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3-Class groups of cubic cyclic function fields

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Abstract: Let F be a global function field over the finite constant field $\mathbb{F}_q$ with $3 \mid q-1$, and let K/F be a cubic cyclic function fields extension with Galois group $G = \mathrm{Gal}(K/F) =< \sigma >$. Denote by $\mathcal{C}(K)$ and $\mathcal{C}(K)_3$ the ideal class group of K and its Sylow 3-subgroup, respectively. Let $\mathcal{C}(K)_3^G = \{[\mathfrak{a}] \in \mathcal{C}(K)_3 | \ \sigma[\mathfrak{a}] = [\mathfrak{a}]\}$ and $\mathcal{C}(K)_3^{1-\sigma} = \{[\mathfrak{a}](\sigma[\mathfrak{a}])^{-1} | \ [\mathfrak{a}] \in \mathcal{C}(K)_3\}$. In this paper, we present a method for computing the 3-rank of the quotient group $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{1-\sigma}/\mathcal{C}(K)_3^{1-\sigma}$. Specifically, when K is a cubic Kummer extension of $\mathbb{F}_q(T)$, we determine explicitly the key factors t, $x_1, \cdots, x_t$, and $[\mathfrak{A}_1], \cdots, [\mathfrak{A}_t]$ in the process of computing the 3-rank of $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{1-\sigma}/\mathcal{C}(K)_3^{1-\sigma}$. Combining this deterministic algorithm along with the structure of class groups for cubic Kummer function fields, the 3-rank of the Sylow 3-subgroup of $\mathcal{C}(K)$ is determined explicitly in this specific case. Examples are given in the last two sections to elucidate our computational method.

Key words: Class group, cubic function fields, genus theory, Artin reciprocity law map

1. Introduction

The structure of the class group of global fields has been investigated intensively by many authors since Gauss first studied the arithmetic of binary quadratic forms. Quadratic and cyclic number fields are mostly addressed by researchers. Genus theory and the Rédei matrix, which were invented respectively by Gauss and Rédei, are the two effective tools to deal with the structure of class groups of quadratic and cyclic number fields. Over the past three decades, Conner and Hurrelbrink's exact hexagon [6] and the generalized Rédei matrix [21] have been combined with class field theory to study the structure of the Sylow subgroups of class groups.

In the seventies of the last century, Gerth began to research the structure of class groups of cyclic number fields and associated density problem. In particular, Gerth studied in great detail the Sylow 3-subgroups of cubic cyclic number fields (see [9]–[10]) and presented analogous results for the 3-rank of the 3-class group of cubic fields to Gauss's for the 2-rank of the 2-class group of quadratic fields. Using Gerth's results on the 3-rank of the class group of cubic number fields, Chen et al. [5], Guo [13], Li and Qin [15], and Zhou [23] studied the 3-ranks of tame kernels of cubic cyclic number fields and associated density problems. Recently, in terms of Gerth's results in the number field case, we presented in [22] the function field analogue of the l-rank of class groups of cyclic function fields by the genus theory and Conner–Hurrelbrink exact hexagon for function fields.

Let F be a global function field over the finite constant field $\mathbb{F}_q$ with $3 \mid q-1$, and let K/F a cyclic extension of degree 3 of global function fields with $G = \mathrm{Gal}(L/K) =< \sigma >$. Denote by $\mathcal{C}(K)$ and $\mathcal{C}(K)_3$ the

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ideal class group of K and its Sylow 3-subgroup, respectively. Let

$$\mathcal{C}(K)_3^G = \{[\mathfrak{a}] \in \mathcal{C}(K)_3 | \ \sigma[\mathfrak{a}] = [\mathfrak{a}]\}$$

and

$$\mathcal{C}(K)_3^{1-\sigma} = \{[\mathfrak{a}](\sigma[\mathfrak{a}])^{-1} | \ [\mathfrak{a}] \in \mathcal{C}(K)_3\}.$$

Genus theory together with the Conner–Hurrelbrink exact hexagon for function fields, the 3-rank, and the structure of the Sylow 3-subgroup of $\mathcal{C}(K)$ were characterized explicitly in [22] when 3 does not divide the class number of F. In this paper, we continue our previous work on the 3-rank of class groups of cubic cyclic function fields. Now we describe the organization of this paper. In Section 2, we present the necessary notation and known results. In Section 3, we give a method for computing the 3-rank of the quotient group $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{1-\sigma}/\mathcal{C}(K)_3^{1-\sigma}$, which is one of the main results of this paper. To be more specific, we prove that the 3-rank of $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{1-\sigma}/\mathcal{C}(K)_3^{1-\sigma}$ equals the rank of matrix $(a_{ij})_{1\leq i,j\leq t}$ where a_{ij} are determined by Artin symbols for cubic Kummer extensions. When K is a cubic Kummer extension of $\mathbb{F}_q(T)$, we determine explicitly in Section 4 the key factors t, $x_1, \cdots, x_t$, and $[\mathfrak{A}_1], \cdots, [\mathfrak{A}_t]$ in the process of computing the rank of matrix $(a_{ij})_{1\leq i,j\leq t}$. It has to be pointed out that the most difficult part of determining the 3-rank of $\mathcal{C}(K)$ is how to compute the 3-rank of $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{1-\sigma}/\mathcal{C}(K)_3^{1-\sigma}$. The computational method for the 3-rank of $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{1-\sigma}/\mathcal{C}(K)_3^{1-\sigma}$ completes the determination of the structure of the Sylow 3-subgroup of $\mathcal{C}(K)$ in this special case. Finally, we conclude this paper with some remarks.

2. Notation and known results

In what follows, we first introduce some terminologies and notation for convenience. Unless otherwise specified, k will denote the rational function field $\mathbb{F}_q(T)$ throughout this paper over a finite field $\mathbb{F}_q$ with q a prime power. Let F be a finite separable extension of k and F^{sep} be a separable closure of F. Denote the set of all prime divisors of F by S_F. For a fixed nonempty finite subset $S_\infty(F)$ of S_F, set $S_0(F) = S_F \setminus S_\infty(F)$. The element in $S_0(F)$ will be called a finite prime divisor of F while one in $S_\infty(F)$ will always be called an infinite prime divisor of F. Denote by $\mathcal{O}_F$ the set of elements of F that are integral at all finite prime divisors. It is well known that $\mathcal{O}_F$ is a Dedekind domain with F as its quotient field. Denote by $I(F)$, $\mathcal{C}(F)$, and $\mathcal{O}_F^*$ respectively the fractional ideal group, ideal class group, and unit group of $\mathcal{O}_F$. Let $h(\mathcal{O}_F) = |\mathcal{C}(F)|$, which is finite, be the class number of $\mathcal{O}_F$. For an ideal $\mathfrak{a} \in I(F)$, we use $[\mathfrak{a}]$ to denote the image of $\mathfrak{a}$ in $\mathcal{C}(F)$. It is worth pointing out here that there is a one-to-one correspondence between prime divisors in $S_0(F)$ and prime ideals in $I(F)$. Without causing confusion, $\mathfrak{p}$ will denote both prime divisor and prime ideal for convenience. For a finite separable extension K/F, let $S_\infty(K)$ be the set of prime divisors of K that are the extensions of $S_\infty(F)$. Then $\mathcal{O}_K$, which is composed of the elements integral at prime divisors in $S_0(K)$, is exactly the integral closure of $\mathcal{O}_F$ in K.

With notation defined as above, we give several definitions that will be used in the following discussion.

Definition 2.1 ([17], [19]) *A finite unramified extension of function fields K/F is called a Hilbert extension with respect to $S_\infty(F)$, or Hilbert extension for simplicity, if all the infinite prime divisors split completely in K. The Hilbert class field of function field F with respect to $S_\infty(F)$, denoted by H_F, is the maximal Hilbert abelian extension of F in F^{sep}.*

The following definition is a function field analogue of the number field case, which was given by Fröhlich [7].

Definition 2.2 ([3], [17], [22]) *The genus field G_F of function field F is the maximal Hilbert abelian extension of F, which is a compositum of F with an abelian function field over k.*

For a global function field $F/\mathbb{F}_q$, it follows readily from the above definitions that G_F is a subfield of H_F. By the class field theory, the Artin reciprocity law map provides us an isomorphism as follows

$$\mathcal{A}_F: \quad \mathcal{C}(F) \to \mathrm{Gal}(H_F/F).$$

Let l be a prime number. This isomorphism tells us that F admits a cyclic Hilbert extension K/F if and only if $l \mid h(\mathcal{O}_F)$. Suppose that K/F is a cyclic Hilbert extension of degree l. Then $K \subset H_F$ and there is an exact sequence as follows:

$$1 \to \mathrm{Gal}(H_F/K) \to \mathrm{Gal}(H_F/F) \to \mathrm{Gal}(K/F) \to 1.$$

Now we focus our attention on the case that F/k is a finite abelian extension. In this case, the genus field G_F of F is exactly the subfield of H_F, which is maximal abelian over k. Note here by Lemma 2.3 in [19] that H_F/k is a Galois extension. These statements mean that the Galois group $\mathrm{Gal}(H_F/G_F)$ is the commutator subgroup of $\mathrm{Gal}(H_F/k)$ and

$$\mathrm{Gal}(G_F/F) \cong \mathrm{Gal}(H_F/F)/\mathrm{Gal}(H_F/G_F).$$

Under the Artin map, $\mathrm{Gal}(H_F/G_F)$ can be identified with a subgroup of $\mathcal{C}(F)$, which is characterized as the principal genus (see Proposition 2.4 in [3] or Lemma 1 in [17]),

$$\{[\mathfrak{a}]^{1-\sigma} \mid \ [\mathfrak{a}] \in \mathcal{C}(F), \quad \sigma \in \mathrm{Gal}(F/k)\},$$

where $[\mathfrak{a}]^{1-\sigma} = [\mathfrak{a}](\sigma[\mathfrak{a}])^{-1}$.

To continue our discussion, we shall take a short detour to some notation. Let $G = <\sigma>$ be a cyclic group of order prime l with a fixed generator σ, and A a finite G-module whose operation is written as multiplication. Set $\mathcal{N} = 1 + \sigma + \cdots + \sigma^{l-1}$ and $\mathcal{I} = 1 - \sigma$. Denote by A_l and ${}_lA$ respectively the Sylow l-subgroup of A and the set of elements in A of order no more than l. Define

$$A^l = \{a^l \mid \ a \in A\},$$

$$A^G = \{a \in A \mid \ \sigma(a) = a\},$$

$$A^{\mathcal{I}} = \{\mathcal{I}(a) = a\sigma(a)^{-1} \mid \ a \in A\}.$$

It is easy to check that A_l, ${}_lA$, A^l, A^G, and $A^{\mathcal{I}}$ are all G-submodules of A. We denote by $r_{l^n}(A)$ the l^n-rank of A, i.e.

$$r_{l^n}(A) := \dim_{\mathbb{Z}/l\mathbb{Z}} \left(A^{l^{n-1}} / A^{l^n} \right).$$

With notation defined as in the previous paragraph, assume from now on that K/F is a cyclic function fields extension of prime degree l with the Galois group $G = \mathrm{Gal}(K/F) = <\sigma>$. It is clear that

$K^*, I(K), \mathcal{C}(K), \mathcal{O}_K^*$ are all G-modules, and the principal genus of $\mathcal{C}(K)$ can be denoted by $\mathcal{C}(K)^{\mathcal{I}}$. The elements in $\mathcal{C}(K)^G$ are called ambiguous ideal classes. The Galois module structure of $\mathcal{C}(K)_l$ and $\mathcal{C}(K)_l^G$, which has been studied intensively by many researchers over a long period of time, is an important and difficult issue in number theory. Let $\mathbb{Z}_l$ be the ring of l-adic integers. Under the map $\sigma \mapsto \zeta_l$, where ζ_l is a primitive lth root of unit, we obtain an isomorphism of discrete valuation rings as follows:

$$\mathbb{Z}_l[\sigma]/(\mathcal{N}) \cong \mathbb{Z}_l[\zeta_l].$$

Thus, if $l \nmid h(\mathcal{O}_F)$, then $\mathcal{C}(K)_l$ is a finite module over $\mathbb{Z}_l[\zeta_l]$ because $\mathcal{N}$ acts trivially on $\mathcal{C}(K)_l$. It is well known that the Galois module structure of $\mathcal{C}(K)_l$ is determined by the dimensions

$$\lambda_i = \dim_{\mathbb{Z}/l\mathbb{Z}} \left(\mathcal{C}(K)_l^{\mathcal{I}^{i-1}} / \mathcal{C}(K)_l^{\mathcal{I}^i} \right), \quad i \geq 1.$$

We assume that $l \nmid h(\mathcal{O}_F)$ in the remainder of this section. Under this condition, it is not hard to check that $\mathcal{C}(K)_l^l \subset \mathcal{C}(K)_l^{\mathcal{I}}$, and furthermore, $\mathcal{C}(K)^l \subset \mathcal{C}(K)^{\mathcal{I}}$ when $\mathcal{C}(F)$ is trivial (see [22]). Notice that quotient groups $\mathcal{C}(K)_l/\mathcal{C}(K)_l^l$, $\mathcal{C}(K)_l/\mathcal{C}(K)_l^{\mathcal{I}}$, and $\mathcal{C}(K)_l^{\mathcal{I}}/\mathcal{C}(K)_l^l$ are all elementary abelian l-groups. This remark implies that

$$r_l(\mathcal{C}(K)_l) = r_l\left(\mathcal{C}(K)_l/\mathcal{C}(K)_l^l\right) = r_l\left(\mathcal{C}(K)_l/\mathcal{C}(K)_l^{\mathcal{I}}\right) + r_l\left(\mathcal{C}(K)_l^{\mathcal{I}}/\mathcal{C}(K)_l^l\right).$$

Accordingly, if $\mathcal{C}(F)$ is trivial, then we observe that

$$\mathcal{C}(K)/\mathcal{C}(K)^{\mathcal{I}} \cong \mathcal{C}(K)_l/\mathcal{C}(K)_l^{\mathcal{I}},$$

$$\mathcal{C}(K)^{\mathcal{I}}/\mathcal{C}(K)^l \cong \mathcal{C}(K)_l^{\mathcal{I}}/\mathcal{C}(K)_l^l,$$

since the action of $\mathcal{I}$ on the non-l-part of $\mathcal{C}(K)$ is invertible. We note that the action of $\mathcal{N}$ on $\mathcal{C}(K)_l$ is trivial, and $\mathcal{N}$ acts trivially also on $\mathcal{C}(K)$ when $\mathcal{C}(F)$ is trivial. These facts tell us that $\mathcal{C}(K)_l^G$ is an elementary abelian l-group, and so is $\mathcal{C}(K)^G$ when $\mathcal{C}(F)$ is trivial. Combining these facts yields the following conclusion.

Lemma 2.3 ([22]) *Let K/F be a cyclic function fields extension of prime degree l with Galois group $G = \mathrm{Gal}(K/F) =< \sigma >$. If $l \nmid h(\mathcal{O}_F)$, then*

$$r_l\left(\mathcal{C}(K)_l^G\right) = r_l\left(\mathcal{C}(K)_l/\mathcal{C}(K)_l^{\mathcal{I}}\right);$$

if $\mathcal{C}(F)$ is trivial, then

$$r_l\left(\mathcal{C}(K)^G\right) = r_l\left(\mathcal{C}(K)/\mathcal{C}(K)^{\mathcal{I}}\right).$$

As mentioned above, when $l = 3$, the Galois module structure of $\mathcal{C}(K)_3$ is determined by the dimensions $\dim_{\mathbb{Z}/3\mathbb{Z}}\left(\mathcal{C}(K)_3/\mathcal{C}(K)_3^{\mathcal{I}}\right)$ and $\dim_{\mathbb{Z}/3\mathbb{Z}}\left(\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}^2}\right)$. However, we note that $\mathcal{C}(K)_3^{\mathcal{I}^2} = \mathcal{C}(K)_3^3$ if $3 \nmid h(\mathcal{O}_F)$. Thus, in order to determine the Galois module structure of $\mathcal{C}(K)_3$, it suffices to describe explicitly the dimensions $\dim_{\mathbb{Z}/3\mathbb{Z}}\left(\mathcal{C}(K)_3/\mathcal{C}(K)_3^{\mathcal{I}}\right)$ and $\dim_{\mathbb{Z}/3\mathbb{Z}}\left(\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^3\right)$. The following result provides the 3-rank of $\mathcal{C}(K)$ under some mild conditions.

Theorem 2.4 ([22]) *Let K/F be a cyclic extension of degree 3 of global function fields with $G = \mathrm{Gal}(L/K) =< \sigma >$. Suppose that $3 \nmid h(\mathcal{O}_F)$, $t = r_3(\mathcal{C}(K)_3^G)$ and $s = r_3\left(\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}\right)$. Then,*

(i) $r_3(\mathcal{C}(K)) = 2t - s$;

(ii) $\mathcal{C}(K)_3$ *is isomorphic to the direct product of an abelian 3-group of rank $2(t-s)$ and an elementary abelian 3-group of rank s, where each element of the elementary abelian 3-group of rank s is an ambiguous ideal class.*

With the help of this result, we shall focus our attention on the 3-rank of cyclic extension of degree 3 in the following sections.

3. 3-Rank of class groups of cubic cyclic function fields

Using the results of the previous section, this section is devoted to studying the structure of the Sylow 3-subgroup of cubic cyclic function fields in the general case.

Unless otherwise specified, suppose that $F/\mathbb{F}_q$ is a global function field with $3 \mid q-1$ and $3 \nmid h(\mathcal{O}_F)$. It follows readily from the Kummer theory that cubic cyclic extension K/F is a Kummer extension. Then there is some element $\alpha \in K^*$ such that $K = F(\sqrt[3]{\alpha})$. Suppose that $G = \mathrm{Gal}(K/F) =< \sigma >$. By the genus theory for function fields mentioned in the previous section, we get an isomorphism induced by the Artin reciprocity law map

$$\mathcal{C}(K)/\mathcal{C}(K)^{\mathcal{I}} \cong \mathrm{Gal}(G_K/K).$$

We note that $\mathcal{I}$ acts as an isomorphism on the non-3-part of $\mathcal{C}(K)$. It can be easily seen that

$$\mathcal{C}(K)_3/\mathcal{C}(K)_3^{\mathcal{I}} \cong \mathcal{C}(K)/\mathcal{C}(K)^{\mathcal{I}}.$$

Thus, it follows from $\mathcal{C}(K)_3^3 \subset \mathcal{C}(K)_3^{\mathcal{I}}$ that $\mathcal{C}(K)/\mathcal{C}(K)^{\mathcal{I}}$ is an elementary abelian 3-group, and this implies in turn that the genus field G_K is a 3-extension of K of degree 3^t in H_K, where $t = r_3\left(\mathcal{C}(K)_3/\mathcal{C}(K)_3^{\mathcal{I}}\right) = r_3\left(\mathcal{C}(K)_3^G\right)$. In view of the Galois theory and Kummer theory, we infer that there are elements $x_1, x_2, \cdots, x_t$ in K, such that

$$G_K = K(\sqrt[3]{x_1}, \sqrt[3]{x_2}, \cdots, \sqrt[3]{x_t}).$$

With the aid of these preparations and the class field theory, we will present explicitly the 3-rank of $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}$, and then describe the structure of the Sylow 3-group of ideal class group of K.

To state the following conclusion, we first introduce some necessary notation. We fix γ as a generator of $\mathbb{F}_q^*$ and put $\eta = \gamma^{\frac{q-1}{3}}$. Then η is a primitive 3-root of unit in F. For a finite abelian function fields extension L/K and an unramified prime ideal $\mathfrak{P} \in I(K)$, denote by $\left(\frac{L/K}{\mathfrak{P}}\right)$ the associated Artin symbol.

Theorem 2.4 tells us that the key to describing the structure of the Sylow 3-subgroup of $\mathcal{C}(K)$ is to compute the 3-ranks of $\mathcal{C}(K)_3^G$ and $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}$. Under some specific conditions, the 3-rank of $\mathcal{C}(K)_3^G$ is relatively easy to determine. The following theorem provides a method for computing the 3-rank of $\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}$ via Artin symbols associated with multiple Kummer extension.

Theorem 3.1 *Let $F/\mathbb{F}_q$ be a global function field with $3 \mid q-1$ and $3 \nmid h(\mathcal{O}_F)$, and let K/F be a cubic cyclic function fields extension with $G = \mathrm{Gal}(K/F) =< \sigma >$. Suppose that $G_K = K(\sqrt[3]{x_1}, \sqrt[3]{x_2}, \cdots, \sqrt[3]{x_t})$ is the genus field of K, where $t = r_3\left(\mathcal{C}(K)_3/\mathcal{C}(K)_3^{\mathcal{I}}\right) = r_3\left(\mathcal{C}(K)_3^G\right)$ and $x_1, x_2, \cdots, x_t \in K$, and ambiguous ideal classes $[\mathfrak{A}_1], [\mathfrak{A}_2], \cdots, [\mathfrak{A}_t] \in \mathcal{C}(K)_3^G$ are a basis for $\mathcal{C}(K)_3^G$.*

Let s be the rank of the matrix $A = (a_{ij})_{1\le i\le t, 1\le j\le t}$, where the rank of A is the rank over $\mathbb{Z}/3\mathbb{Z}$, and $a_{ij} \in \mathbb{Z}/3\mathbb{Z}$ are determined by the following relation:

$$\frac{\tau_{ij}(\sqrt[3]{x_i})}{\sqrt[3]{x_i}} = \eta^{a_{ij}}, \quad \tau_{ij} = \left(\frac{K(\sqrt[3]{x_i})/K}{\mathfrak{A}_j}\right).$$

Then $s = r_3\left(\mathcal{C}(K)_3^G \mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}\right)$.

One remark is in order about this theorem before finishing the proof of it. When s is computed by the method of this theorem, it follows from Thorem 2.4 that $r_3(\mathcal{C}(K)) = 2t - s$, and the Sylow 3-subgroup of $\mathcal{C}(K)$ is isomorphic to the direct product of an abelian 3-group of rank $2(t-s)$ and an elementary abelian 3-group of rank s, where each element of the elementary abelian 3-group of rank s is an ambiguous ideal class.

Proof of Theorem 3.1. To ease notation, we denote by K_i the field $K(\sqrt[3]{x_i})$ for $1 \le i \le t$. Now we begin the proof of the following equation:

$$\mathrm{rank}(A) = r_3\left(\mathcal{C}(K)_3^G \mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}\right).$$

We first define the following composite homomorphism:

$$\Phi: \quad \mathcal{C}(K)_3^G \to \mathcal{C}(K)_3/\mathcal{C}(K)_3^{\mathcal{I}} \cong \mathrm{Gal}(G_K/K),$$

where the first map is induced by the inclusion $\mathcal{C}(K)_3^G \subset \mathcal{C}(K)_3$, and the isomorphism is induced by the Artin map. Thus, we get that for any ideal class $[\mathfrak{A}] \in I(K)$,

$$\Phi([\mathfrak{A}]) = \left(\frac{G_K/K}{\mathfrak{A}}\right).$$

We note that Φ is a linear map of $\mathbb{Z}/3\mathbb{Z}$-vector spaces from $\mathcal{C}(K)_3^G$ to $\mathrm{Gal}(G_K/K)$, and $\ker\Phi = \mathcal{C}(K)_3^G \cap \mathcal{C}(K)_3^{\mathcal{I}}$.

Since $G_K = K_1K_2\cdots K_t$ and $K_i \cap K_j = K$ for $1 \le i \ne j \le t$, we get an isomorphism as follows:

$$\begin{aligned}\Psi: \mathrm{Gal}(G_K/K) &\to \mathrm{Gal}(K_1/K) \times \mathrm{Gal}(K_2/K) \times \cdots \times \mathrm{Gal}(K_t/K),\\ \phi &\mapsto (\phi|_{K_1}, \phi|_{K_2}, \cdots, \phi|_{K_t}),\end{aligned}$$

where $\phi|_{K_i}$ denotes the restriction of $\phi \in \mathrm{Gal}(G_K/K)$ to K_i. Notice that $K_i/K, i = 1, \cdots, t$, are all Kummer extensions. It follows from the Kummer theory that there is an isomorphism for every $1 \le i \le t$

$$\chi_i: \ \mathrm{Gal}(K_i/K) \to \mathbb{Z}/3\mathbb{Z}, \ \varphi \mapsto b_\varphi,$$

where b_φ satisfies the relation $\frac{\varphi(\sqrt[3]{x_i})}{\sqrt[3]{x_i}} = \eta^{b_\varphi}$. Therefore, we can take the composite of the above three homomorphisms to establish the following map:

$$\Pi=\left(\prod_{i=1}^{t}\chi_i\right)\circ\Psi\circ\Phi:\mathcal{C}(K)_3^G\to\underbrace{\mathbb{Z}/3\mathbb{Z}\times\cdots\times\mathbb{Z}/3\mathbb{Z}}_{t},$$

$$[\mathfrak{A}]\mapsto\left(b_{\left(\frac{K_1/K}{\mathfrak{A}}\right)},\cdots,b_{\left(\frac{K_t/K}{\mathfrak{A}}\right)}\right).$$

It is obvious that the composite Π is a linear map of $\mathbb{Z}/3\mathbb{Z}$-vector spaces with the same kernel as Φ since $\prod_{i=1}^{t}\chi_i$ and Ψ are both isomorphisms.

By the definition of the matrix A, we get that it is exactly the matrix of linear map Π with respect to the basis $[\mathfrak{A}_1],[\mathfrak{A}_2],\cdots,[\mathfrak{A}_t]$ of $\mathcal{C}(K)_3^G$. Thus, the following equation can be established easily by the facts from linear space

$$\begin{aligned}r_3\left(\mathcal{C}(K)_3^G\cap\mathcal{C}(K)_3^{\mathcal{I}}\right)=\dim_{\mathbb{Z}/3\mathbb{Z}}(\ker\Pi)&=t-\dim_{\mathbb{Z}/3\mathbb{Z}}(\mathrm{im}\Pi)\\&=t-\mathrm{rank}(A).\end{aligned}\tag{1}$$

Using the second isomorphism theorem of groups, we have

$$\mathcal{C}(K)_3^G/\mathcal{C}(K)_3^G\cap\mathcal{C}(K)_3^{\mathcal{I}}\cong\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}.$$

Combining this isomorphism with equation (1) yields that

$$\begin{aligned}\mathrm{rank}(A)&=t-r_3\left(\mathcal{C}(K)_3^G\cap\mathcal{C}(K)_3^{\mathcal{I}}\right)\\&=r_3\left(\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}\right),\end{aligned}$$

which is just what we wanted to prove, and this completes our proof. □

Remark 3.2 *With notation as in the above theorem, it is clear that* $s\le t$, *and* $\mathcal{C}(K)_3=\mathcal{C}(K)_3^G$, *i.e. all elements in* $\mathcal{C}(K)_3$ *are ambiguous ideal classes, if* $s=t$. *As for the equation* $r_3(\mathcal{C}(K))=2t-s$, *we can approach it using another method as follows. We first observe that if* $3\nmid h(\mathcal{O}_F)$, *then*

$$\mathcal{C}(K)_3^3=\mathcal{C}(K)_3^{\mathcal{I}^2},$$

and

$$r_3(\mathcal{C}(K))=r_3\left(\mathcal{C}(K)_3/\mathcal{C}(K)_3^3\right)=r_3\left(\mathcal{C}(K)_3/\mathcal{C}(K)_3^{\mathcal{I}}\right)+r_3\left(\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^3\right).\tag{2}$$

However, it is easy to see that

$$|\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}^2}|=|\mathcal{C}(K)_3^G\cap\mathcal{C}(K)_3^{\mathcal{I}}|.\tag{3}$$

$\mathcal{C}(K)_3^G/\mathcal{C}(K)_3^G\cap\mathcal{C}(K)_3^{\mathcal{I}}\cong\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}$ *together with (3) yields that*

$$\begin{aligned}s&=r_3\left(\mathcal{C}(K)_3^G\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}\right)\\&=r_3\left(\mathcal{C}(K)_3^G/\mathcal{C}(K)_3^G\cap\mathcal{C}(K)_3^{\mathcal{I}}\right)\\&=r_3\left(\mathcal{C}(K)_3^G\right)-r_3\left(\mathcal{C}(K)_3^G\cap\mathcal{C}(K)_3^{\mathcal{I}}\right).\end{aligned}\tag{4}$$

It follows from (2) and (4) that $r_3(\mathcal{C}(K)) = 2t - s$. *Although the way to the* 3*-rank of* $\mathcal{C}(K)$ *presented here is simpler compared with that in Theorem 2.4, we should point out that we cannot obtain the results of the second part of Theorem 2.4 from this simple method.*

From the above theorem, we see that one should find explicitly the number t, the elements $x_1, x_2, \cdots, x_t$, and the ambiguous ideal classes $[\mathfrak{A}_1], [\mathfrak{A}_2], \cdots, [\mathfrak{A}_t]$, which is not easy work in general, when one applies the above theorem to a concrete case. However, when we focus our attention on the cubic Kummer function fields over a rational function field, the computations mentioned above become relatively easy. In the remainder of this paper, we consider the cubic Kummer function fields over $k = \mathbb{F}_q(T)$ with $3 \mid q-1$, show how to determine explicitly t, $x_1, \cdots, x_t$ and $[\mathfrak{A}_1], \cdots, [\mathfrak{A}_t]$, and present explicitly the 3-rank of cubic Kummer function fields over k in some special cases.

4. Kummer function fields over rational function field

Let $S_\infty(k) = \{\infty = \frac{1}{T}\}$. Then the integral domain of k with respect to $S_\infty(k)$ is the polynomial ring $\mathcal{O}_k = \mathbb{F}_q[T]$, $\mathcal{O}_k^* = \mathbb{F}_q^*$, and $\mathcal{C}(k)$ is trivial. It is worth noting here that the finite prime divisor of k corresponds to the monic irreducible polynomial of positive degree in $\mathcal{O}_k$. Suppose that $K = k(\sqrt[3]{D})$ is a cubic Kummer function field, where D is a 3-power free polynomial in $\mathcal{O}_k$ of positive degree. It is not hard to check that two cubic Kummer function fields $k(\sqrt[3]{D})$ and $k(\sqrt[3]{D'})$ are equal if and only if $D' = x^3 D^i$ for some $x \in k^*$ and positive integer $1 \le i \le 2$.

As for the cubic Kummer function field $K = k(\sqrt[3]{D})$ with the Galois group $G = \mathrm{Gal}(K/k) =< \sigma >$, let $D = a\prod_{i=1}^{g} P_i^{e_i}$, where $a \in \mathbb{F}_q^*$, P_i are monic irreducible polynomials and $1 \le e_i \le 2, 1 \le i \le g$. In fact, we may take a as an element in $\mathbb{F}_q^*/\left(\mathbb{F}_q^*\right)^3$. We can assume that the leading coefficient of D is either 1 or γ, i.e. $a \in \{1, \gamma\}$, where γ is a fixed generator of $\mathbb{F}_q^*$. In order to go on with the discussion, we determine firstly how the prime divisors of k ramify in K. The following conclusion, which can be verified easily, is well known (see Lemma 3 in [17]).

Lemma 4.1 *With notation defined as in the previous paragraph, the finite prime divisor* P *of* k *ramifies in* K *if and only if* $P \mid D$. *With respect to the infinite prime divisor* ∞, *if* $3 \nmid \deg D$, *then* ∞ *ramifies totally in* K; *if* $3 \mid \deg D$ *and* D *is monic modulo* $\left(\mathbb{F}_q^*\right)^3$, *then* ∞ *splits completely in* K; *otherwise,* ∞ *is inert in* K.

In terms of this lemma, we can compute $t = r_3\left(\mathcal{C}(K)_3/\mathcal{C}(K)_3^{\mathcal{I}}\right) = r_3\left(\mathcal{C}(K)_3^G\right)$ as follows (see Corollary 2.2.10 in [1] or Corollary 3.6 in [22]).

Lemma 4.2 *Let* $K = k(\sqrt[3]{D})/k$ *be a cubic Kummer extension with* $D = a\prod_{i=1}^{g} P_i^{e_i}$. *Suppose that* κ *is equal to 0 or 1 accordingly as* ∞ *splits completely in* K *or not. Then,*

$$t = r_3\left(\mathcal{C}(K)_3^G\right) = g + \kappa - 1 - \log_3\left([\mathbb{F}_q^* : \mathbb{F}_q^* \cap N_{K/k}(K^*)]\right),$$

where $N_{K/k}$ *denotes the norm map from* K *to* k.

The following lemma characterizes when the generator γ of $\mathbb{F}_q^*$ can be a norm of some elements in K^* (see Lemma 3.4 in [2]).

Lemma 4.3 *Let* $K = k(\sqrt[3]{D})$ *with* $D = a\prod_{i=1}^{g} P_i^{e_i}$. *If* $3 \mid \deg P_i$ *for every* $1 \le i \le g$, *then* $\mathbb{F}_q^* \cap N_{K/k}(K^*) = \mathbb{F}_q^*$; *otherwise* $\mathbb{F}_q^* \cap N_{K/k}(K^*) = \left(\mathbb{F}_q^*\right)^3$.

We observe that $\left(\mathbb{F}_q^*\right)^3 \subset N_{K/k}(\mathcal{O}_K^*) \subset \mathbb{F}_q^* \cap N_{K/k}(K^*) \subset \mathbb{F}_q^*$ and $[\mathbb{F}_q^* : \left(\mathbb{F}_q^*\right)^3] = 3$. Combining this fact along with the above three lemmas, we can determine t explicitly as follows.

Proposition 4.4 *Let* $K = k(\sqrt[3]{D})$ *with* $D = a\prod_{i=1}^{g} P_i^{e_i}$, *where* $a \in \{1, \gamma\}$ *and* $1 \le e_i \le 2, i = 1, \cdots, g$. *For* $t = r_3\left(\mathcal{C}(K)_3^G\right)$, *we have*

(i) *if* $a = 1$, $3 \mid \deg D$ *and* $3 \mid \deg P_i$ *for every* $1 \le i \le g$, *then* $t = g - 1$;

(ii) *if* $a = 1$, $3 \mid \deg D$ *and* $3 \nmid \deg P_i$ *for some* P_i, *then* $t = g - 2$;

(iii) *if* $a = \gamma$, $3 \mid \deg D$ *and* $3 \mid \deg P_i$ *for every* $1 \le i \le g$, *then* $t = g$;

(iv) *if* $a = \gamma$, $3 \mid \deg D$ *and* $3 \nmid \deg P_i$ *for some* P_i, *then* $t = g - 1$;

(v) *if* $3 \nmid \deg D$, *then then* $t = g - 1$.

It has to be pointed out that the conclusion of the above proposition, which is established from a different point of view compared with Peng's, is a special case of Lemma 5 in [17]. Actually, Peng [17] described explicitly the genus field of Kummer function fields (see also Theorem 2.5 in [20]). For the sake of completeness, we present the result here without proof.

Lemma 4.5 *With notation defined as above, suppose that* $3 \nmid \deg P_1, \cdots, \deg P_m$, *and* $3 \mid \deg P_{m+1}, \cdots, \deg P_g$. *We have:*

(i) *if* $a = 1$, $3 \mid \deg D$ *and* $m = 0$, *then* $G_K = k(\sqrt[3]{P_1}, \cdots, \sqrt[3]{P_g})$;

(ii) *if* $a = 1$, $3 \mid \deg D$ *and* $m > 1$, *then*

$$G_K = k(\sqrt[3]{P_1 P_2^{a_2}}, \cdots, \sqrt[3]{P_1 P_m^{a_m}}, \sqrt[3]{P_{m+1}}, \cdots, \sqrt[3]{P_g}),$$

where a_i *are integers in* $\{1, 2\}$ *such that* $3 \mid \deg P_1 + a_i \deg P_i$, $i = 2, \cdots, m$;

(iii) *if* $a = \gamma$, $3 \mid \deg D$ *and* $m = 0$, *then* $G_K = k(\sqrt[3]{\gamma}, \sqrt[3]{P_1}, \cdots, \sqrt[3]{P_g})$;

(iv) *if* $a = \gamma$, $3 \mid \deg D$ *and* $m > 1$, *then*

$$G_K = k(\sqrt[3]{\gamma}, \sqrt[3]{P_1 P_2^{a_2}}, \cdots, \sqrt[3]{P_1 P_m^{a_m}}, \sqrt[3]{P_{m+1}}, \cdots, \sqrt[3]{P_g}),$$

where a_i *are defined as the case* (ii);

(v) *if* $a = 1$ *and* $3 \nmid \deg D$, *then* $G_K = k(\sqrt[3]{P_1}, \cdots, \sqrt[3]{P_g})$;

(vi) *if* $a=\gamma$ *and* $3\nmid \deg D$, *then*

$$G_K = k(\sqrt[3]{\gamma^{b_1}P_1},\cdots,\sqrt[3]{\gamma^{b_m}P_m},\sqrt[3]{P_{m+1}},\cdots,\sqrt[3]{P_g}),$$

where b_i *are integers in* $\{1,2\}$ *such that* $3\mid \deg P_i - b_i\deg D$, $i=1,\cdots,m$.

Combining Proposition 4.4 along with Lemma 4.5, we get explicitly the number t and elements $x_1,\cdots,x_t$ for the Kummer function field $K=k(\sqrt[3]{D})$. We summarize these results as follows for ease of use.

Proposition 4.6 *Let* $K=k(\sqrt[3]{D})$ *with* $D=a\prod_{i=1}^{g}P_i^{e_i}$, *where* $a\in\{1,\gamma\}$ *and* $1\le e_i\le 2, i=1,\cdots,g$, *and* $t=r_3\left(\mathcal{C}(K)_3^G\right)$. *Suppose that* $3\nmid \deg P_1,\cdots,\deg P_m$, *and* $3\mid \deg P_{m+1},\cdots,\deg P_g$. *We get that:*

(i) *if* $a=1$, $3\mid\deg D$ *and* $m=0$, *then* $t=g-1$ *and* $G_K=K(\sqrt[3]{P_1},\cdots,\sqrt[3]{P_{g-1}})$;

(ii) *if* $a=1$, $3\mid\deg D$ *and* $m>1$, *then* $t=g-2$ *and*

$$G_K = K(\sqrt[3]{P_1P_2^{a_2}},\cdots,\sqrt[3]{P_1P_{m-1}^{a_{m-1}}},\sqrt[3]{P_{m+1}},\cdots,\sqrt[3]{P_g}),$$

where a_i *are integers in* $\{1,2\}$ *such that* $3\mid\deg P_1+a_i\deg P_i$, $i=2,\cdots,m-1$;

(iii) *if* $a=\gamma$, $3\mid\deg D$ *and* $m=0$, *then* $t=g$ *and* $G_K=K(\sqrt[3]{P_1},\cdots,\sqrt[3]{P_g})$;

(iv) *if* $a=\gamma$, $3\mid\deg D$ *and* $m>1$, *then* $t=g-1$ *and*

$$G_K = K(\sqrt[3]{P_1P_2^{a_2}},\cdots,\sqrt[3]{P_1P_m^{a_m}},\sqrt[3]{P_{m+1}},\cdots,\sqrt[3]{P_g}),$$

where a_i *are integers in* $\{1,2\}$ *such that* $3\mid\deg P_1+a_i\deg P_i$, $i=2,\cdots,m$;

(v) *if* $a=1$ *and* $3\nmid\deg D$, *then* $t=g-1$ *and* $G_K=K(\sqrt[3]{P_1},\cdots,\sqrt[3]{P_{g-1}})$;

(vi) *if* $a=\gamma$ *and* $3\nmid\deg D$, *then* $t=g-1$ *and*

$$G_K = K(\sqrt[3]{\gamma^{b_1}P_1},\cdots,\sqrt[3]{\gamma^{b_{m-1}}P_{m-1}},\sqrt[3]{P_{m+1}},\cdots,\sqrt[3]{P_g}),$$

where b_i *are integers in* $\{1,2\}$ *such that* $3\mid\deg P_i-b_i\deg D$, $i=1,\cdots,m-1$.

Finally, we need to describe explicitly the basis of $\mathcal{C}(K)_3^G$ for cubic Kummer function field K, which is our main goal of the rest of this section. In fact, Wittmann determined explicitly in [20] the generators of $\mathcal{C}(K)_3^G$ by considering the long exact cohomological sequence attached to a classical exact sequence.

For the cubic Kummer function field $K=k(\sqrt[3]{D})$ with $D=a\prod_{i=1}^{g}P_i^{e_i}$, denote by $\mathfrak{p}_1,\cdots,\mathfrak{p}_g$ the prime ideals of $\mathcal{O}_K$ lying above the finite prime ideals $P_1\mathcal{O}_k,\cdots,P_g\mathcal{O}_k$ of k, which are exactly the ramified finite prime divisors in K/k. Suppose that $\mathfrak{a}$ is an integral ideal in $\mathcal{O}_K$ such that all the prime ideals dividing it split completely in K/k, and satisfies $\sigma\mathfrak{a}=\beta\mathfrak{a}$ where $\beta\in K^*$ with $N_{K/k}(\beta)\in\mathbb{F}_q^*\backslash\left(\mathbb{F}_q^*\right)^3$. We refer to the remark of Corollary 2.4 in [20] for the reason that we can make this assumption. With the help of Corollary 2.4 in [20], we can describe the basis of $\mathcal{C}(K)_3^G$ as follows.

Proposition 4.7 *With notation defined as above, we can obtain*

(i) *if* $a=1$, $3\mid\deg D$ *and* $m=0$, *then* $[\mathfrak{p}_1],\cdots,[\mathfrak{p}_{g-2}],[\mathfrak{a}]$ *is a basis for* $\mathcal{C}(K)_3^G$;

(ii) *if* $a=1$, $3\mid\deg D$ *and* $m>1$, *then* $[\mathfrak{p}_1],\cdots,[\mathfrak{p}_{g-2}]$ *is a basis for* $\mathcal{C}(K)_3^G$;

(iii) *if* $a=\gamma$, $3\mid\deg D$ *and* $m=0$, *then* $[\mathfrak{p}_1],\cdots,[\mathfrak{p}_{g-1}],[\mathfrak{a}]$ *is a basis for* $\mathcal{C}(K)_3^G$;

(iv) *if* $a=\gamma$, $3\mid\deg D$ *and* $m>1$, *then* $[\mathfrak{p}_1],\cdots,[\mathfrak{p}_{g-1}]$ *is a basis for* $\mathcal{C}(K)_3^G$;

(v) *if* $3\nmid\deg D$, *then* $[\mathfrak{p}_1],\cdots,[\mathfrak{p}_{g-1}]$ *is a basis for* $\mathcal{C}(K)_3^G$.

In view of Proposition 4.6 and 4.7, we can now determine explicitly in this special case the rank of A mentioned in Theorem 3.1 by computing the appropriate power residue symbols. Since the procedure of computations is similar, we only give the details for case (i) in Proposition 4.6.

In order to do the computation, we need to recall what the lth power residue symbol in $\mathbb{F}_q[T]$. Let $P\in\mathbb{F}_q[T]$ be an irreducible polynomial and l a prime number with $l\mid q-1$. For $A\in\mathbb{F}_q[T]$ with $P\nmid A$, define the lth power residue symbol $\left(\frac{A}{P}\right)_l$ as the unique element in $\mathbb{F}_q^*$ such that

$$A^{(q^{\deg P}-1)/l}\equiv\left(\frac{A}{P}\right)_l \pmod{P}.$$

If $P\mid A$, define $\left(\frac{A}{P}\right)_l=0$. We can extend the definition of the lth power residue symbol to the case that P is replaced with an arbitrary $0\neq B\in\mathbb{F}_q[T]$. For polynomial $B=\prod_{i=1}^m P_i^{n_i}\in\mathbb{F}_q[T]$, define $\left(\frac{A}{B}\right)_l$ as

$$\left(\frac{A}{B}\right)_l=\prod_{i=1}^m\left(\frac{A}{P_i}\right)_l^{n_i}.$$

For further details and the properties of the lth power residue symbol, we refer to Chapter 3 in [19].

With notation defined as in the previous two propositions, we now compute matrix A in Theorem 3.1. In order to simplify the calculation and demonstrate the procedure, we make an assumption in case (i) that every prime ideal $\mathfrak{p}_i$ splits completely in $K(\sqrt[3]{P_i})/K$.

In case (i), $t=g-1$, $G_K=K(\sqrt[3]{P_1},\cdots,\sqrt[3]{P_{g-1}})$, and $[\mathfrak{p}_1],\cdots,[\mathfrak{p}_{g-2}],[\mathfrak{a}]$ is a basis for $\mathcal{C}(K)_3^G$. Denote by $\mathfrak{P}_i$ the prime ideal of $K(\sqrt[3]{P_i})$ lying above $\mathfrak{p}_i$ for $1\le i\le g-1$. For $1\le i\neq j\le g-2$, since P_i and P_j are prime to each other,

$$\frac{\left(\frac{K(\sqrt[3]{P_i})/K}{\mathfrak{p}_j}\right)\sqrt[3]{P_i}}{\sqrt[3]{P_i}}\equiv\sqrt[3]{P_i}^{N\mathfrak{p}_j-1}=P_i^{(q^{\deg P_j}-1)/3}\equiv\left(\frac{P_i}{P_j}\right)_3 \pmod{\mathfrak{P}_j}. \tag{5}$$

This congruence implies that

$$\eta^{a_{ij}}=\left(\frac{P_i}{P_j}\right)_3,\quad 1\le i\neq j\le g-2. \tag{6}$$

Similarly, we have

$$\eta^{a_{g-1\ j}}=\left(\frac{P_{g-1}}{P_j}\right)_3,\quad 1\le j\le g-2,\quad \eta^{a_{i\ g-1}}=\left(\frac{P_i}{\mathfrak{a}}\right)_3,\quad 1\le i\le g-1. \tag{7}$$

Thus, for $1 \le i \ne j \le g-1$, a_{ij} and a_{g-1g-1} can be determined by the above three equations (6) and (7). Now there are only the diagonal entries $a_{ii}, 1 \le i \le g-2$ that need to be determined. As it is well known that if $\mathfrak{p}_i$ splits completely in $K(\sqrt[3]{P_i})/K$, then the Artin symbol $\left(\frac{K(\sqrt[3]{P_i})/K}{\mathfrak{p}_i}\right)$ is an identity, and thus $a_{ii} = 0$. Hence, matrix A is determined completely in this case.

Some remarks are in order before giving an example to illustrate our computational method. We note here that the congruence

$$\left(\frac{K(\sqrt[3]{P_i})/K}{\mathfrak{p}_i}\right)\sqrt[3]{P_i} \equiv \sqrt[3]{P_i}^{N\mathfrak{p}_i} (\mathrm{mod}\ \mathfrak{P}_i)$$

holds also for P_i and $\mathfrak{p}_i$. However, we cannot divide both sides of the above congruence by $\sqrt[3]{P_i}$ as (5) because $\eta^{a_{ii}} \not\equiv P_i^{(N\mathfrak{p}_i-1)/3} (\mathrm{mod}\ \mathfrak{P}_i)$. Thus, the above method for computing a_{ij} for $i \ne j$ cannot be applied here to determine a_{ii} when the assumption that every prime ideal $\mathfrak{p}_i$ splits completely in $K(\sqrt[3]{P_i})/K$ is removed. We point out that the computation of $\left(\frac{K(\sqrt[3]{P_i})/K}{\mathfrak{p}_i}\right)\sqrt[3]{P_i}/\sqrt[3]{P_i}$ is complicated and associated with the local Hilbert symbol for local fields extension $K_{\mathfrak{p}_i}(\sqrt[3]{P_i})/K_{\mathfrak{p}_i}$ whose Galois group is isomorphic to the decomposition group of $\mathfrak{p}_i$.

Example 4.1 *Let $k = \mathbb{F}_7(T)$, $P_1 = T$, $P_2 = T^2+2$, $P_3 = T^2+3T+4$, $D = P_1P_2P_3$, and $K = k(\sqrt[3]{D})$. Then $3 \nmid \deg D$, and this means by Proposition 4.6 and 4.7 in turn that $t = 3$, $G_K = K(\sqrt[3]{P_1}, \sqrt[3]{P_2})$, and $[\mathfrak{p}_1], [\mathfrak{p}_2]$ is a basis for $\mathcal{C}(K)_3^G$. We observe that 3 is a generator of $\mathbb{F}_7^*$ and 2 is a primitive 3-root of unit in k.*

It is easy to check by the lth power reciprocity law that $\left(\frac{P_1}{P_2}\right)_3 = \left(\frac{P_2}{P_1}\right)_3 = 4$. By the relation

$$2^{a_{12}} = \left(\frac{P_1}{P_2}\right)_3 = 4 = \left(\frac{P_2}{P_1}\right)_3 = 2^{a_{21}},$$

$a_{12} = a_{21} = 2$. In what follows, we compute a_{11}. Notice that P_1 is ramified totally in K/k and unramified in $k(\sqrt[3]{P_2P_3})/k$, and $K(\sqrt[3]{P_1}) = k(\sqrt[3]{P_1}, \sqrt[3]{P_2P_3})$. This implies that the ramified index of P_1 in $K(\sqrt[3]{P_1})/k$ is 3, and implies in turn that the prime ideal $\mathfrak{p}_1$ of K lying above P_1 is unramified in $K(\sqrt[3]{P_1})/K$. Furthermore, we can assert by the Kummer theorem that $\mathfrak{p}_1$ splits completely in $K(\sqrt[3]{P_1})/K$. This tells us that the Artin symbol $\left(\frac{K(\sqrt[3]{P_1})/K}{\mathfrak{p}_1}\right)$ is an identity, and thus $a_{11} = 0$. No matter what value a_{22} takes, the determinant of A is nonzero in $\mathbb{Z}/3\mathbb{Z}$, and thus the rank of A is 2. Therefore, $s = 2$.

By the remark of Theorem 3.1, the 3-rank of $\mathcal{C}(K)$ is equal to 2, and the Sylow 3-subgroup $\mathcal{C}(K)_3$ of $\mathcal{C}(K)$ is an elementary abelian group. This implies clearly that

$$\mathcal{C}(K)_3 \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

5. Further remarks

Generally speaking, characterizing explicitly the structure of class groups of global fields is an important and difficult issue in number theory. In this paper, we concentrate our attention on the 3-rank of cubic cyclic function fields. For the global function field $F/\mathbb{F}_q$ with $3 \mid q-1$, we determined in Theorem 3.1 the 3-rank of the ideal class group of cubic cyclic function field K by our previous results for arbitrary cyclic extensions.

From the proof of our theorem, we see that the difficulties in computing $r_3(\mathcal{C}(K))$ lie in the determination of $s = r_3\left(\mathcal{C}(K)_3^G \mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}\right)$. With the help of the class field theory, we convert this problem into calculating a matrix, which is determined by the Artin symbol for Kummer function fields. Furthermore, our theorem tells us also that the Sylow 3-group of $\mathcal{C}(K)$ is isomorphic to the direct product of an abelian 3-group and an elementary abelian 3-group, and this allows us to understand more clearly the structure of $\mathcal{C}(K)$. However, we should also point out that the computation of matrix A in our theorem is not an ordinary thing. In fact, Wittmann and Bae's result on $\lambda_2 = \dim_{\mathbb{Z}/l\mathbb{Z}}\left(\mathcal{C}(K)_l^{\mathcal{I}}/\mathcal{C}(K)_l^{\mathcal{I}^2}\right)$ (see [2] and [20]) can be used to calculate s. We describe in detail in the remainder of this section how to make use of Wittmann and Bae's result to determine s for the Kummer function fields over a rational function field.

Note that

$$|\mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}^2}| = |\mathcal{C}(K)_3^G \cap \mathcal{C}(K)_3^{\mathcal{I}}|, \tag{8}$$

and

$$\mathcal{C}(K)_3^G/\mathcal{C}(K)_3^G \cap \mathcal{C}(K)_3^{\mathcal{I}} \cong \mathcal{C}(K)_3^G \mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}. \tag{9}$$

Since $\mathcal{C}(k)$ is trivial, all groups involved above are elementary abelian 3-group. Combining (8) with (9) yields that

$$\begin{aligned} s &= r_3\left(\mathcal{C}(K)_3^G \mathcal{C}(K)_3^{\mathcal{I}}/\mathcal{C}(K)_3^{\mathcal{I}}\right) \\ &= r_3\left(\mathcal{C}(K)_3^G\right) - r_3\left(\mathcal{C}(K)_3^G \cap \mathcal{C}(K)_3^{\mathcal{I}}\right) \\ &= t - \lambda_2. \end{aligned}$$

Hence, with the aid of computation of λ_2 provided by Wittmann and Bae, we can determine explicitly the 3-rank of $\mathcal{C}(K)$ also. However, as mentioned before, this only tells us the rank of $\mathcal{C}(K)$; we cannot know more clearly in this way the structure of $\mathcal{C}(K)$ as Theorem 2.4 described.

Finally, we conclude our paper with an example by the method of Theorem 2.4 given along with the computation of λ_2 provided by Wittmann and Bae.

Example 5.1 *Let $k = \mathbb{F}_7(T)$, $P_1 = T$, $P_2 = T^2 + 2$, $P_3 = T^2 + 3T + 5$, $P_4 = T^3 + T + 1$, $D = P_1^2 P_2 P_3 P_4$, and $K = k(\sqrt[3]{D})$. This corresponds to case (ii) in Proposition 4.6, and thus $t = 2$. Bae's computation in [2] implies that $\lambda_2 = 0$. Combining this fact with arguments mentioned in this section yields that $s = t - \lambda_2 = 2$. It follows from Theorem 2.4 that the 3-rank of $\mathcal{C}(K)$ is equal to 2, and the Sylow 3-subgroup $\mathcal{C}(K)_3$ of $\mathcal{C}(K)$ is an elementary abelian group. This implies in turn that*

$$\mathcal{C}(K)_3 \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

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