varphi \neq \emptyset$. Thus, $P\varphi \in \pi_B(\beta)$. Hence, $(\pi_A(\alpha))\varphi \subseteq \pi_B(\beta)$. □

The following lemma is used for characterizing the $\mathcal{L}$-relation on $T(X,E,F)$.

Lemma 3.8 *Let $\alpha, \beta \in T(X,E,F)$. Then the following statements are equivalent.*

(i) $\alpha = \lambda\beta$ for some $\lambda \in T(X,E,F)$.

(ii) For each $A \in X/E$, there exists $B \in X/F$ such that $A\alpha \subseteq B\beta$.

(iii) There exists EF-admissible $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

Proof (i) ⇒ (ii) Assume that $\alpha = \lambda\beta$ for some $\lambda \in T(X,E,F)$. Let $A \in X/E$. Then by Lemma 2.1, we have $A\lambda \subseteq B$ for some $B \in X/F$. By assumption, we obtain that $A\alpha = A\lambda\beta \subseteq B\beta$.

(ii) ⇒ (iii) To show that $X\alpha \subseteq X\beta$, let $y \in X\alpha$. Then $x\alpha = y$ for some $x \in X$. Thus, $x \in A$ for some $A \in X/E$. By (ii), there exists $B \in X/F$ such that

$$y = x\alpha \in A\alpha \subseteq B\beta \subseteq X\beta.$$

Therefore, $X\alpha \subseteq X\beta$. For each $P \in \pi(\alpha)$, we have $P\alpha_* = x\alpha \in X\alpha \subseteq X\beta$ for all $x \in P$. Define $\varphi : \pi(\alpha) \to \pi(\beta)$ by

$$P\varphi = (P\alpha_*)\beta^{-1} \text{ for all } P \in \pi(\alpha).$$

Then φ is well defined. Let $A \in X/E$ and let $I_A = \{i \in X\alpha : i\alpha^{-1} \cap A \neq \emptyset\}$. For each $i \in I_A$, we let $P_i := i\alpha^{-1}$. Then

$$\pi_A(\alpha) = \{P_i : i \in I_A\} \text{ and } i = P_i\alpha_* \text{ for all } i \in I_A.$$

Let $i \in I_A$. By (ii), we have $i \in A\alpha \subseteq B\beta$ for some $B \in X/F$. Then $B \cap P_i\varphi = B \cap (P_i\alpha_*)\beta^{-1} = B \cap i\beta^{-1} \neq \emptyset$. Hence, φ is EF-admissible by Proposition 3.7. Finally, we will show that $\alpha_* = \varphi\beta_*$. Let $P \in \pi(\alpha)$ and $p \in P$. Then $p\alpha \in X\alpha \subseteq X\beta$ and so $p\alpha = x\beta$ for some $x \in X$. Thus, $x \in (p\alpha)\beta^{-1} = (P\alpha_*)\beta^{-1} = P\varphi$. Therefore,

$$P\alpha_* = p\alpha = x\beta = P\varphi\beta_*,$$

as required.

(iii) ⇒ (i) Suppose that $\varphi : \pi(\alpha) \to \pi(\beta)$ is EF-admissible such that $\alpha_* = \varphi\beta_*$. Let $A \in X/E$. Then $(\pi_A(\alpha))\varphi \subseteq \pi_B(\beta)$ for some $B \in X/F$. For each $x \in A$, we let $P_x = (x\alpha)\alpha^{-1} \in \pi_A(\alpha)$. By assumption and Proposition 3.7, we have $P_x\varphi \cap B \neq \emptyset$. We choose $b_x \in P_x\varphi \cap B$. Define $\lambda_A : A \to X$ by

$$x\lambda_A = b_x \text{ for all } x \in A.$$

Let $\lambda \in T(X)$ be such that $\lambda|_A = \lambda_A$ for all $A \in X/E$. Since X/E is a partition of X, λ is well defined. Obviously, $\lambda \in T(X,E,F)$. Let $x \in X$. Then $x \in A$ for some $A \in X/E$. By Proposition 3.7, there is $B \in X/F$ such that $x\lambda = x\lambda|_A = b_x \in P_x\varphi \cap B$ where $P_x \in \pi_A(\alpha)$. Since $\alpha_* = \varphi\beta_*$, we obtain that

$$x\alpha = P_x\alpha_* = P_x\varphi\beta_* = b_x\beta = x\lambda\beta.$$

Hence, $\alpha = \lambda\beta$. □

Using Lemma 3.8, we can establish the next result.

Theorem 3.9 *Let $\alpha, \beta \in T(X,E,F)$. Then the following statements are equivalent.*

(i) $(\alpha, \beta) \in \mathcal{L}$.

(ii) For each $A \in X/E$, there exist $B, C \in X/F$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(iii) There exists an EF^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

As an immediate consequence of Theorem 3.9, we have the following.

Corollary 3.10 *Let $\alpha, \beta \in T(X,E,F)$ be such that $(\alpha, \beta) \in \mathcal{L}$. Then the following statements hold.*

(i) For each $A \in X/E$, there exist $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(ii) For each $A \in X/F$, there exist $B, C \in X/F$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(iii) There is an E^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

(iv) There is an F^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

(v) $X\alpha = X\beta$.

Now we can determine $\mathcal{L}$ for two regular elements of $T(X,E,F)$. As an immediate consequence of Proposition 2.13 and Theorem 3.9, we obtain:

Theorem 3.11 *Let α and β be regular elements of $T(X,E,F)$. Then $(\alpha, \beta) \in \mathcal{L}$ if and only if $X\alpha = X\beta$.*

To describe the $\mathcal{J}$-relation on $T(X,E,F)$, we first give the following lemma.

Lemma 3.12 *Let $\alpha, \beta \in T(X,E,F)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T(X,E,F)$ if and only if there exists $\varphi : X\beta \to X$ satisfying the following:*

(i) for each $x, y \in X\beta$, $(x,y) \in E$ implies that $(x\varphi, y\varphi) \in F$ and

(ii) for each $A \in X/E$, there exists $B \in X/F$ such that $A\alpha \subseteq (B\beta)\varphi$.

Proof Suppose that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T(X,E,F)$. Let $\varphi = \mu|_{X\beta}$ and let $x, y \in X\beta$ be such that $(x,y) \in E$. Then since $\mu \in T(X,E,F)$, we have

$$(x\varphi, y\varphi) = (x\mu|_{X\beta}, y\mu|_{X\beta}) = (x\mu, y\mu) \in F.$$

Let $A \in X/E$. By Lemma 2.1, there exists $B \in X/F$ such that $A\lambda \subseteq B$. Thus, $A\alpha = A\lambda\beta\mu \subseteq B\beta\mu = B\beta\mu|_{X\beta} = (B\beta)\varphi$.

Conversely, assume that there exists $\varphi : X\beta \to X$ satisfying the conditions (i) and (ii). Let $A \in X/E$ be such that $A \cap X\beta \neq \emptyset$. By (i), $(A \cap X\beta)\varphi \subseteq B$ for some $B \in X/F$. Fix some $b_A \in B$ and define $\mu_A : A \to B$ by

$$x\mu_A = \begin{cases} x\varphi & \text{if } x \in X\beta, \\ b_A & \text{otherwise.} \end{cases}$$

Let $\mu : X \to X$ be defined by

$$\mu|_A = \begin{cases} \mu_A & \text{if } A \cap X\beta \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/E$ and C_A is a constant map from A into X. Since X/E is a partition of X, it follows that μ is well defined. From (i), we have $\mu \in T(X,E,F)$.

For each $A \in X/E$, by (ii) we choose and fix $B_A \in X/F$ such that $A\alpha \subseteq (B_A\beta)\varphi$. Let $x \in A$. Then we choose and fix $b_x \in B_A$ such that $x\alpha = (b_x\beta)\varphi$. Define $\lambda : X \to X$ by $x\lambda = b_x$ for all $x \in X$. Then $\lambda \in T(X,E,F)$. Furthermore, for $x \in X$,

$$x\lambda\beta\mu = b_x\beta\mu = (b_x\beta)\varphi = x\alpha,$$

which implies that $\alpha = \lambda\beta\mu$, as desired. □

Lemma 3.12 is useful to obtain this result.

Theorem 3.13 *Let $\alpha, \beta \in T(X,E,F)$. Then $(\alpha,\beta) \in \mathcal{J}$ if and only if there exist $\varphi : X\beta \to X$ and $\psi : X\alpha \to X$ satisfying the following:*

(i) for each $x, y \in X\beta$, $(x,y) \in E$ implies that $(x\varphi, y\varphi) \in F$,

(ii) for each $x, y \in X\alpha$, $(x,y) \in E$ implies that $(x\psi, y\psi) \in F$, and

(iii) for each $A \in X/E$, there exist $B, C \in X/F$ such that $A\alpha \subseteq (B\beta)\varphi$ and $A\beta \subseteq (C\alpha)\psi$.

Next, to describe the $\mathcal{D}$-relation on $T(X,E,F)$, the following corollary follows from Theorem 3.3 and Proposition 2.3.

Corollary 3.14 *Let $\alpha, \beta \in T(X,E,F)$. If $(\alpha,\beta) \in \mathcal{R}$, then $\pi(\alpha) = \pi(\beta)$ and $F(\alpha) = F(\beta)$.*

Theorem 3.15 *Let $\alpha, \beta \in T(X,E,F)$. Then $(\alpha,\beta) \in \mathcal{D}$ if and only if there exist an EF^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ and a bijection $\psi : X\alpha \to X\beta$ satisfying the following:*

(i) for each $x, y \in X\alpha$, $(x,y) \in E$ implies that $(x\psi, y\psi) \in F$,

(ii) for each $x, y \in X\beta$, $(x,y) \in E$ implies that $(x\psi^{-1}, y\psi^{-1}) \in F$, and

(iii) $\alpha_*\psi = \varphi\beta_*$.

Proof Suppose that $(\alpha, \beta) \in \mathcal{D}$. Then $(\alpha, \gamma) \in \mathcal{R}$ and $(\gamma, \beta) \in \mathcal{L}$ for some $\gamma \in T(X, E, F)$. By Corollaries 3.14, and 3.10(v), we have $\pi(\alpha) = \pi(\gamma)$ and $X\beta = X\gamma$, respectively. Since $(\alpha, \gamma) \in \mathcal{R}$, by Theorem 3.5, there exists a bijection $\psi : X\alpha \to X\beta$ satisfying (i), (ii), and

$$\gamma = \alpha\psi.$$

Let $P \in \pi(\gamma) = \pi(\alpha)$ and $x \in P$. Then $P\gamma_* = x\gamma = x\alpha\psi = P\alpha_*\psi$. Thus, $\gamma_* = \alpha_*\psi$. Since $(\gamma, \beta) \in \mathcal{L}$, by Theorem 3.9, there exists an EF^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that

$$\gamma_* = \varphi\beta_*.$$

Hence, $\alpha_*\psi = \varphi\beta_*$ and the assertion follows.

Conversely, assume that $\varphi : \pi(\alpha) \to \pi(\beta)$ is an EF^*-admissible bijection and $\psi : X\alpha \to X\beta$ is a bijection satisfying the conditions (i), (ii), and (iii). Define $\gamma \in T(X)$ by $x\gamma = (x\alpha)\psi$ for all $x \in X$. Then $\gamma \in T(X, E, F)$ by (i) and

$$\gamma = \alpha\psi.$$

Next, we will show that $\pi(\alpha) = \pi(\gamma)$. Let $y \in X\alpha$. Then $\{y\psi\} = (y\alpha^{-1})\alpha\psi = (y\alpha^{-1})\gamma$. Thus, $y\alpha^{-1} \subseteq (y\alpha^{-1})\gamma\gamma^{-1} \subseteq (y\psi)\gamma^{-1} \in \pi(\gamma)$. Hence, $\pi(\alpha) \preceq \pi(\gamma)$. On the other hand, let $z \in X\gamma$. Then $\{z\psi^{-1}\} = (z\gamma^{-1})\gamma\psi^{-1} = (z\gamma^{-1})\alpha\psi\psi^{-1} = (z\gamma^{-1})\alpha id_{X\alpha} = (z\gamma^{-1})\alpha$. Thus, $z\gamma^{-1} \subseteq (z\psi^{-1})\alpha^{-1} \in \pi(\alpha)$ and hence $\pi(\gamma) \preceq \pi(\alpha)$. Consequently, $\pi(\alpha) = \pi(\gamma)$. Let $P \in \pi(\gamma)$ and $x \in P$. Then

$$P\gamma_* = x\gamma = x\alpha\psi = P\alpha_*\psi,$$

and this implies that $\gamma_* = \alpha_*\psi$. By (iii), we obtain that $\gamma_* = \alpha_*\psi = \varphi\beta_*$. By Theorem 3.9, we have that $(\gamma, \beta) \in \mathcal{L}$. It follows from Corollary 3.10(v) that $X\gamma = X\beta$. This implies that $\psi : X\alpha \to X\gamma$ such that $\gamma = \alpha\psi$. From (i) and (ii), it follows from Theorem 3.5 that $(\alpha, \gamma) \in \mathcal{R}$. Hence, $(\alpha, \beta) \in \mathcal{D}$, as required. □

In order to describe Green's relation $\mathcal{D}$ for regular elements of $T(X, E, F)$, we observe the following.

Lemma 3.16 *Let α and β be regular elements of $T(X, E, F)$. Suppose that $\psi : X\alpha \to X\beta$ is a bijection satisfying the following:*

(i) for all $x, y \in X\alpha$, $(x, y) \in E$ implies that $(x\psi, y\psi) \in F$ and

(ii) for all $x, y \in X\beta$, $(x, y) \in E$ implies that $(x\psi^{-1}, y\psi^{-1}) \in F$.

Then there exists an EF^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_*\psi = \varphi\beta_*$.

Proof Define $\varphi : \pi(\alpha) \to \pi(\beta)$ by

$$P\varphi = (P\alpha_*\psi)\beta_*^{-1} \text{ for all } P \in \pi(\alpha).$$

Obviously, φ is well defined and $\varphi\beta_* = \alpha_*\psi$. Notice that α_*, β_*^{-1} and ψ are all bijection, and so also is φ. Thus, what remains is to verify that φ is EF^*-admissible. Let $A \in X/E$. Then $B' = A\alpha \subseteq B$ for some $B \in X/E$ by Corollary 2.2(ii). By (i), we have that $C' = B'\psi \subseteq B\psi \subseteq C$ for some $C \in X/F$. By regularity of β and Corollary 2.9(i), we can write

$$C' \subseteq C \cap X\beta \subseteq D\beta$$

for some $D \in X/F$. We assert that $(\pi_A(\alpha))\varphi \subseteq \pi_D(\beta)$. In fact, if $P \in \pi_A(\alpha)$, then $P\alpha_* \in A\alpha = B'$. Hence,

$$P\alpha_*\psi \in B'\psi = C' \subseteq D\beta$$

and $P\varphi \cap D = (P\alpha_*\psi)\beta_*^{-1} \cap D \neq \emptyset$, which implies that $P\varphi \in \pi_D(\beta)$ and the assertion holds. Hence, φ is EF-admissible. Similarly, φ^{-1} is EF-admissible and the conclusion follows. □

As an immediate consequence of Theorem 3.15 and Lemma 3.16, we have the next result.

Theorem 3.17 *Let α and β be regular elements of $T(X,E,F)$. Then $(\alpha,\beta) \in \mathcal{D}$ if and only if there exists a bijection $\psi : X\alpha \to X\beta$ satisfying the following:*

(i) for all $x,y \in X\alpha$, $(x,y) \in E$ implies that $(x\psi, y\psi) \in F$ and

(ii) for all $x,y \in X\beta$, $(x,y) \in E$ implies that $(x\psi^{-1}, y\psi^{-1}) \in F$.

Acknowledgment

The authors would like to express gratitude to the Science Achievement Scholarship of Thailand (SAST) for the full scholarship to one of the authors and support in academic activities.

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