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Spectral expansion for the singular Dirac system with impulsive conditions

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Abstract: In this work, we study the one-dimensional Dirac system on a whole line with impulsive conditions. We construct a spectral function of this system. Using the spectral function, we establish a Parseval equality and spectral expansion formula for such a system.

Key words: Dirac operator, impulsive conditions, singular point, spectral function, Parseval equality, spectral expansion

1. Introduction

Many problems of engineering interest are governed by partial differential equations. When we seek a solution of a partial differential equation by separation of variables, it leads to the problem of expanding an arbitrary function as a series of eigenfunctions. The method relies on the completeness of the eigenfunctions corresponding to one of the variables. Thus, spectral expansion theorems are important in mathematics. Using several methods, the eigenfunction expansion is obtained, including the methods of integral equations, contour integration, and finite difference (see [19], [32]).

The Dirac operators play an important role in the theory of relativistic quantum mechanics because fundamental physics of relativistic quantum mechanics was formulated by the Dirac operators. For example, they predict the existence of a positron and elucidate the origin of spin $1/2$ of an electron. We refer the reader to [30].

Discontinuous (or impulsive) boundary value problems have been extensively investigated in recent years (see [1–16, 18, 20–27, 29, 31, 33–41]). Many researchers have investigated these problems due to their significant applications in various fields of science and engineering, such as in radio science (see [22]), the theory of heat and mass transfer (see [21]), and geophysics (see [18]).

On the other hand, direct or inverse spectral problems for Dirac operators were studied in [1, 4, 6, 7, 12, 14, 19, 23, 34, 36, 39, 40]. In [13], Hıra and Altınışık investigated asymptotic behavior of eigenvalues and eigenfunctions of the discontinuous Dirac system, which includes an eigenvalue parameter in a transmission condition. In [31], Tharwat and Bhrawy computed the eigenvalues of discontinuous regular Dirac systems with transmission conditions at the point of discontinuity numerically. In [15], Kablan and Özden studied a Dirac system with transmission condition and eigenparameter in boundary conditions. They investigated the existence of the solution and some spectral properties of this problem.

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In this work, we construct a spectral function and obtain a Parseval equality and a spectral expansion theorem for Dirac systems with impulsive conditions on the whole line.

2. Main results

Let us consider the Dirac system

$$\tau(y) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{dy(x)}{dx} + B(x)\, y(x) = \lambda y(x), \ x \in J := J_1 \cup J_2, \tag{1}$$

where λ is a complex spectral parameter and

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \ B(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix},$$

$J_1 := [a,c)$, $J_2 := (c,b]$, $-\infty < a < c < b < +\infty$. We assume that the points a, b, and c are regular for the differential expression τ. p and r are real-valued, Lebesgue measurable functions on J and $p, r \in L^1(J_k)$ $(k = 1, 2)$. The point c is regular if $p, r \in L^1[c-\epsilon, c+\epsilon]$ for some $\epsilon > 0$.

We will consider the Dirac system (1) with the boundary conditions

$$\begin{aligned} &y_2(a)\cos\gamma + y_1(a)\sin\gamma = 0, \ \gamma \in \mathbb{R} := (-\infty, \infty), \\ &y_2(b)\cos\alpha + y_1(b)\sin\alpha = 0, \ \alpha \in \mathbb{R}, \end{aligned} \tag{2}$$

and impulsive (or transmission) conditions

$$y(c+) = Cy(c-), \ C \in M_2(\mathbb{R}), \ \det C = \delta > 0, \tag{3}$$

where $M_2(\mathbb{R})$ denotes the 2×2 matrices with entries from $\mathbb{R}$.

We introduce the direct sum Hilbert space $\mathcal{H} = L^2(J_1; E) \dot{+} L^2(J_2; E)$ (where $E := \mathbb{R}^2$) of vector-valued functions with values in $\mathbb{R}^2$ and with the inner product

$$\langle u, v \rangle_{\mathcal{H}} := \int_a^c (u(x), v(x))_E\, dx + \beta \int_c^b (u(x), v(x))_E\, dx, \ \beta = \frac{1}{\delta},$$

where

$$\begin{aligned} u(x) &= \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, v(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}, \\ u_1(x) &= \begin{cases} u_1^{(1)}(x), & x \in J_1 \\ u_1^{(2)}(x), & x \in J_2 \end{cases}, \ u_2(x) = \begin{cases} u_2^{(1)}(x), & x \in J_1 \\ u_2^{(2)}(x), & x \in J_2 \end{cases}, \\ v_1(x) &= \begin{cases} v_1^{(1)}(x), & x \in J_1 \\ v_1^{(2)}(x), & x \in J_2 \end{cases}, \ v_2(x) = \begin{cases} v_2^{(1)}(x), & x \in J_1 \\ v_2^{(2)}(x), & x \in J_2 \end{cases}, \end{aligned}$$

$$(u(x), v(x))_E = u_1(x)\, v_1(x) + u_2(x)\, v_2(x).$$

We will denote by

$$\begin{aligned}\phi_1(x,\lambda) &= \begin{pmatrix}\phi_{11}(x,\lambda)\\ \phi_{12}(x,\lambda)\end{pmatrix}, \phi_{11}(x,\lambda)=\begin{cases}\phi_{11}^{(1)}(x), & x\in J_1\\ \phi_{11}^{(2)}(x), & x\in J_2\end{cases},\\ \phi_{12}(x,\lambda) &= \begin{cases}\phi_{12}^{(1)}(x), & x\in J_1\\ \phi_{12}^{(2)}(x), & x\in J_2\end{cases},\end{aligned}$$

and

$$\begin{aligned}\phi_2(x,\lambda) &= \begin{pmatrix}\phi_{21}(x,\lambda)\\ \phi_{22}(x,\lambda)\end{pmatrix},\ \phi_{21}(x,\lambda)=\begin{cases}\phi_{21}^{(1)}(x), & x\in J_1\\ \phi_{21}^{(2)}(x), & x\in J_2\end{cases},\\ \phi_{22}(x,\lambda) &= \begin{cases}\phi_{22}^{(1)}(x), & x\in J_1\\ \phi_{22}^{(2)}(x), & x\in J_2\end{cases},\end{aligned}$$

the solution of the Dirac system $\tau(y)=\lambda y(x)\,,\ x\in\Omega_1\cup\Omega_2$, which satisfies the initial conditions

$$\begin{aligned}&\phi_{11}^{(1)}(a,\lambda)=-\cos\gamma,\ \phi_{12}^{(1)}(a,\lambda)=\sin\gamma,\\ &\phi_{21}^{(2)}(b,\lambda)=\cos\alpha,\ \phi_{22}^{(2)}(b,\lambda)=-\sin\alpha,\end{aligned} \tag{4}$$

and impulsive conditions (3).

Now we will prove that system (1) with conditions (2)–(3) has a compact resolvent operator and thus it has a purely discrete spectrum. Furthermore, we get a Parseval equality for this problem.

Let us define

$$G(x,t,\lambda)=\begin{cases}\frac{\phi_2(x,\lambda)\phi_1^T(t,\lambda)}{W(\phi_1,\phi_2)}, & a\le t<x,\ x\ne c,t\ne c\\ \frac{\phi_1(x,\lambda)\phi_2^T(t,\lambda)}{W(\phi_1,\phi_2)}, & x<t\le b,\ x\ne c,t\ne c,\end{cases}$$

which is called *Green's matrix* of the problem (1)–(3) (see [19] and [12]).

Definition 1 *A matrix-valued function $M(x,t)$ in E of two variables with $a\le x,t\le b$ is called the Hilbert–Schmidt kernel if*

$$\int_a^b\int_a^b \|M(x,t)\|_E^2\,dxdt<+\infty.$$

Theorem 2 ([28]) *If*

$$\sum_{i,k=1}^{\infty}|a_{ik}|^2<+\infty, \tag{5}$$

then the operator A defined by the formula $A\{y_i\}=\{z_i\}$, where

$$z_i=\sum_{k=1}^{\infty}a_{ik}y_k,\ \ (i=1,2,...), \tag{6}$$

is compact in the sequence space l^2.

Theorem 3 *Without loss of generality, we can assume that* $\lambda = 0$ *is not an eigenvalue of the problem (1)–(3). Then* $G(x,t)$ *defined by*

$$G(x,t) := G(x,t,0) = \begin{cases} \frac{\phi_2(x)\phi_1^T(t)}{W(\phi_1,\phi_2)}, & a \leq t < x, \ x \neq c, t \neq c \\ \frac{\phi_1(t)\phi_2^T(t)}{W(\phi_1,\phi_2)}, & x < t \leq b, \ x \neq c, t \neq c \end{cases} \tag{7}$$

is a Hilbert–Schmidt kernel.

Proof By the upper half of formula (7), we have

$$\int_a^b dx \int_a^x \|G(x,t)\|_E^2 \, dt < +\infty,$$

and by the lower half of (7), we have

$$\int_a^b dx \int_x^b \|G(x,t)\|_E^2 \, dt < +\infty$$

since the inner integral exists and is a linear combination of the products $\phi_{ij}(x)\phi_{k,l}(t)$ $(i,j,k,l = 1,2)$, and these products belong to $L^2(a,b) \times L^2(a,b)$ because each of the factors belongs to $L^2(a,b)$. Then we obtain

$$\int_a^b \int_a^b \|G(x,t)\|_E^2 \, dxdt < +\infty. \tag{8}$$

□

Theorem 4 *The operator* $\mathbf{K}$ *defined by the formula*

$$g(x) = (\mathbf{K}f)(x) := \int_a^b G(x,t) f(t) \, dt$$

is compact and self-adjoint in space $\mathcal{H}$.

Proof Let $\phi_i = \phi_i(x)$ $(i = 1,2,...)$ be a complete, orthonormal basis of $\mathcal{H}$. Since $G(x,t)$ is a Hilbert–Schmidt kernel, we can define

$$\begin{aligned} y_i &= \langle f, \phi_i \rangle_{\mathcal{H}} = \int_a^c (f(t), \phi_i(t))_E \, dt + \beta \int_c^b (f(t), \phi_i(t))_E \, dt, \\ z_i &= \langle g, \phi_i \rangle_{\mathcal{H}} = \int_a^c (g(t), \phi_i(t))_E \, dt + \beta \int_c^b (g(t), \phi_i(t))_E \, dt, \\ a_{ik} &= \int_a^b \int_a^b (G(x,t)\phi_i(x), \phi_k(t))_E \, dxdt \ (i,k = 1,2,...). \end{aligned}$$

Then $\mathcal{H}$ is mapped isometrically to l^2. Consequently, our integral operator transforms into the operator defined by formula (6) in the space l^2 by this mapping and condition (8) is translated into condition (5). By Theorem 2, this operator is compact. Therefore, the original operator $\mathbf{K}$ is compact.

Let $f,g\in\mathcal{H}$. As $G(x,t)=G^{T}(t,x)$ and $G(x,t)$ is a matrix-valued function in E defined on $J\times J$, we have

$$\begin{aligned}\langle \mathbf{K}f,g\rangle_{\mathcal{H}} &= \int_a^c ((\mathbf{K}f)(x),g(x))_E\,dx+\beta\int_c^b ((\mathbf{K}f)(x),g(x))_E\,dx \\ &= \int_a^c\int_a^c (G(x,t)f(t),g(x))_E\,dtdx \\ &\quad +\beta\int_c^b\int_c^b (G(x,t)f(t),g(x))_E\,dtdx \\ &= \int_a^c (f(t),\int_a^c G(t,x)g(x))_E dxdt \\ &\quad +\beta\int_c^b (f(t),\int_c^b G(t,x)g(x))_E dxdt=\langle f,\mathbf{K}g\rangle_{\mathcal{H}}.\end{aligned}$$

Thus, we have proved that $\mathbf{K}$ is self-adjoint in $\mathcal{H}$. □

From Theorem 4 and the Hilbert–Schmidt theorem [17], there is an orthonormal system $\{\varphi_n\}$ $(n\in\mathbb{Z}:=\{0,\pm1,\pm2,...\})$ of eigenvectors of the self-adjoint problem (1)–(3) with corresponding nonzero eigenvalues λ_n $(n\in\mathbb{Z})$ such that

$$\int_a^c \|f(x)\|_E^2\,dx+\beta\int_c^b \|f(x)\|_E^2\,dx=\sum_{n=-\infty}^{\infty}a_n^2, \tag{9}$$

which is called the *Parseval equality*, where $f\in\mathcal{H}$ and $a_n=\langle f,\varphi_n\rangle_{\mathcal{H}}$ $(n\in\mathbb{Z})$.

We will denote by

$$\begin{aligned}\psi_1(x,\lambda) &= \begin{pmatrix}\psi_{11}(x,\lambda)\\ \psi_{12}(x,\lambda)\end{pmatrix},\psi_{11}(x,\lambda)=\begin{cases}\psi_{11}^{(1)}(x), & x\in\Omega_1\\ \psi_{11}^{(2)}(x), & x\in\Omega_2\end{cases}, \\ \psi_{12}(x,\lambda) &= \begin{cases}\psi_{12}^{(1)}(x), & x\in\Omega_1\\ \psi_{12}^{(2)}(x), & x\in\Omega_2\end{cases},\end{aligned}$$

(where $\Omega_1:=(-\infty,c)$, $\Omega_2:=(c,\infty)$) and

$$\begin{aligned}\psi_2(x,\lambda) &= \begin{pmatrix}\psi_{21}(x,\lambda)\\ \psi_{22}(x,\lambda)\end{pmatrix},\psi_{21}(x,\lambda)=\begin{cases}\psi_{21}^{(1)}(x), & x\in\Omega_1\\ \psi_{21}^{(2)}(x), & x\in\Omega_2\end{cases}, \\ \psi_{22}(x,\lambda) &= \begin{cases}\psi_{22}^{(1)}(x), & x\in\Omega_1\\ \psi_{22}^{(2)}(x), & x\in\Omega_2\end{cases},\end{aligned}$$

the solution of the Dirac system $\tau(y)=\lambda y(x)$ $(x\in\Omega_1\cup\Omega_2)$, which satisfies the initial conditions

$$\begin{aligned}&\psi_{11}^{(1)}(d,\lambda)=1,\ \psi_{12}^{(1)}(d,\lambda)=0,\\ &\psi_{21}^{(1)}(d,\lambda)=0,\ \psi_{22}^{(1)}(d,\lambda)=1,\ a<d<c,\end{aligned} \tag{10}$$

and impulsive conditions (3).

Let λ_n $(n \in \mathbb{Z})$ be the eigenvalues and y_n $(n \in \mathbb{Z})$ be the corresponding eigenfunctions of the self-adjoint problem (1)–(3), where

$$\begin{aligned} y_n(x) &= \begin{pmatrix} y_{n_1}(x) \\ y_{n_2}(x) \end{pmatrix}, \\ y_{n_1}(x) &= \begin{cases} y_{n_1}^{(1)}(x), & x \in J_1 \\ y_{n_1}^{(2)}(x), & x \in J_2 \end{cases}, \; y_{n_2}(x) = \begin{cases} y_{n_2}^{(1)}(x), & x \in J_1 \\ y_{n_2}^{(2)}(x), & x \in J_2. \end{cases} \end{aligned}$$

Since the solutions $\psi_1(x,\lambda)$ and $\psi_2(x,\lambda)$ of system (1) are linearly independent, we get

$$y_n(x) = c_n\psi_1(x,\lambda_n) + d_n\psi_2(x,\lambda_n) \;\; (n \in \mathbb{Z}).$$

There is no loss of generality in assuming that $|c_n| \leq 1$ and $|d_n| \leq 1$ $(n \in \mathbb{Z})$. Now let us set

$$\alpha_n^2 = \int_a^c \|y_n(x)\|_E^2 \, dx + \beta \int_c^b \|y_n(x)\|_E^2 \, dx \; (n \in \mathbb{Z}).$$

Let

$$f(.) = \begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix} \in \mathcal{H},$$

$$f_1(x) = \begin{cases} f_1^{(1)}(x), & x \in J_1 \\ f_1^{(2)}(x), & x \in J_2 \end{cases}, \quad f_2(x) = \begin{cases} f_2^{(1)}(x), & x \in J_1 \\ f_2^{(2)}(x), & x \in J_2. \end{cases}$$

If we apply the Parseval equality (9) to $f(x)$, then we obtain

$$\begin{aligned} &\int_a^c \|f(x)\|_E^2 \, dx + \beta \int_c^b \|f(x)\|_E^2 \, dx \\ = &\sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n^2} \left\{ \int_a^c (f(x), y_n(x))_E \, dx + \beta \int_c^b (f(x), y_n(x))_E \, dx \right\}^2 \\ = &\sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n^2} \left\{ \begin{array}{c} \int_a^c (f(x), c_n\psi_1(x,\lambda_n) + d_n\psi_2(x,\lambda_n))_E \, dx \\ +\beta \int_c^b (f(x), c_n\psi_1(x,\lambda_n) + d_n\psi_2(x,\lambda_n))_E \, dx \end{array} \right\}^2 \\ = &\sum_{n=-\infty}^{\infty} \frac{c_n^2}{\alpha_n^2} \left\{ \begin{array}{c} \int_a^c (f(x), \psi_1(x,\lambda_n))_E \, dx \\ +\beta \int_c^b (f(x), \psi_1(x,\lambda_n))_E \, dx \end{array} \right\}^2 \\ &+2 \sum_{n=-\infty}^{\infty} \frac{c_n d_n}{\alpha_n^2} \left\{ \begin{array}{c} \int_a^c (f(x), \psi_1(x,\lambda_n))_E \, dx \\ +\beta \int_c^b (f(x), \psi_1(x,\lambda_n))_E \, dx \end{array} \right\} \\ &\times \left\{ \begin{array}{c} \int_a^c (f(x), \psi_2(x,\lambda_n))_E \, dx \\ +\beta \int_c^b (f(x), \psi_2(x,\lambda_n))_E \, dx \end{array} \right\} \\ &+ \sum_{n=-\infty}^{\infty} \frac{d_n^2}{\alpha_n^2} \left\{ \begin{array}{c} \int_a^c (f(x), \psi_2(x,\lambda_n))_E \, dx \\ +\beta \int_c^b (f(x), \psi_2(x,\lambda_n))_E \, dx \end{array} \right\}^2. \end{aligned} \tag{11}$$

Now we will define the step function $\mu_{ij,[a,b]}$ $(i,j=1,2)$ on $\mathbb{R}$ by

$$\mu_{11,[a,b]}(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_n<0}\frac{c_n^2}{\alpha_n^2}, & \text{for } \lambda\leq 0\\ \sum_{0\leq\lambda_n<\lambda}\frac{c_n^2}{\alpha_n^2} & \text{for } \lambda>0,\end{cases}$$

$$\mu_{12,[a,b]}(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_n<0}\frac{c_nd_n}{\alpha_n^2}, & \text{for } \lambda\leq 0\\ \sum_{0\leq\lambda_n<\lambda}\frac{c_nd_n}{\alpha_n^2} & \text{for } \lambda>0,\end{cases}$$

$$\mu_{12,[a,b]}(\lambda)=\mu_{21,[a,b]}(\lambda),$$

$$\mu_{22,[a,b]}(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_n<0}\frac{d_n^2}{\alpha_n^2}, & \text{for } \lambda\leq 0\\ \sum_{0\leq\lambda_n<\lambda}\frac{d_n^2}{\alpha_n^2} & \text{for } \lambda>0.\end{cases}$$

From (11), we obtain

$$\int_a^c \|f(x)\|_E^2\,dx+\beta\int_c^b \|f(x)\|_E^2\,dx=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)F_j(\lambda)\,d\mu_{ij,[a,b]}(\lambda), \tag{12}$$

where

$$F_1(\lambda) = \int_a^c (f(x),\psi_1(x,\lambda))_E\,dx+\beta\int_c^b (f(x),\psi_1(x,\lambda))_E\,dx,$$

$$F_2(\lambda) = \int_a^c (f(x),\psi_2(x,\lambda))_E\,dx+\beta\int_c^b (f(x),\psi_2(x,\lambda))_E\,dx.$$

Now we will prove a lemma, but first we recall some definitions.

A function f defined on an interval $[a_1,b_1]$ is said to be of *bounded variation* if there is a constant $C>0$ such that

$$\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\leq C$$

for every partition

$$a_1=x_0<x_1<...<x_n=b_1 \tag{13}$$

of $[a_1,b_1]$ by points of subdivision $x_0,x_1,...,x_n$.

Let f be a function of bounded variation. Then, by the *total variation* of f on $[a_1,b_1]$, denoted by $\overset{b_1}{\underset{a_1}{V}}(f)$, we mean the quantity

$$\overset{b_1}{\underset{a_1}{V}}(f):=\sup\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|,$$

where the least upper bound is taken over all (finite) partitions (13) of the interval $[a_1,b_1]$ (see [17]).

Lemma 5 *There exists a positive constant* $\Lambda = \Lambda(\xi)$, $\xi > 0$ *such that*

$$\overset{\xi}{\underset{-\xi}{V}} \left\{ \mu_{ij,[a,b]}(\lambda) \right\} < \Lambda \ \ (i,j = 1,2), \tag{14}$$

where Λ *does not depend on* b.

Proof From (10), we have

$$\psi_{ij}^{(1)}(c_0, \lambda) = \delta_{ij},$$

where δ_{ij} $(i,j = 1,2)$ is the Kronecker delta. The functions $\psi_{ij}^{(1)}(x,\lambda)$ $(i,j = 1,2)$ are continuous both with respect to $x \in [a,c)$ and $\lambda \in \mathbb{R}$. Thus, for any $\varepsilon > 0$ there exists a $c_0 < k < c$ such that

$$\left| \psi_{ij}^{(1)}(x,\lambda) - \delta_{ij} \right| < \varepsilon, \ \varepsilon > 0, \ |\lambda| < \xi, \ x \in [c_0, k]. \tag{15}$$

Let

$$f_k(x) = \begin{pmatrix} f_{k1}(x) \\ f_{k2}(x) \end{pmatrix}$$

be a nonnegative vector-valued function such that $f_{k1}(x)$ vanishes outside the interval $[c_0, k]$ with

$$\int_{c_0}^{k} f_{k1}(x)\, dx = 1, \tag{16}$$

and $f_{k2}(x) = 0$. Set

$$\begin{aligned} F_{ik}(\lambda) &= \int_{c_0}^{k} (f_k(x), \psi_i)_E \, dx \\ &= \int_{c_0}^{k} f_{k1}(x)\, \psi_{i1}^{(1)}(x,\lambda)\, dx \quad (i = 1,2). \end{aligned}$$

Using (15) and (16), we obtain

$$|F_{1k}(\lambda) - 1| < \varepsilon, \ |F_{2k}(\lambda)| < \varepsilon, \ |\lambda| < \xi. \tag{17}$$

Now, by applying the Parseval equality (12) to $f_k(x)$, we get

$$\begin{aligned} \int_{c_0}^{k} f_{k1}^2(x)\, dx \ &\geq \int_{-\xi}^{\xi} F_{1k}^2(\lambda)\, d\mu_{11,[a,b]}(\lambda) + 2\int_{-\xi}^{\xi} F_{1k}(\lambda) F_{2k}(\lambda)\, d\mu_{12,[a,b]}(\lambda) \\ &+ \int_{-\xi}^{\xi} F_{2k}^2(\lambda)\, d\mu_{22,[a,b]}(\lambda) \geq \int_{-\xi}^{\xi} F_{1k}^2(\lambda)\, d\mu_{11,[a,b]}(\lambda) - 2\int_{-\xi}^{\xi} |F_{1k}(\lambda)|\, |F_{2k}(\lambda)| \left| d\mu_{12,[a,b]}(\lambda) \right|. \end{aligned}$$

From (17), we have

$$\begin{aligned} \int_{c_0}^{k} f_{k1}^2(x)\, dx \ &> \int_{-\xi}^{\xi} (1-\varepsilon)^2\, d\mu_{11,[a,b]}(\lambda) - 2\int_{-\xi}^{\xi} \varepsilon(1+\varepsilon) \left| d\mu_{12,[a,b]}(\lambda) \right| \\ &= (1-\varepsilon)^2 \left(\mu_{11,[a,b]}(\xi) - \mu_{11,[a,b]}(-\xi) \right) - 2\varepsilon(1+\varepsilon) \overset{\xi}{\underset{-\xi}{V}} \left\{ \mu_{12,[a,b]}(\lambda) \right\}. \end{aligned}$$

Since

$$\underset{-\xi}{\overset{\xi}{V}}\left\{\mu_{12,[a,b]}(\lambda)\right\} \leq \frac{1}{2}\left[\begin{array}{c}\mu_{11,[a,b]}(\xi)-\mu_{11,[a,b]}(-\xi) \\ +\mu_{22,[a,b]}(\xi)-\mu_{22,[a,b]}(-\xi)\end{array}\right], \tag{18}$$

we get

$$\begin{aligned}\int_{c_0}^{k} f_{k1}^2(x)\,dx \quad &> \quad (1-3\varepsilon)\left\{\mu_{11,[a,b]}(\xi)-\mu_{11,[a,b]}(-\xi)\right\} \\ &\quad -\varepsilon(1+\varepsilon)\left\{\mu_{22,[a,b]}(\xi)-\mu_{22,[a,b]}(-\xi)\right\}. \end{aligned} \tag{19}$$

Let

$$g_k(x)=\left(\begin{array}{c} g_{k1}(x) \\ g_{k2}(x)\end{array}\right)$$

be a nonnegative vector-valued function such that $g_{k2}(x)$ vanishes outside the interval $[c_0,k]$ with $\int_{c_0}^{k} g_{k2}(x)\,dx=1$, and $g_{k1}(x)=0$. Similar arguments apply to the function $g_k(x)$, and we obtain

$$\begin{aligned}\int_{c_0}^{k} g_{k2}^2(x)\,dx \quad &> \quad (1-3\varepsilon)\left\{\mu_{22,[a,b]}(\xi)-\mu_{22,[a,b]}(-\xi)\right\} \\ &\quad -\varepsilon(1+\varepsilon)\left\{\mu_{11,[a,b]}(\xi)-\mu_{11,[a,b]}(-\xi)\right\}. \end{aligned} \tag{20}$$

If we add the inequalities (19) and (20), then we get

$$\int_{c_0}^{k}\left\{f_{k1}^2(x)+g_{k2}^2(x)\right\}dx>\left(1-4\varepsilon-\varepsilon^2\right)\left\{\begin{array}{c}\mu_{11,[a,b]}(\xi)-\mu_{11,[a,b]}(-\xi) \\ +\mu_{22,[a,b]}(\xi)-\mu_{22,[a,b]}(-\xi)\end{array}\right\}.$$

If we choose $\varepsilon>0$ such that $1-4\varepsilon-\varepsilon^2>0$, then we obtain the assertion of the lemma for the functions $\mu_{11,[a,b]}(-\xi)$ and $\mu_{22,[a,b]}(-\xi)$ relying on their monotonicity. From (18), we have the assertion of the lemma for the function $\mu_{12,[a,b]}(-\xi)$. □

Now we recall Helly's theorems.

Theorem 6 ([17]) *Let* $(w_n)_{n\in\mathbb{N}}$ $(\mathbb{N}:=\{1,2,...\}$ *be a uniformly bounded sequence of real nondecreasing functions on a finite interval* $a_0\leq\lambda\leq b_0$. *Then there exists a subsequence* $(w_{n_k})_{k\in\mathbb{N}}$ *and a nondecreasing function* w *such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda)=w(\lambda),\ a_0\leq\lambda\leq b_0.$$

Theorem 7 ([17]) *Assume that* $(w_n)_{n\in\mathbb{N}}$ *is a real, uniformly bounded sequence of nondecreasing functions on a finite interval* $a_0\leq\lambda\leq b_0$, *and suppose that*

$$\lim_{n\to\infty} w_n(\lambda)=w(\lambda),\ a_0\leq\lambda\leq b_0.$$

If f *is any continuous function on* $a_0\leq\lambda\leq b_0$, *then*

$$\lim_{n\to\infty}\int_{a_0}^{b_0} f(\lambda)\,dw_n(\lambda)=\int_{a_0}^{b_0} f(\lambda)\,dw(\lambda).$$

Now let ϱ be any nondecreasing function on $-\infty < \lambda < \infty$. Denote by $L^2_\varrho(\mathbb{R})$ the Hilbert space of all functions $f : \mathbb{R} \to \mathbb{R}$ that are measurable with respect to the Lebesgue–Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} f^2(\lambda)\, d\varrho(\lambda) < \infty,$$

with the inner product

$$(f,g)_\varrho := \int_{-\infty}^{\infty} f(\lambda)\, g(\lambda)\, d\varrho(\lambda).$$

We introduce the direct sum Hilbert space $H := L^2(\Omega_1; E) \dot{+} L^2(\Omega_2; E)$, $(\Omega_1 = (-\infty, c),\ \Omega_2 = (c, \infty))$ with the inner product

$$\langle f,g\rangle_H := \int_{-\infty}^{c} (f(x), g(x))_E\, dx + \beta \int_{c}^{\infty} (f(x), g(x))_E\, dx,$$

where

$$\begin{aligned}
f(x) &= \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},\ g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, \\
f_1(x) &= \begin{cases} f_1^{(1)}(x), & x \in \Omega_1 \\ f_1^{(2)}(x), & x \in \Omega_2 \end{cases},\quad f_2(x) = \begin{cases} f_2^{(1)}(x), & x \in \Omega_1 \\ f_2^{(2)}(x), & x \in \Omega_2 \end{cases}, \\
g_1(x) &= \begin{cases} g_1^{(1)}(x), & x \in \Omega_1 \\ g_1^{(2)}(x), & x \in \Omega_2 \end{cases},\quad g_2(x) = \begin{cases} g_2^{(1)}(x), & x \in \Omega_1 \\ g_2^{(2)}(x), & x \in \Omega_2. \end{cases}
\end{aligned}$$

The main results of this paper are the following three theorems.

Theorem 8 *Let $f \in H$. Then there exist monotonic functions $\mu_{11}(\lambda)$ and $\mu_{22}(\lambda)$, which are bounded over every finite interval, and a function $\mu_{12}(\lambda)$, which is of bounded variation over every finite interval with the property*

$$\int_{-\infty}^{c} \|f(x)\|_E^2\, dx + \beta \int_{c}^{\infty} \|f(x)\|_E^2\, dx = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\, F_j(\lambda)\, d\mu_{ij}(\lambda), \tag{21}$$

which is called the Parseval equality, where

$$F_i(\lambda) = \lim_{n\to\infty} \left\{ \begin{array}{c} \int_{-n}^{c} (f(x), \psi_i(x,\lambda))_E\, dx \\ +\beta \int_{c}^{n} (f(x), \psi_i(x,\lambda))_E\, dx \end{array} \right\} \quad (i = 1, 2).$$

We note that the matrix-valued function $\mu = (\mu_{ij})_{i,j=1}^2$ $(\mu_{12} = \mu_{21})$ is called a *spectral function* for the singular Dirac system $\tau(y) = \lambda y(x)$ $(x \in \Omega_1 \cup \Omega_2)$ with the impulsive conditions (3).

Proof Assume that the vector-valued function $f_n(x) = \begin{pmatrix} f_{1n}(x) \\ f_{2n}(x) \end{pmatrix}$,

$$f_{1n}(x) = \begin{cases} f_{1n}^{(1)}(x), & x \in \Omega_1 \\ f_{1n}^{(2)}(x), & x \in \Omega_2 \end{cases},\quad f_{2n}(x) = \begin{cases} f_{2n}^{(1)}(x), & x \in \Omega_1 \\ f_{2n}^{(2)}(x), & x \in \Omega_2 \end{cases},$$

satisfies the following conditions:

1) $f_n(x)$ vanishes outside the interval $[-n,c)\cup(c,n]$, where $a<-n<c<n<b$.

2) The vector-valued functions $f_n(x)$ and $f_n'(x)$ are continuous.

3) $f_n(x)$ satisfies the conditions given by (2)–(3).

If we apply the Parseval equality to $f_n(x)$, then we get

$$\int_{-n}^{c}\|f_n(x)\|_E^2\,dx+\beta\int_c^n\|f_n(x)\|_E^2\,dx$$

$$=\sum_{k=-\infty}^{\infty}\frac{1}{\alpha_k^2}\left\{\begin{array}{c}\int_a^c(f_n(x),y_k(x))_E\,dx\\+\beta\int_c^b(f_n(x),y_k(x))_E\,dx\end{array}\right\}^2. \tag{22}$$

Then, via integrating by parts, we obtain

$$\begin{aligned}
&\int_a^c(f_n(x),y_k(x))_E\,dx+\beta\int_c^b(f_n(x),y_k(x))_E\,dx\\
=&\ \frac{1}{\lambda_k}\int_a^c f_{1n}^{(1)}(x)\left[-y_{k2}^{(1)\prime}(x)+p(x)y_{k1}^{(1)}(x)\right]dx\\
&+\frac{1}{\lambda_k}\beta\int_c^b f_{1n}^{(2)}(x)\left[-y_{k2}^{(2)\prime}(x)+p(x)y_{k1}^{(2)}(x)\right]dx\\
&+\frac{1}{\lambda_k}\int_a^c f_{2n}^{(1)}(x)\left[y_{k1}^{(1)\prime}(x)+r(x)y_{k2}^{(1)}(x)\right]dx\\
&+\frac{1}{\lambda_k}\beta\int_c^b f_{2n}^{(2)}(x)\left[y_{k1}^{(2)\prime}(x)+r(x)y_{k2}^{(2)}(x)\right]dx\\
=&\ \frac{1}{\lambda_k}\int_a^c\left[-f_{2n}^{(1)\prime}(x)+p(x)f_{1n}^{(1)}(x)\right]y_{k1}^{(1)}(x)\,dx\\
&+\frac{1}{\lambda_k}\beta\int_c^b\left[-f_{2n}^{(2)\prime}(x)+p(x)f_{1n}^{(2)}(x)\right]y_{k1}^{(2)}(x)\,dx\\
&+\frac{1}{\lambda_k}\int_a^c\left[f_{1n}^{(1)\prime}(x)+r(x)f_{2n}^{(1)}(x)\right]y_{k2}^{(1)}(x)\,dx\\
&+\frac{1}{\lambda_k}\beta\int_c^b\left[f_{1n}^{(2)\prime}(x)+r(x)f_{2n}^{(2)}(x)\right]y_{k2}^{(2)}(x)\,dx.
\end{aligned}$$

Thus, we have

$$\sum_{|\lambda_k|\geq s}\frac{1}{\alpha_k^2}\left\{\begin{array}{c}\int_{-n}^{c}(f_n(x),y_k(x))_E\,dx\\+\beta\int_c^n(f_n(x),y_k(x))_E\,dx\end{array}\right\}^2$$

$$\leq \frac{1}{s^2}\sum_{|\lambda_k|\geq s}\frac{1}{\alpha_k^2}\left\{\begin{array}{c} \int_{-n}^{c}\left[-f_{2n}^{(1)\prime}(x)+p(x)f_{1n}^{(1)}(x)\right]y_{k1}^{(1)}(x)\,dx \\ +\beta\int_{c}^{n}\left[-f_{2n}^{(2)\prime}(x)+p(x)f_{1n}^{(2)}(x)\right]y_{k1}^{(2)}(x)\,dx \\ +\int_{-n}^{c}\left[f_{1n}^{(1)\prime}(x)+r(x)f_{2n}^{(1)}(x)\right]y_{k2}^{(1)}(x)\,dx \\ +\beta\int_{c}^{n}\left[f_{1n}^{(2)\prime}(x)+r(x)f_{2n}^{(2)}(x)\right]y_{k2}^{(2)}(x)\,dx \end{array}\right\}^2$$

$$\leq \frac{1}{s^2}\sum_{k=-\infty}^{\infty}\frac{1}{\alpha_k^2}\left\{\begin{array}{c} \int_{-n}^{c}\left[-f_{2n}^{(1)\prime}(x)+p(x)f_{1n}^{(1)}(x)\right]y_{k1}^{(1)}(x)\,dx \\ +\beta\int_{c}^{n}\left[-f_{2n}^{(2)\prime}(x)+p(x)f_{1n}^{(2)}(x)\right]y_{k1}^{(2)}(x)\,dx \\ +\int_{-n}^{c}\left[f_{1n}^{(1)\prime}(x)+r(x)f_{2n}^{(1)}(x)\right]y_{k2}^{(1)}(x)\,dx \\ +\beta\int_{c}^{n}\left[f_{1n}^{(2)\prime}(x)+r(x)f_{2n}^{(2)}(x)\right]y_{k2}^{(2)}(x)\,dx \end{array}\right\}^2$$

$$= \frac{1}{s^2}\left\{\begin{array}{c} \int_{-n}^{c}\left[-f_{2n}^{(1)\prime}(x)+p(x)f_{1n}^{(1)}(x)\right]^2 dx \\ +\beta\int_{c}^{n}\left[-f_{2n}^{(2)\prime}(x)+p(x)f_{1n}^{(2)}(x)\right]^2 dx \\ +\int_{-n}^{c}\left[f_{1n}^{(1)\prime}(x)+r(x)f_{2n}^{(1)}(x)\right]^2 dx \\ +\beta\int_{c}^{n}\left[f_{1n}^{(2)\prime}(x)+r(x)f_{2n}^{(2)}(x)\right]^2 dx \end{array}\right\}.$$

By using (22), we obtain

$$\left|\int_{-n}^{c}\|f_n(x)\|_E^2\,dx+\beta\int_{c}^{n}\|f_n(x)\|_E^2\,dx-\sum_{-N\leq\lambda_k\leq N}\frac{1}{\alpha_k^2}\left\{\langle f_n(.),y_k(.)\rangle_H\right\}^2\right|$$

$$\leq \sum_{-N\leq\lambda_k\leq N}\frac{1}{\alpha_k^2}\left\{\langle f_n(.),(c_k\psi_1(.,\lambda_k)+d_k\psi_2(.,\lambda_k))\rangle_H\right\}^2$$

$$=\int_{-N}^{N}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{in}(\lambda)\,d\mu_{ij,[a,b]}(\lambda),$$

where

$$F_{in}(\lambda)=\langle f_n(.),\psi_i(.,\lambda)\rangle_H\ \ (i=1,2).$$

Consequently, we get

$$\begin{aligned}
&\left|\begin{array}{c}\int_{-n}^{c}\|f_n(x)\|_E^2\,dx+\beta\int_c^n\|f_n(x)\|_E^2\,dx\\ -\int_{-N}^{N}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{in}(\lambda)\,d\mu_{ij,[a,b]}(\lambda)\end{array}\right|\\
&\le\tfrac{1}{N^2}\int_{-n}^{c}\left[-f_{2n}^{(1)\prime}(x)+p(x)\,f_{1n}^{(1)}(x)\right]^2dx\\
&+\beta\tfrac{1}{N^2}\int_{c}^{n}\left[-f_{2n}^{(2)\prime}(x)+p(x)\,f_{1n}^{(2)}(x)\right]^2dx\\
&+\tfrac{1}{N^2}\int_{-n}^{c}\left[f_{1n}^{(1)\prime}(x)+r(x)\,f_{2n}^{(1)}(x)\right]^2dx\\
&+\tfrac{1}{N^2}\beta\int_{c}^{n}\left[f_{1n}^{(2)\prime}(x)+r(x)\,f_{2n}^{(2)}(x)\right]^2dx.
\end{aligned}\tag{23}$$

By Lemma 5 and Theorems 6 and 7, we can find sequences $\{a_s\}$ and $\{b_s\}$ $(a_s\to-\infty$ and $b_s\to+\infty,\ s\to\infty)$ such that the functions $\mu_{ij,[a_s,b_s]}(\lambda)$ converge to a function $\mu_{ij}(\lambda)$ $(i,j=1,2)$. Passing to the limit with respect to $\{a_s\}$ and $\{b_s\}$ in (23), we have

$$\begin{aligned}
\left|\begin{array}{c}\int_{-n}^{c}\|f_n(x)\|_E^2\,dx+\beta\int_c^n\|f_n(x)\|_E^2\,dx\\ -\int_{-N}^{N}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{jn}(\lambda)\,d\mu_{ij}(\lambda)\end{array}\right| \le\ &\frac{1}{N^2}\int_{-n}^{c}\left[-f_{2n}^{(1)\prime}(x)+p(x)\,f_{1n}^{(1)}(x)\right]^2dx\\
&+\beta\frac{1}{N^2}\int_{c}^{n}\left[-f_{2n}^{(2)\prime}(x)+p(x)\,f_{1n}^{(2)}(x)\right]^2dx\\
&+\frac{1}{N^2}\int_{-n}^{c}\left[f_{1n}^{(1)\prime}(x)+r(x)\,f_{2n}^{(1)}(x)\right]^2dx\\
&+\frac{1}{N^2}\beta\int_{c}^{n}\left[f_{1n}^{(2)\prime}(x)+r(x)\,f_{2n}^{(2)}(x)\right]^2dx.
\end{aligned}$$

As $N\to\infty$, we get

$$\int_{-n}^{c}\|f_n(x)\|_E^2\,dx+\beta\int_c^n\|f_n(x)\|_E^2\,dx=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{jn}(\lambda)\,d\mu_{ij}(\lambda)\,.$$

Now let

$$f(x)=\left\{\begin{array}{ll}f^{(1)}(x), & x\in\Omega_1\\ f^{(2)}(x), & x\in\Omega_2\end{array}\right.,$$

$f(.)\in H$. Choose vector-valued functions

$$f_\eta(x)=\left\{\begin{array}{ll}f_\eta^{(1)}(x), & x\in\Omega_1\\ f_\eta^{(2)}(x), & x\in\Omega_2\end{array}\right.,$$

satisfying conditions 1–3 and such that

$$\lim_{\eta\to\infty}\int_{-\infty}^{c}\left\|f^{(1)}(x)-f_\eta^{(1)}(x)\right\|_E^2dx+\beta\lim_{\eta\to\infty}\int_{c}^{\infty}\left\|f^{(2)}(x)-f_\eta^{(2)}(x)\right\|_E^2dx=0.$$

Let

$$F_{i\eta}(\lambda) = \int_{-\infty}^{c} \left(f_\eta^{(1)}(x), \psi_i(x,\lambda)\right)_E dx$$

$$+\beta \int_{c}^{\infty} \left(f_\eta^{(2)}(x), \psi_i(x,\lambda)\right)_E dx \quad (i=1,2).$$

Then we have

$$\int_{-\infty}^{c} \left\|f_\eta^{(1)}(x)\right\|_E^2 dx + \beta \int_{c}^{\infty} \left\|f_\eta^{(2)}(x)\right\|_E^2 dx = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{i\eta}(\lambda) F_{j\eta}(\lambda) d\mu_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{c} \left\|f_{\eta_1}^{(1)}(x) - f_{\eta_2}^{(1)}(x)\right\|_E^2 dx + \beta \int_{c}^{\infty} \left\|f_{\eta_1}^{(2)}(x) - f_{\eta_2}^{(2)}(x)\right\|_E^2 dx \to 0$$

as $\eta_1, \eta_2 \to \infty$, we have

$$\int_{-\infty}^{\infty} \sum_{i=1}^{2} \left(F_{i\eta_1}(\lambda) F_{j\eta_1}(\lambda) - F_{i\eta_2}(\lambda) F_{j\eta_2}(\lambda)\right) d\mu_{ij}(\lambda)$$

$$= \int_{-\infty}^{c} \left\|f_{\eta_1}^{(1)}(x) - f_{\eta_2}^{(1)}(x)\right\|_E^2 dx + \beta \int_{c}^{\infty} \left\|f_{\eta_1}^{(2)}(x) - f_{\eta_2}^{(2)}(x)\right\|_E^2 dx \to 0$$

as $\eta_1, \eta_2 \to \infty$. Therefore, there is a limit function $F_i \ (i=1,2)$ that satisfies

$$\int_{-\infty}^{c} \left\|f^{(1)}(x)\right\|_E^2 dx + \beta \int_{c}^{\infty} \left\|f^{(2)}(x)\right\|_E^2 dx = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda) d\mu_{ij}(\lambda),$$

by the completeness of the space $L_\mu^2(\mathbb{R})$.

Now we will show that the sequence $(K_{\eta i}) \ (i=1,2)$ given by

$$K_{\eta i}(\lambda) = \int_{-\eta}^{c} (f^{(1)}(x), \psi_i(x,\lambda))_E dx + \int_{c}^{\eta} (f^{(2)}(x), \psi_i(x,\lambda))_E dx \ (i=1,2)$$

converges as $\eta \to \infty$ to $F_i \ (i=1,2)$ in the metric of the space $L_\mu^2(\mathbb{R})$. Let g be another function in H. By similar arguments, $G(\lambda)$ can be defined by g. It is obvious that

$$\int_{-\infty}^{c} \left\|f^{(1)}(x) - g^{(1)}(x)\right\|_E^2 dx + \beta \int_{c}^{\infty} \left\|f^{(2)}(x) - g^{(2)}(x)\right\|_E^2 dx$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left\{(F_i(\lambda) - G_i(\lambda))(F_j(\lambda) - G_j(\lambda))\right\} d\mu_{ij}(\lambda).$$

Let

$$g(x) = \begin{cases} f(x), & x \in [-\eta, c) \cup (c, \eta] \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left\{ \left(F_i(\lambda) - K_{\eta i}(\lambda) \right) \left(F_j(\lambda) - K_{\eta j}(\lambda) \right) \right\} d\mu_{ij}(\lambda)$$

$$= \int_{-\infty}^{-\eta} \left\| f^{(1)}(x) \right\|_E^2 dx + \beta \int_{\eta}^{\infty} \left\| f^{(2)}(x) \right\|_E^2 dx \to 0 \ \ (\eta \to \infty),$$

which proves that $(K_{\eta i})$ converges to F_i $(i = 1, 2)$ in $L_\mu^2(\mathbb{R})$ as $\eta \to \infty$. □

Theorem 9 *Suppose that the functions*

$$f(x) = \begin{cases} f^{(1)}(x), & x \in \Omega_1 \\ f^{(2)}(x), & x \in \Omega_2 \end{cases}, \quad g(x) = \begin{cases} g^{(1)}(x), & x \in \Omega_1 \\ g^{(2)}(x), & x \in \Omega_2 \end{cases},$$

$f, g \in H,$ *and* $F_i(\lambda),$ $G_i(\lambda)$ $(i = 1, 2)$ *are their Fourier transforms. Then we have*

$$\int_{-\infty}^{c} \left(f^{(1)}(x), g^{(1)}(x) \right)_E dx + \beta \int_{c}^{\infty} \left(f^{(2)}(x), g^{(2)}(x) \right)_E dx$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) G_j(\lambda) d\mu_{ij}(\lambda),$$

which is called the generalized Parseval equality.

Proof It is clear that $F \mp G$ are transforms of $f \mp g$. Therefore, we have

$$\int_{-\infty}^{c} \left\| f^{(1)}(x) + g^{(1)}(x) \right\|_E^2 dx + \beta \int_{c}^{\infty} \left\| f^{(2)}(x) + g^{(2)}(x) \right\|_E^2 dx$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(F_i(\lambda) + G_i(\lambda) \right) \left(F_j(\lambda) + G_j(\lambda) \right) d\mu_{ij}(\lambda)$$

and

$$\int_{-\infty}^{c} \left\| f^{(1)}(x) - g^{(1)}(x) \right\|_E^2 dx + \beta \int_{c}^{\infty} \left\| f^{(2)}(x) - g^{(2)}(x) \right\|_E^2 dx$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(F_i(\lambda) - G_i(\lambda) \right) \left(F_j(\lambda) - G_j(\lambda) \right) d\mu_{ij}(\lambda).$$

By subtracting one of these equalities from the other one, we obtain the desired result. □

Theorem 10 *Let*

$$f(x) = \begin{cases} f^{(1)}(x), & x \in \Omega_1 \\ f^{(2)}(x), & x \in \Omega_2 \end{cases}, \ f \in H.$$

Then the integrals

$$\int_{-\infty}^{\infty} F_i(\lambda) \psi_j(x, \lambda) d\mu_{ij}(\lambda) \ \ (i, j = 1, 2)$$

converge in H. Consequently, we have

$$f(x)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)\,\psi_j(x,\lambda)\,d\mu_{ij}(\lambda),$$

which is called the expansion theorem.

Proof Take any function $f_s\in H$ and any positive number s, and set

$$f_s(x)=\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\,\psi_j(x,\lambda)\,d\mu_{ij}(\lambda),$$

where

$$f_s(x)=\begin{cases} f_s^{(1)}(x), & x\in\Omega_1\\ f_s^{(2)}(x), & x\in\Omega_2.\end{cases}$$

Let

$$g(x)=\begin{cases} g^{(1)}(x), & x\in\Omega_1\\ g^{(2)}(x), & x\in\Omega_2\end{cases},\ g\in H$$

be a vector-valued function that is equal to zero outside the finite interval $[-\tau,c)\cup(c,\tau]$, where $\tau\geq s$. Thus, we obtain

$$\begin{aligned}
&\int_{-\tau}^{c}\left(f_s^{(1)}(x),g^{(1)}(x)\right)_E dx+\beta\int_{c}^{\tau}\left(f_s^{(2)}(x),g^{(2)}(x)\right)_E dx\\
=\;&\int_{-\tau}^{c}\left(\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\,\psi_j(x,\lambda)\,d\mu_{ij}(\lambda),g^{(1)}(x)\right)_E dx\\
&+\beta\int_{c}^{\tau}\left(\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\,\psi_j(x,\lambda)\,d\mu_{ij}(\lambda),g^{(2)}(x)\right)_E dx\\
=\;&\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\left\{\begin{array}{c}\int_{-\tau}^{c}\left(g^{(1)}(x),\psi_j(x,\lambda)\right)_E dx\\ +\beta\int_{c}^{\tau}\left(g^{(2)}(x),\psi_j(x,\lambda)\right)_E dx\end{array}\right\}\\
=\;&\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\,G_j(\lambda)\,d\mu_{ij}(\lambda).
\end{aligned}\tag{24}$$

From Theorem 9, we get

$$\int_{-\infty}^{c}\left(f^{(1)}(x),g^{(1)}(x)\right)_E dx+\beta\int_{c}^{\infty}\left(f^{(2)}(x),g^{(2)}(x)\right)_E dx$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)\,G_j(\lambda)\,d\mu_{ij}(\lambda).\tag{25}$$

By (24) and (25), we have

$$\langle f - f_s, g \rangle_H = \int_{|\lambda|>s} \sum_{i,j=1}^{2} F_i(\lambda) G_j(\lambda) \, d\mu_{ij}(\lambda).$$

Applying this equality to the function

$$g(x) = \begin{cases} f(x) - f_s(x), & x \in [-s, c) \cup (c, s] \\ 0, & \text{otherwise,} \end{cases}$$

we get

$$\|f - f_s\|_H^2 = \int_{|\lambda|>s} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda) \, d\mu_{ij}(\lambda).$$

Letting $s \to \infty$ yields the expansion result. □

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