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Limit behaviors of nonoscillatory solutions of three-dimensional time scale systems

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Abstract: In this article, we investigate the oscillatory behavior of a three-dimensional system of dynamic equations on an unbounded time scale. A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers. An example is given to illustrate some of the results.

Key words: Three-dimensional dynamical system, time scales, fixed points, existence of nonoscillatory solutions, classification

1. Introduction

In this paper, we study the nonlinear system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = q(t)g(z(t)) \\ z^{\Delta}(t) = -r(t)h(x(t)) \end{cases} \tag{1}$$

on $[t_0, \infty)_{\mathbb{T}}$ such that $t_0 \in \mathbb{T}$ and $t_0 \geq 0$, where $p, q \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$, $r \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, and

$$\int_{t_0}^{\infty} p(s)\, \Delta s = \infty = \int_{t_0}^{\infty} q(s)\, \Delta s. \tag{2}$$

We also assume that $f, g, h \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing functions such that $uf(u) > 0$, $ug(u) > 0$ and $uh(u) > 0$ for $u \neq 0$. Here we only consider unbounded time scales, and by $t \geq t_0$, we mean $t \in [t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Classifications of nonoscillatory solutions for some other versions of system (1) are also considered in [8–11].

The theory of time scales was initiated by Stefan Hilger in his PhD thesis [6] in 1988. The main purpose was to unify and extend continuous and discrete cases in one comprehensive theory. Since 1988, there has been much research in many areas of time scales including the classification and existence of dynamical systems. For an introduction to the theory of time scales, we refer readers to the books written by Bohner and Peterson [2, 3].

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By a solution of (1), we mean a collection of functions, where $x, y, z \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}), T \geq t_0$ and (x, y, z) satisfies system (1) for all large $t \geq T$. A solution (x, y, z) of system (1) is said to be proper if

$$\sup\{|x(s)|, |y(s)|, |z(s)| : s \in [t,\infty)_{\mathbb{T}}\} > 0$$

for $t \geq t_0$. A proper solution (x, y, z) of (1) is said to be nonoscillatory if the component functions x, y, and z are all nonoscillatory, i.e. either eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

Suppose that N is a set of all nonoscillatory solutions (x, y, z) of system (1). In [1], Akin et al. showed that any nonoscillatory solution (x, y, z) of system (1) belongs to one of the following classes:

$$N^+ := \{(x, y, z) \in N : \quad \operatorname{sgn} x(t) = \operatorname{sgn} y(t) = \operatorname{sgn} z(t), \quad t \geq t_0\}$$

$$N^- := \{(x, y, z) \in N : \quad \operatorname{sgn} x(t) = \operatorname{sgn} z(t) \neq \operatorname{sgn} y(t), \quad t \geq t_0\}.$$

In the literature, solutions in N^+ and N^- are known as *Type* (a) and *Type* (b) solutions, respectively. The following lemma describes the long-term behavior of two of the components of a nonoscillatory solution.

Lemma 1.1 *[1, Lemma 4.2] Assume that (x, y, z) is a nonoscillatory solution in N^-. Then*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0.$$

The following lemma gives us the criteria for relative compactness.

Lemma 1.2 *[5, Lemma 2.2] Suppose that $X \subseteq BC[t_0,\infty)_{\mathbb{T}}$ is bounded and uniformly Cauchy. Further, suppose that X is equicontinuous on $[t_0, t_1]_{\mathbb{T}}$ for any $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then X is relatively compact.*

We also give Schauder's fixed point theorem, proved by Schauder in the 1930s, and Knaster's fixed point theorem; see [7, 12], respectively.

Theorem 1.3 (Knaster's fixed point theorem) *If $(M, \leq)$ is a complete lattice and $T : M \to M$ is order-preserving (also called monotone or isotone), then T has a fixed point. In fact, the set of fixed points of T is a complete lattice.*

Theorem 1.4 (Schauder's fixed point theorem) *Let M be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose that $T : M \to M$ is a compact operator. Then T has a fixed point.*

In the next section, we examine the solutions in each class N^+ and N^-. We used fixed point theorems to establish our results. We provide an example of one of the results and close with open problems.

2. Classification and existence in N^+ and N^-

This section deals with the existence of nonoscillatory solutions of system (1) by using well-known fixed point theorems such as the Knaster's fixed point theorem and Schauder's fixed point theorem. For the sake of simplicity in our main results, set

$$I_1 = \int_{t_0}^{\infty} p(t) f\left(k_1 - \int_t^{\infty} q(s) g\left(k_2 + k_3 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$

$$I_2 = \int_{t_0}^{\infty} q(t) g\left(k_4 + \int_t^{\infty} r(s) h\left(k_5 \int_{t_0}^{s} p(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$

$$I_3 = \int_{t_0}^{\infty} r(t) h\left(\int_{t_0}^{t} p(s) f\left(k_6 \int_{t_0}^{s} q(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$

$$I_4 = \int_{t_0}^{\infty} p(t) f\left(\int_{t_0}^{t} q(s) g\left(k_7 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$

$$I_5 = \int_{t_0}^{\infty} p(t) f\left(\int_t^{\infty} q(s) g\left(k_8 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$

$$I_6 = \int_{t_0}^{\infty} p(t) f\left(k_9 \int_{t_0}^{t} q(s)\Delta s\right)\Delta t,$$

$$I_7 = \int_{t_0}^{\infty} q(t) g\left(\int_t^{\infty} r(s) h\left(k_{10} \int_s^{\infty} p(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$

$$I_8 = \int_{t_0}^{\infty} q(t) g\left(k_{11} + k_{12} \int_t^{\infty} r(s)\Delta s\right)\Delta t,$$

$$I_9 = \int_{t_0}^{\infty} r(t) h\left(k_{13} \int_{t_0}^{t} p(s)\Delta s\right)\Delta t,$$

$$R(t,\infty) = \int_t^{\infty} r(s)\Delta s$$

for all nonnegative real numbers k_i where $1 \le i \le 13$.

2.1. Existence in N^+

Suppose that (x,y,z) is a nonoscillatory solution of system (1) in N^+ such that $x > 0$ eventually. (The case $x < 0$ can be shown similarly.) Then by the equations of system (1), we have that x and y are positive increasing functions, and z is a positive decreasing function. We conclude that $x \to c_1$ or $x \to \infty$, $y \to c_2$ or $y \to \infty$, and $z \to c_3$ or $z \to 0$, where $0 < c_1, c_2, c_3 < \infty$. Consequently, in light of this information, we have the following subclasses:

$$N^+_{B,B,B} := \left\{(x,y,z) \in N^+ : \lim_{t\to\infty} |x(t)| = c_1,\ \lim_{t\to\infty} |y(t)| = c_2,\ \lim_{t\to\infty} |z(t)| = c_3\right\}$$

$$N^+_{B,B,0} := \left\{(x,y,z) \in N^+ : \lim_{t\to\infty} |x(t)| = c_1,\ \lim_{t\to\infty} |y(t)| = c_2,\ \lim_{t\to\infty} |z(t)| = 0\right\}$$

$$N^+_{B,\infty,B} := \left\{(x,y,z) \in N^+ : \lim_{t\to\infty} |x(t)| = c_1,\ \lim_{t\to\infty} |y(t)| = \infty,\ \lim_{t\to\infty} |z(t)| = c_3\right\}$$

$$N^+_{B,\infty,0} := \left\{(x,y,z) \in N^+ : \lim_{t\to\infty} |x(t)| = c_1,\ \lim_{t\to\infty} |y(t)| = \infty,\ \lim_{t\to\infty} |z(t)| = 0\right\}$$

$$N^+_{\infty,B,B} := \left\{(x,y,z) \in N^+ : \lim_{t\to\infty} |x(t)| = \infty,\ \lim_{t\to\infty} |y(t)| = c_2,\ \lim_{t\to\infty} |z(t)| = c_3\right\}$$

$$N^+_{\infty,B,0} := \left\{(x,y,z) \in N^+ : \lim_{t\to\infty} |x(t)| = \infty,\ \lim_{t\to\infty} |y(t)| = c_2,\ \lim_{t\to\infty} |z(t)| = 0\right\}$$

$$N^+_{\infty,\infty,B} := \left\{(x,y,z) \in N^+ : \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty,\ \lim_{t\to\infty} |z(t)| = c_3\right\}$$

$$N^+_{\infty,\infty,0} := \left\{(x,y,z) \in N^+ : \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty,\ \lim_{t\to\infty} |z(t)| = 0\right\}.$$

Throughout this paper, without loss of generality, we assume the first component function x of (x,y,z) is eventually positive. Our first result is:

Theorem 2.1 *Suppose* $R(t_0,\infty) < \infty$*. If* $I_1 < \infty$ *and* $I_8 < \infty$ *for all positive constants* $k_1, k_2, k_3, k_{11}, k_{12}$*, then* $N^+_{B,B,B} \neq \emptyset$*.*

Proof Assume $I_1 < \infty$ and $I_8 < \infty$ for all $k_1, k_2, k_3, k_{11}, k_{12} > 0$. Choose $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} p(t) f\left(k_1 - \int_t^{\infty} q(s) g\left(k_2 + k_3 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2}$$

and

$$\int_{t_1}^{\infty} q(s) g\left(k_{11} + k_{12}\int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s < k_1,$$

where $k_3 = k_{12} = h\left(\frac{1}{2}\right) > 0$ and $k_2 = k_{11}$ for $t \geq t_1$.

Let $\mathbb{X}$ be the set of all continuous and bounded functions with the norm $\|x\| = \sup_{t\geq t_1} |x(t)|$. Then $\mathbb{X}$ is a Banach space [4]. Define a subset Ω of $\mathbb{X}$ such that

$$\Omega := \left\{x \in \mathbb{X} : \quad \frac{1}{2} \leq x(t) \leq 1, \quad t \geq t_1\right\}$$

and an operator $Fx : \mathbb{X} \to \mathbb{X}$ by

$$(Fx)(t) = \frac{1}{2} + \int_{t_1}^{t} p(s) f\left(k_1 - \int_s^{\infty} q(u) g\left(k_2 + \int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\Delta s$$

for $t \geq t_1$. First, for every $x \in \Omega$, $\|x\| = \sup_{t\geq t_1} |x(t)|$, we have $\frac{1}{2} \leq \|x(t)\| \leq 1$ for $t \geq t_1$, which implies Ω is bounded. For showing that Ω is closed, it is enough to show that it includes all limit points. Let x_n be a sequence in Ω converging to x as $n \to \infty$. Then $\frac{1}{2} \leq x_n(t) \leq 1$ for $t \geq t_1$. Taking the limit of x_n as $n \to \infty$, we have $\frac{1}{2} \leq x(t) \leq 1$ for $t \geq t_1$, which implies $x \in \Omega$. Since x_n is any sequence in Ω, it follows that Ω is closed. Now let us show that Ω is also convex. For $x_1, x_2 \in \Omega$, and $\alpha \in [0,1]$, we have

$$\frac{1}{2} = \frac{\alpha}{2} + (1-\alpha)\frac{1}{2} \leq \alpha x_1 + (1-\alpha)x_2 \leq \alpha + (1-\alpha) = 1,$$

where $\frac{1}{2} \leq x_1, x_2 \leq 1$, i.e. Ω is convex. Also,

$$\frac{1}{2} \leq (Fx)(t) \leq \frac{1}{2} + \int_{t_1}^{t} p(s) f\left(k_1 - \int_s^\infty q(u) g\left(k_2 + h\left(\frac{1}{2}\right)\int_u^\infty r(\tau)\Delta\tau\right)\Delta u\right)\Delta s$$
$$\leq 1,$$

i.e. $F : \Omega \to \Omega$. Let us now show that F is continuous on Ω. Let $\{x_n\}$ be a sequence in Ω such that $x_n \to x \in \Omega$ as $n \to \infty$. Then

$$\begin{aligned}
&|(Fx_n - Fx)(t)| \\
&\leq \int_{t_1}^{t} p(s) \left| f\left(k_1 - \int_s^\infty q(u) g\left(k_2 + \int_u^\infty r(\tau) h(x_n(\tau))\Delta\tau\right)\Delta u\right) \right. \\
&\left. - f\left(k_1 - \int_s^\infty q(u) g\left(k_2 + \int_u^\infty r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\right| \Delta s.
\end{aligned}$$

Then the continuity of f, g, and h and the Lebesgue dominated convergence theorem imply that F is continuous on Ω. Finally, since

$$(Fx)^{\Delta}(t) = p(t) f\left(k_1 - \int_t^\infty q(u) g\left(k_2 + \int_u^\infty r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right) < \infty,$$

we have that F is relatively compact by the mean value theorem and Arzelà–Ascoli theorem. Thus, by Theorem 1.4, we have that there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. Then by taking the derivative of $\bar{x}$, we obtain

$$\bar{x}^{\Delta}(t) = p(t) f\left(k_1 - \int_t^\infty q(u) g\left(k_2 + \int_u^\infty r(\tau) h(\bar{x}(\tau))\Delta\tau\right)\Delta u\right), \quad t \geq t_1.$$

Setting

$$\bar{y}(t) := k_1 - \int_t^\infty q(u) g\left(k_2 + \int_u^\infty r(\tau) h(\bar{x}(\tau))\Delta\tau\right)\Delta u$$

for $k_1 > 0$ and taking its derivative yields

$$\bar{y}^{\Delta}(t) = q(t) g\left(k_2 + \int_t^\infty r(\tau) h(\bar{x}(\tau))\Delta\tau\right), \quad t \geq t_1.$$

Finally, differentiating

$$\bar{z}(t) := k_2 + \int_t^\infty r(\tau) h(\bar{x}(\tau))\Delta\tau$$

gives

$$\bar{z}^{\Delta}(t) = -r(t) h(\bar{x}(t)), \quad t \geq t_1.$$

Consequently, $(\bar{x}, \bar{y}, \bar{z})$ is a solution of system (1). As $t \to \infty$, we have that $\bar{x}(t) \to c_1$, $\bar{y}(t) \to k_1$ and $\bar{z}(t) \to k_2$, where $0 < c_1 < \infty$, i.e. $N^{+}_{B,B,B} \neq \emptyset$. □

The following theorem can be proven very similarly to Theorem 2.1. Therefore, the proof is left to the reader.

Theorem 2.2 *Suppose $R(t_0,\infty)<\infty$. If $I_1<\infty$ and $I_8<\infty$ for $k_2=k_{11}=0$ and for all $k_1,k_3,k_{12}>0$, then $N^+_{B,B,0}\neq\emptyset$.*

We now consider the case when $x(t)$ diverges.

Theorem 2.3 *If both I_2 and I_9 are finite for $k_4=0$ and for all $k_5,k_{13}>0$, then $N^+_{\infty,B,0}\neq\emptyset$.*

Proof Suppose that $I_2<\infty$ and $I_9<\infty$ for $k_4=0,k_5,k_{13}>0$. Then choose $t_1\geq t_0$ so large that

$$\int_{t_1}^{\infty} q(t)g\left(\int_t^{\infty} r(s)h\left(k_5\int_{t_1}^{s} p(\tau)\Delta\tau\right)\Delta s\right)\Delta t<\frac{1}{2},$$

where $k_5=f(1)>0$. Let $\mathbb{X}$ be a partially ordered Banach space of all real-valued continuous functions with the norm $\|y\|=\sup_{t\geq t_1}|y(t)|$ and the usual pointwise ordering $\leq$. Define a subset Ω of $\mathbb{X}$ such that

$$\Omega:=\left\{y\in\mathbb{X}:\quad \frac{1}{2}\leq y(t)\leq 1,\quad t\geq t_1\right\}$$

and an operator $Fy:\mathbb{X}\to\mathbb{X}$ by

$$(Fy)(t)=\frac{1}{2}+\int_{t_1}^{t} q(s)g\left(\int_s^{\infty} r(u)h\left(\int_{t_1}^{u} p(\tau)f(y(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

First, note that $(\Omega,\leq)$ is a complete lattice. Indeed, $\inf B\in\Omega$ and $\sup B\in\Omega$ for any subset B of Ω. Now, since

$$\begin{aligned}\frac{1}{2}&\leq (Fy)(t)\\ &\leq \frac{1}{2}+\int_{t_1}^{t} q(s)g\left(\int_s^{\infty} r(u)h\left(f(1)\int_{t_1}^{u} p(\tau)\Delta\tau\right)\Delta u\right)\Delta s\leq 1\end{aligned}$$

for all $t\geq t_1$, $F:\Omega\to\Omega$. For $y_1\leq y_2$, where $y_1,y_2\in\Omega$, we can show $Fy_1\leq Fy_2$ since f, g, and h are nondecreasing mappings. Therefore, by Theorem 1.3, there exists $\bar{y}\in\Omega$ such that $\bar{y}=F\bar{y}>0$ eventually. Differentiating $\bar{y}$ yields

$$\bar{y}^{\Delta}(t)=q(t)g\left(\int_t^{\infty} r(u)h\left(\int_{t_1}^{u} p(\tau)f(\bar{y}(\tau))\Delta\tau\right)\Delta u\right). \tag{3}$$

Set

$$\bar{z}(t):=\int_t^{\infty} r(u)h\left(\int_{t_1}^{u} p(\tau)f(\bar{y}(\tau))\Delta\tau\right)\Delta u$$

for $t\geq t_1$. Differentiating $\bar{z}(t)$ gives

$$\bar{z}^{\Delta}(t)=-r(t)h\left(\int_{t_1}^{t} p(\tau)f(\bar{y}(\tau))\Delta\tau\right)\Delta u. \tag{4}$$

Finally, by setting

$$\bar{x}(t) = \int_{t_1}^{t} p(\tau) f(\bar{y}(\tau))\Delta\tau > 0$$

for $t \geq t_1$, and taking its derivative, we have

$$\bar{x}^{\Delta}(t) = p(t) f(\bar{y}(t)).$$

Note that the above equation is the first of system (1), and (3) and (4) are the second and third of (1), respectively.

Now let us examine the limit behavior of $\bar{x}, \bar{y}$, and $\bar{z}$. Since

$$\bar{x}(t) \geq f\left(\frac{1}{2}\right) \int_{t_1}^{t} p(\tau)\Delta\tau,$$

$\bar{x}(t) \to \infty$ as $t \to \infty$. The fact that $\bar{y} \in \Omega$ gives that it has a finite limit as $t \to \infty$. Finally, because

$$\int_{t}^{\infty} r(u) h\left(\int_{t_1}^{u} p(\tau) f\left(\frac{1}{2}\right)\Delta\tau\right)\Delta u \leq \bar{z}(t) \leq \int_{t}^{\infty} r(u) h\left(\int_{t_1}^{u} p(\tau) f(1)\Delta\tau\right)\Delta u,$$

and $I_9 < \infty$ for $k_{13} > 0$, we obtain $\bar{z}(t) \to 0$ as $t \to \infty$. Hence, $N^{+}_{\infty,B,0} \neq \emptyset$. □

The following theorem can be proven in a similar fashion to that of Theorem 2.3.

Theorem 2.4 *If $I_2 < \infty$ and $I_9 < \infty$ for all $k_4, k_5, k_{13} > 0$, then $N^{+}_{\infty,B,B} \neq \emptyset$.*

Next we consider the case when both $x(t)$ and $y(t)$ diverge and $z(t)$ converges to a positive real number.

Theorem 2.5 *If $I_3 < \infty$ and $I_6 = \infty$ for all $k_6, k_9 > 0$, then $N^{+}_{\infty,\infty,B} \neq \emptyset$.*

Proof Suppose that $I_3 < \infty$ and $I_6 = \infty$ for $k_6, k_9 > 0$. Then choose $t_1 \geq t_0$ sufficiently large that

$$\int_{t_1}^{\infty} r(t) h\left(\int_{t_1}^{t} p(s) f\left(k_6 \int_{t_0}^{s} q(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{4},$$

where $k_6 = g(\frac{1}{2})$ and $k_9 = g(\frac{1}{4})$. Let $\mathbb{X}$ be a partially ordered Banach space of real-valued continuous functions with the norm $\|z\| = \sup_{t \geq t_1} |z(t)|$ and the usual pointwise ordering $\leq$. Define a subset Ω of $\mathbb{X}$ such that

$$\Omega := \left\{ z \in \mathbb{X} : \quad \frac{1}{4} \leq z(t) \leq \frac{1}{2}, \quad t \geq t_1 \right\}$$

and an operator $Fz : \mathbb{X} \to \mathbb{X}$ by

$$(Fz)(t) = \frac{1}{4} + \int_{t}^{\infty} r(s) h\left(\int_{t_1}^{s} p(u) f\left(\int_{t_1}^{u} q(\tau) g(z(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

By a similar process as in Theorem 2.3, we can show that $F : \Omega \to \Omega$ is an increasing mapping and $(\Omega, \leq)$ is a complete lattice. Then, by Theorem 1.3, there exists a $\bar{z} \in \Omega$ such that $\bar{z} = F\bar{z}$. For $t \geq t_1$, set

$$\bar{x}(t) := \int_{t_1}^{t} p(u) f\left(\int_{t_1}^{u} q(\tau) g(\bar{z}(\tau))\Delta\tau\right)\Delta u$$

and

$$\bar{y}(t) := \int_{t_1}^{t} q(\tau)g(\bar{z}(\tau))\Delta\tau.$$

Then

$$\begin{aligned} \bar{z}^{\Delta}(t) &= -r(t)h(\bar{x}(t)), \\ \bar{y}^{\Delta}(t) &= q(t)g(\bar{z}(t)), \\ \bar{x}^{\Delta}(t) &= p(t)f\left(\int_{t_1}^{t} q(\tau)g(\bar{z}(\tau))\,\Delta\tau\right) = p(t)f(\bar{y}(t)). \end{aligned}$$

Consequently $(\bar{x}, \bar{y}, \bar{z})$ is a solution of system (1). Finally, by taking the limit of $\bar{x}, \bar{y}$, and $\bar{z}$ as t approaches infinity, we have $N^{+}_{\infty,\infty,B} \neq \emptyset$.

□

We continue in the case when $z(t)$ converges to 0.

Theorem 2.6 *Suppose $R(t_0, \infty) < \infty$. If $I_3 < \infty$ and $I_4 = I_8 = \infty$ for all positive constants k_6, k_7, k_{12}, and $k_{11} = 0$, then $N^{+}_{\infty,\infty,0} \neq \emptyset$.*

Proof Suppose $I_3 < \infty$ and $I_4 = I_8 = \infty$ for $k_6, k_7, k_{12} > 0$, $k_{11} = 0$. Then we can choose $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} r(t)h\left(\int_{t_1}^{t} p(s)f\left(k_6\int_{t_0}^{s} q(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2}$$

and

$$\int_{t_1}^{\infty} p(s)f\left(\int_{t_1}^{s} q(\tau)g\left(k_7\int_{\tau}^{\infty} r(v)\Delta v\right)\Delta\tau\right)\Delta s > 1, \quad t \geq t_1,$$

where $k_6 = g(\frac{1}{2})$ and $k_7 = k_{12} = h(1)$. Let $\mathbb{X}$ be a partially ordered Banach space of real-valued continuous functions with the norm $\|y\| = \sup_{t\geq t_1} |y(t)|$ and the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega := \left\{z \in \mathbb{X} : \quad h(1)\int_{t}^{\infty} r(s)\Delta s \leq z(t) \leq \frac{d_1}{2}, \quad t \geq t_1\right\}.$$

Let us define an operator $T : \mathbb{X} \to \mathbb{X}$ such that

$$(Tz)(t) = \int_{t}^{\infty} r(s)h\left(\int_{t_1}^{s} p(u)f\left(\int_{t_1}^{u} q(\tau)g(z(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

The remainder of the proof can be done as in Theorem 2.5 by using the fact $I_4 = I_8 = \infty$, and therefore, $N^{+}_{\infty,\infty,0} \neq \emptyset$. □

2.2. Existence in N^{-}

This section represents the limit behavior of nonoscillatory solutions of system (1) along with the existence of such solutions in N^{-}. Suppose that (x, y, z) is a nonoscillatory solution of system (1) in N^{-} such that $x > 0$ eventually. By the same discussion in the previous subsection and by Lemma 1.1, one has the following lemma:

Lemma 2.7 *Assume that (x, y, z) is a nonoscillatory solution of system (1) in N^-. Then (x, y, z) belongs to one of the following subclasses:*

$$N^-_{B,0,0} := \left\{(x, y, z) \in N^- : \lim_{t\to\infty} |x(t)| = c_1 \ \ \lim_{t\to\infty} |y(t)| = 0, \ \lim_{t\to\infty} |z(t)| = 0\right\}$$
$$N^-_{0,0,0} := \left\{(x, y, z) \in N^- : \lim_{t\to\infty} |x(t)| = 0, \ \lim_{t\to\infty} |y(t)| = 0, \ \lim_{t\to\infty} |z(t)| = 0\right\},$$

where $0 < c_1 < \infty$.

The first result of this section considers the case when each of the component solutions converges.

Theorem 2.8 *Suppose $R(t_0, \infty) < \infty$. If $I_5 < \infty$ and $I_8 < \infty$ for all $k_8 = k_{12} > 0$ and $k_{11} = 0$, then $N^-_{B,0,0} \neq \emptyset$, provided that f is an odd function.*

Proof Suppose that $I_5 < \infty$ and $I_8 < \infty$ for all $k_8 = k_{12} > 0$ and $k_{11} = 0$. Then choose $k_8, k_{12} > 0$, and $t_1 \geq t_0$ sufficiently large such that

$$\int_{t_1}^{\infty} p(t) f\left(\int_t^{\infty} q(s) g\left(k_8 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2},$$

where $k_8 = h\left(\frac{3}{2}\right)$. Let $\mathbb{X}$ be a partially ordered Banach space of real-valued continuous functions with the norm $\|x\| = \sup_{t\geq t_1} |x(t)|$ and the usual pointwise ordering $\leq$. Define a subset Ω of $\mathbb{X}$ such that

$$\Omega := \left\{x \in \mathbb{X} : \quad 1 \leq x(t) \leq \frac{3}{2}, \quad t \geq t_1\right\}$$

and an operator $Fx : \mathbb{X} \to \mathbb{X}$ by

$$(Fx)(t) = 1 + \int_t^{\infty} p(s) f\left(\int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

One can show that F is an increasing mapping into itself and $(\Omega, \leq)$ is a complete lattice. Therefore, by Theorem 1.3, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. It follows that $\bar{x}(t) > 0$ for $t \geq t_1$ and converges to 1 as t approaches infinity. Also,

$$\bar{x}^{\Delta}(t) = -p(t) f\left(\int_t^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(\bar{x}(\tau))\Delta\tau\right)\Delta u\right), \quad t \geq t_1.$$

Now for $t \geq t_1$, set

$$\bar{y}(t) = -\int_t^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(\bar{x}(\tau))\Delta\tau\right)\Delta u$$

and

$$\bar{z}(t) = \int_t^{\infty} r(\tau) h(\bar{x}(\tau))\Delta\tau.$$

Then, since f is odd, we have

$$\begin{aligned}\bar{x}^{\Delta}(t) &= p(t)f(\bar{y}(t)),\\ \bar{y}^{\Delta}(t) &= q(t)g(\bar{z}(t)),\\ \bar{z}^{\Delta}(t) &= -r(t)h(\bar{x}(t)).\end{aligned}$$

Consequently, $(\bar{x}, \bar{y}, \bar{z})$ is a solution of system (1). Since both $\bar{y}(t)$ and $\bar{z}(t)$ converge to 0 as t approaches infinity, $N^{-}_{B,0,0} \neq \emptyset$. □

3. Example

We give an example to illustrate one of our theoretical claims. Recall:

Theorem 3.1 *[2, Theorem 1.79 (ii)] If $[a,b]$ consists of only isolated points and $a < b$, then*

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t).$$

The example illustrates Theorem 2.3.

Example 1 *Let $\mathbb{T} = 3^{\mathbb{N}}, k_5 = 1 = k_{13}$, and consider the following system:*

$$\begin{cases}\Delta_3 x(t) = \left(\frac{t}{t-1}\right)^{\frac{1}{3}} y^{\frac{1}{3}}(t)\\ \Delta_3 y(t) = \frac{1}{3t^{\frac{1}{5}}} z^{\frac{3}{5}}(t)\\ \Delta_3 z(t) = -\frac{26}{54t^{\frac{21}{5}}} x^{\frac{1}{5}}(t),\end{cases} \tag{5}$$

where

$$\Delta_3 k(t) = \frac{k(\sigma(t)) - k(t)}{\mu(t)} \quad \text{for} \quad \sigma(t) = 3t \quad \text{and} \quad \mu(t) = 2t, \quad t \in \mathbb{T}.$$

First we show that (2) holds. If $s = 3^m$ and $t = 3^n$, $m, n \in \mathbb{N}$, we have

$$\int_3^{\infty} p(s)\Delta s = \lim_{t\to\infty} \int_3^t p(s)\Delta s = 2 \lim_{n\to\infty} \sum_{s=3}^{\rho(3^n)} \left(\frac{s^4}{s-1}\right)^{\frac{1}{3}} > 2 \lim_{n\to\infty} \sum_{m=1}^{n-1} 3^m = \infty.$$

Similarly one can obtain $\int_3^{\infty} q(s)\Delta s = \infty$.

Now we consider I_2. With $\tau = 3^m$ and $s = 3^n$, $m, n \in \mathbb{N}$, we have

$$\int_3^s \left(\frac{\tau}{\tau - 1}\right)^{\frac{1}{3}} \Delta\tau = 2\sum_{m=1}^{n-1} \left(\frac{3^{4m}}{3^m - 1}\right)^{\frac{1}{3}} < 2\sum_{m=1}^{n-1} (3^m)^{\frac{4}{3}}$$

since $3^m - 1 > 1$ on $\mathbb{N}$. We claim that

$$\sum_{m=1}^{n-1} (3^m)^{\frac{4}{3}} < (3^n)^{\frac{4}{3}}.$$

The sum formula for a finite geometric series, $1-3^{\frac{4}{3}}<0$, *and* $\left(3^{\frac{4}{3}}\right)^{1-n}-1<1$ *for* $n\in\mathbb{N}$ *yield*

$$0\leq\frac{\left(3^{\frac{4}{3}}\right)^{1-n}-1}{1-3^{\frac{4}{3}}}<1.$$

Thus, the claim indeed holds, and consequently we have

$$\int_3^s\left(\frac{\tau}{\tau-1}\right)^{\frac{1}{3}}\Delta\tau<2s^{\frac{4}{3}}. \tag{6}$$

Also, we obtain

$$\begin{aligned}\int_t^T r(s)h\left(\int_3^s p(\tau)\Delta\tau\right)\Delta s &< \int_t^T \frac{26}{54}\frac{1}{s^{\frac{21}{5}}}(2s^{\frac{4}{3}})^{\frac{1}{5}}\Delta s\\ &=\frac{26\cdot 2^{\frac{6}{5}}}{54}\sum_{s\in[t,T)_{3^{\mathbb{N}}}}\frac{1}{s^{\frac{44}{15}}}\\ &<2\sum_{s\in[t,T)_{3^{\mathbb{N}}}}\frac{1}{s^{\frac{44}{15}}}\end{aligned}$$

by (6). *Therefore, as* $T\to\infty$, *we obtain*

$$\sum_{s\in[t,\infty)_{3^{\mathbb{N}}}}\frac{1}{s^{\frac{44}{15}}}=\alpha\cdot\frac{1}{t^{\frac{44}{15}}}, \tag{7}$$

where $\alpha=1-\frac{1}{3^{\frac{44}{15}}}$. *Finally, with* $t=3^m$ *and* $T=3^n$, $m,n\in\mathbb{N}$, *we have*

$$\begin{aligned}\int_{t_0}^T q(t)g\left(\int_t^\infty r(s)h\left(\int_{t_0}^s p(\tau)\Delta\tau\right)\Delta s\right)\Delta t &< \frac{(2\alpha)^{\frac{3}{5}}}{3}\int_3^T\frac{1}{t^{\frac{1}{5}}}\left(\frac{1}{t^{\frac{44}{15}}}\right)^{\frac{3}{5}}\Delta t\\ &=\frac{(2\alpha)^{\frac{3}{5}}}{3}\int_3^T\frac{1}{t^{\frac{49}{25}}}\Delta t\\ &=\frac{(2\alpha)^{\frac{3}{5}}}{3}\sum_{m=1}^{n-1}2\frac{1}{(3^m)^{\frac{49}{25}}}3^m\\ &=\frac{2(2\alpha)^{\frac{3}{5}}}{3}\sum_{m=1}^{n-1}\left(\frac{1}{3^{\frac{24}{25}}}\right)^m\end{aligned}$$

by (7). *Since the above integral converges as* T *approaches infinity, we have* $I_2<\infty$. *By using a similar discussion and* (7), *it is shown that* $I_9<\infty$. *One can also show that* $(t,1-\frac{1}{t},\frac{1}{t^3})$ *is a nonoscillatory solution of system* (5). *Hence,* $N^+_{\infty,B,0}\neq\emptyset$ *by Theorem 2.3.*

4. Conclusion and open problems

In this paper, we considered a three-dimensional time scale system of first-order dynamic equations. We established some sufficient conditions for the existence of the nonoscillatory solutions of the system using the Schauder fixed point theorem and the Knaster fixed point theorem. While we were able to determine the operators for a fixed point theorem for most subclasses of N^+, the operators needed for $N^+_{B,\infty,B}$ and $N^+_{B,\infty,0}$ are still unknown; as an open problem, one can determine the conditions to guarantee that $N^+_{B,\infty,B} \neq \emptyset$ and $N^+_{B,\infty,0} \neq \emptyset$. Similarly, existence of nonoscillatory solutions of (1) in $N^-_{0,0,0}$ remains an open problem for interested readers.

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