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On the higher derivatives of the inverse tangent function

Mohamed Amine BOUTICHE*, **Mourad RAHMANI**
Faculty of Mathematics, USTHB, Algiers, Algeria

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Abstract: In this paper, we find explicit formulas for higher-order derivatives of the inverse tangent function. More precisely, we study polynomials that are induced from the higher-order derivatives of $\arctan(x)$. Successively, we give generating functions, recurrence relations, and some particular properties for these polynomials. Connections to Chebyshev, Fibonacci, Lucas, and matching polynomials are established.

Key words: Explicit formula, derivative polynomial, inverse tangent function, Chebyshev polynomial, matching polynomial

1. Introduction

The problem of establishing closed formulas for the n-derivative of the function $\arctan(x)$ is not straightforward and has been proved to be important for deriving rapidly convergent series for π [2, 3, 14]. Recently, many authors investigated the aforementioned problem and derived simple explicit closed-form higher derivative formulas for some classes of functions. In [1, 6, 8] and references therein, the authors found explicit forms of the derivative polynomials of the hyperbolic, trigonometric tangent, cotangent, and secant functions. Several new closed formulas for higher-order derivatives have been established for trigonometric and hyperbolic functions in [19], tangent and cotangent functions in [16], and arc-sine functions in [17].

We note from entries 1.1.7(3) and 1.1.7(4) in chapter 1 of Brychkov's handbook [7, p. 14] that the higher-order derivatives of $\arctan(x)$ can be expressed in terms of Chebyshev polynomials as follows:

$$\begin{cases} \dfrac{d^{2n}}{dx^{2n}}(\arctan(ax)) = (-1)^n (2n-1)! a^{2n+1} x \left(1+a^2x^2\right)^{-n-1/2} U_{2n-1}\left(\frac{1}{\sqrt{1+a^2x^2}}\right) \ (n \geq 1) \\ \dfrac{d^{2n+1}}{dx^{2n+1}}(\arctan(ax)) = (-1)^n (2n)! a^{2n+1} \left(1+a^2x^2\right)^{-n-1/2} T_{2n+1}\left(\frac{1}{\sqrt{1+a^2x^2}}\right) \ (n \geq 0) \end{cases}.$$

In the present work and in order to simplify the above formulas, we study polynomials that are induced from the higher-order derivatives of $\arctan(x)$. Then our main result is

$$\frac{d^n}{dx^n}(\arctan(ax)) = \frac{a^n (n-1)!}{(1+a^2x^2)^{\frac{n+1}{2}}} U_{n-1}\left(-\frac{ax}{\sqrt{1+a^2x^2}}\right) \ (n \geq 1),$$

where U_n is the nth Chebyshev polynomial of the second kind. In the rest of paper, without loss of generality, we assume $a = 1$.

*Correspondence: mboutiche@usthb.dz
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2. The fundamental properties of the alpha and beta polynomials

We consider the problem of finding the nth derivative of $\arctan(x)$. It is easy to see that there exists a real sequence of polynomials

$$P_n(x) = (-1)^n n! \operatorname{Im}((x+i)^{n+1})$$

such that

$$\begin{aligned}\frac{d^n}{dx^n}(\arctan x) &= \frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{2i}\left(\frac{1}{x-i}-\frac{1}{x+i}\right)\right] \\ &= \frac{d^{n-1}}{dx^{n-1}}\left[\operatorname{Im}\left(\frac{1}{x-i}\right)\right] \\ &= \frac{P_{n-1}(x)}{(1+x^2)^n}, \qquad (1)\end{aligned}$$

where $\operatorname{Im}(z)$ denotes the imaginary part of z.

By differentiation (1) with respect to x, we get the recursion relation [14]

$$P_0(x) = 1, \; P_{n+1}(x) = \left(1+x^2\right) P_n'(x) - 2(n+1) x P_n(x). \qquad (2)$$

An explicit expression of $P_n(x)$ is obtained by using the binomial formula

$$P_n(x) = (-1)^n n! \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n+1}{2k+1} x^{n-2k}, \qquad (3)$$

where $\lfloor x \rfloor$ denotes the integral part of x, that is, the greatest integer not exceeding x. We may rewrite

$$\begin{aligned}\beta_n(x) &:= (-1)^n \frac{P_n(x)}{n!} \\ &= \operatorname{Im}((x+i)^{n+1}) \qquad (4) \\ &= \sum_{k=0}^{n} \binom{n+1}{k+1} \cos\left(\frac{k\pi}{2}\right) x^{n-k}.\end{aligned}$$

In particular, we have

$$\beta_n(1) = 2^{\frac{n+1}{2}} \cos\left((n-1)\frac{\pi}{4}\right) = 2^{\frac{n+1}{2}} \sin\left((n+1)\frac{\pi}{4}\right).$$

In 1755, Euler derived the well-known formula [11, p. 39]

$$\arctan(x) = \sum_{n\geq 0} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}}.$$

As an immediate application of (4), we obtain another expansion of the inverse tangent function.

Theorem 1 *We have*

$$\arctan(x) = \sum_{n\geq 0} \frac{\beta_n(x)}{n+1} \frac{x^{n+1}}{(1+x^2)^{n+1}}.$$

Proof From (1) and [14, p. 228, Eq. (9)]

$$\arctan(x) = \sum_{n\geq 1} (-1)^{n+1} \frac{d^n}{dx^n} \arctan(x) \; \frac{x^n}{n!},$$

we get the desired result. □

Now we give some fundamental results concerning $\beta_n(x)$.

Theorem 2 (Generating function) *The ordinary generating function of $\beta_n(x)$ is given by*

$$f_x(z) = \sum_{n\geq 0} \beta_n(x) z^n = \frac{1}{1 - 2xz + (1+x^2) z^2} \quad (x \in \mathbb{R}; |z| < 1) \tag{5}$$

Proof We have

$$\begin{aligned} f_x(z) &= \sum_{n\geq 0} (xz)^n \sum_{k\geq 0} \binom{n+1}{k+1} \cos\left(\frac{k\pi}{2}\right) (-x)^{-k} \\ &= \sum_{n\geq 0} (xz)^n \operatorname{Re}\left(\sum_{k\geq 0} \binom{n+1}{k+1} \left(-\frac{i}{x}\right)^k \right) \\ &= \sum_{n\geq 0} (xz)^n \operatorname{Re}\left(\frac{i}{x^n} (x-i)^{n+1} - ix \right) \\ &= \frac{1}{2} \sum_{n\geq 0} z^n \left(i(x-i)^{n+1} - ix(x+i)^{n+1} \right) \\ &= \frac{1}{2} i(x-i) \sum_{n\geq 0} (z(x-i))^n - \frac{1}{2} i(x+i) \sum_{n\geq 0} (z(x+i))^n \\ &= \frac{1}{2} \left(\frac{i(x-i)}{1 - z(x-i)} - \frac{i(x+i)}{1 - z(x+i)} \right). \end{aligned}$$

Thus, the proof of the theorem is completed. □

Theorem 3 (Generating function) *The exponential generating function of $\beta_n(x)$ is given by*

$$\sum_{n\geq 0} \beta_n(x) \frac{z^n}{n!} = (\cos(z) + x\sin(z)) e^{xz}. \tag{6}$$

Proof From (4), we have

$$\begin{aligned} \sum_{n\geq 0} \operatorname{Im}\left((x+i)^{n+1} \right) \frac{z^n}{n!} &= \operatorname{Im}\left((x+i) \sum_{n\geq 0} \frac{((x+i)z)^n}{n!} \right) \\ &= \operatorname{Im}\left((x+i) \exp((x+i)z) \right) \\ &= e^{xz} \operatorname{Im}((x+i)e^{iz}) \\ &= e^{xz} (\cos z + x \sin z). \end{aligned}$$

Thus, the proof of the theorem is completed. □

Theorem 4 (Recurrence relation) *The* $\beta_n(x)$ *satisfy the following three-term recurrence relation:*

$$\beta_{n+1}(x) = 2x\beta_n(x) - \left(1+x^2\right)\beta_{n-1}(x), \tag{7}$$

with initial conditions $\beta_0(x) = 1$ *and* $\beta_1(x) = 2x$.

Proof By differentiation (5) with respect to z, we obtain

$$\left(1 - 2xz + \left(1+x^2\right)z^2\right)\frac{\partial}{\partial z}f_x(z) = \left(2x - 2\left(1+x^2\right)z\right)f_x(z),$$

or equivalently

$$\left(1 - 2xz + \left(1+x^2\right)z^2\right)\sum_{n\geq 0} n\beta_n(x) z^{n-1} = \left(2x - 2\left(1+x^2\right)z\right)\sum_{n\geq 0}\beta_n(x) z^n.$$

After some rearrangement, we get

$$\sum_{n\geq 0}(n+1)\beta_{n+1}(x) z^n = \sum_{n\geq 0}\left(2x(n+1)\beta_n(x) - \left(1+x^2\right)(n+1)\beta_{n-1}(x)\right)z^n.$$

Equating the coefficient of z^n, we get the result. □

The first few $\beta_n(x)$ are listed in Eq. (8).

$$\begin{aligned}
&\beta_0(x) = 1,\\
&\beta_1(x) = 2x,\\
&\beta_2(x) = 3x^2 - 1,\\
&\beta_3(x) = 4x^3 - 4x,\\
&\beta_4(x) = 5x^4 - 10x^2 + 1,\\
&\beta_5(x) = 6x^5 - 20x^3 + 6x.
\end{aligned} \tag{8}$$

Theorem 5 *The leading coefficient of* x^n *in* $\beta_n(x)$ *is* $n+1$ *and the following result holds true:*

$$\beta_n(-x) = (-1)^n \beta_n(x). \tag{9}$$

Proof From (4) we may rewrite $\beta_n(x)$ as

$$\beta_n(x) = (n+1)x^n - \frac{1}{6}n\left(n^2-1\right)x^{n-2} + \cdots,$$

in which the leading coefficient of x^n in $\beta_n(x)$ is $n+1$. On the other hand, since

$$f_{-x}(-z) = f_x(z)$$

$$\sum_{n\geq 0}\beta_n(-x)(-z)^n = \sum_{n\geq 0}\beta_n(x) z^n.$$

Comparing these two series, we get (9). □

Remark 1 *Using (9) we can write*

$$P_n(x) = n!\beta_n(-x), \tag{10}$$

and the exponential generating function of $P_n(x)$ is given by

$$\sum_{n\geq 0} P_n(x)\frac{z^n}{n!} = \frac{1}{1+2xz+(1+x^2)z^2}.$$

and (2) becomes

$$\beta_{n+1}(x) = 2x\beta_n(x) - \frac{1+x^2}{n+1}\beta_n'(x). \tag{11}$$

Theorem 6 *For $n \geq 1$, we have*

$$\frac{d}{dx}\beta_n(x) = (n+1)\beta_{n-1}(x). \tag{12}$$

Proof By differentiation of $\beta_n(x)$ with respect to x, we obtain

$$\begin{aligned}\frac{d}{dx}\beta_n(x) &= (n+1)\operatorname{Im}\left((x+i)^n\right) \\ &= (n+1)\beta_{n-1}(x).\end{aligned}$$

□

Theorem 7 (Differential Equation) *$\beta_n(x)$ satisfies the linear second order ODE*

$$\left(1+x^2\right)\beta_n''(x) - 2nx\beta_n'(x) + n(n+1)\beta_n(x) = 0 \tag{13}$$

Proof By differentiating (11) and using (12), we find (13). □

Remark 2 *It is well known that the classical orthogonal polynomials are characterized by being solutions of the differential equation*

$$A(x)\gamma_n''(x) + B(x)\gamma_n'(x) + \lambda_n\gamma_n(x) = 0,$$

where A and B are independent of n and λ_n is independent of x. It is obvious that the $\beta_n(x)$ are nonclassical orthogonal polynomials.

Using matrix notation, (7) can be written as

$$\begin{pmatrix}\beta_{r+1}(x) & \beta_{r+2}(x)\end{pmatrix} = \begin{pmatrix}\beta_r(x) & \beta_{r+1}(x)\end{pmatrix}\begin{pmatrix}0 & -\left(1+x^2\right) \\ 1 & 2x\end{pmatrix}.$$

Therefore

$$\begin{pmatrix}\beta_{n+r}(x) & \beta_{n+r+1}(x)\end{pmatrix} = \begin{pmatrix}\beta_r(x) & \beta_{r+1}(x)\end{pmatrix}\begin{pmatrix}0 & -\left(1+x^2\right) \\ 1 & 2x\end{pmatrix}^n$$

for $n \geq 0$. Letting $r = 0$, we get

$$\begin{pmatrix}\beta_n(x) & \beta_{n+1}(x)\end{pmatrix} = \begin{pmatrix}1 & 2x\end{pmatrix}\begin{pmatrix}0 & -\left(1+x^2\right) \\ 1 & 2x\end{pmatrix}^n.$$

Theorem 8 *We have*

$$\beta_n(x) = \begin{pmatrix} 1 & 2x \end{pmatrix} \begin{pmatrix} 0 & -\left(1+x^2\right) \\ 1 & 2x \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It follows from the general theory of determinant [18] that $\beta_n(x)$ is the following $n \times n$ determinant:

$$\beta_n(x) = \begin{vmatrix} 2x & -\left(1+x^2\right) & 0 & \cdots & 0 \\ -1 & 2x & -\left(1+x^2\right) & & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ & & \ddots & \ddots & -\left(1+x^2\right) \\ 0 & \cdots & 0 & -1 & 2x \end{vmatrix}.$$

In order to compute the above determinant, we recall that the Chebyshev polynomials $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad \text{when} \quad x = \cos\theta,$$

and can also be written as determinant identity

$$U_n(x) = \begin{vmatrix} 2x & 1 & 0 & \cdots & 0 \\ 1 & 2x & 1 & & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ & & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 2x \end{vmatrix}. \tag{14}$$

The next lemma is used in the proof of Theorem 9

Lemma 1 *For a, b, c nonzero, we have*

$$\begin{vmatrix} b & c & 0 & \cdots & 0 \\ a & b & c & & \vdots \\ 0 & a & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & c \\ 0 & \cdots & 0 & a & b \end{vmatrix} = \left(\sqrt{ac}\right)^n U_n\left(\frac{b}{2\sqrt{ac}}\right). \tag{15}$$

Proof From (14), we have

$$\left(\sqrt{ac}\right)^n U_n\left(\frac{b}{2\sqrt{ac}}\right) = \begin{vmatrix} b & \sqrt{ac} & 0 & \cdots & 0 \\ \sqrt{ac} & b & \sqrt{ac} & & \vdots \\ 0 & \sqrt{ac} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \sqrt{ac} \\ 0 & \cdots & 0 & \sqrt{ac} & b \end{vmatrix}.$$

Now, by the symmetrization process [4], we get the result. □

Theorem 9 *For $n \geq 1$, we have*

$$\frac{d^n}{dx^n}(\arctan(x)) = \frac{(n-1)!}{(1+x^2)^n}\operatorname{Im}((i-x)^n)$$
$$= \frac{(n-1)!}{(1+x^2)^{\frac{n+1}{2}}}U_{n-1}\left(\frac{-x}{\sqrt{1+x^2}}\right),$$

where U_n is the nth Chebyshev polynomial of the second kind.

Proof We apply Lemma 1 with $a=-1, b=2x$, and $c=-\left(1+x^2\right)$ to obtain

$$\beta_n(x) = \left(\sqrt{1+x^2}\right)^n U_n\left(\frac{x}{\sqrt{1+x^2}}\right). \tag{16}$$

From (1) and (10), we get the desired result. □

Corollary 1 *We have*

$$\begin{cases} \dfrac{d^{2n}}{dx^{2n}}(\arctan(x)) = (-1)^n(2n-1)!x\left(1+x^2\right)^{-n-1/2}U_{2n-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) & (n \geq 1) \\ \dfrac{d^{2n+1}}{dx^{2n+1}}(\arctan(x)) = (-1)^n(2n)!\left(1+x^2\right)^{-n-1/2}T_{2n+1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) & (n \geq 0) \end{cases}. \tag{17}$$

Proof Formula (17) is an immediate consequence of Theorem 9, upon considering even and odd cases for n and using the relations

$$U_{2n-1}\left(\frac{-x}{\sqrt{1+x^2}}\right) = (-1)^n xU_{2n-1}\left(\frac{1}{\sqrt{1+x^2}}\right),$$
$$U_{2n}\left(\frac{-x}{\sqrt{1+x^2}}\right) = (-1)^n\sqrt{1+x^2}T_{2n+1}\left(\frac{1}{\sqrt{1+x^2}}\right),$$

where T_n is the nth Chebyshev polynomial of the first kind. □

Corollary 2 *For $n \geq 1$, we have*

$$\frac{d^n}{dx^n}\left(\tanh^{-1}(x)\right) = \frac{(n-1)!}{2(1-x^2)^n}((x+1)^n-(x-1)^n)$$
$$= \frac{1}{i^{n-1}}\frac{(n-1)!}{(1-x^2)^{\frac{n+1}{2}}}U_{n-1}\left(\frac{ix}{\sqrt{1-x^2}}\right)$$

Proof Since $\tanh^{-1}(x) = \frac{1}{i}\arctan(ix)$, we have

$$\frac{d^n}{dx^n}\left(\tanh^{-1}(x)\right) = \frac{(-1)^n}{i^{n+1}}\frac{P_{n-1}(ix)}{(1-x^2)^n}$$
$$= \frac{1}{i^{n-1}}\frac{P_{n-1}(-ix)}{(1-x^2)^n}.$$

Thus, the proof of the Corollary is completed. □

Theorem 10 *The roots of $\beta_n(x)$ of degree $n \geq 1$ have n simple zeros in $\mathbb{R}$ at*

$$x_k = \cot\left(\frac{k\pi}{n+1}\right), \text{ for each } k = 1, \ldots, n. \tag{18}$$

Proof Since the zeros of $U_n(z)$ are

$$z_k = \cos\left(\frac{k\pi}{n+1}\right), k = 1, \ldots, n,$$

it follows from (16) and by setting

$$z_k = \frac{x_k}{\sqrt{1+x_k^2}}$$

that the zeros of $\beta_n(x)$ are given by (18). □

It is well known that for any sequence of monic polynomials $p_n(x)$ whose degrees increase by one from one member to the next they satisfy an extended recurrence relation [10]

$$p_{n+1}(x) = xp_n(x) - \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} p_{n-j}(x),$$

and the zeros of $p_n(x)$ are the eigenvalues of the $n \times n$ Hessenberg matrix of the coefficients $\begin{bmatrix} n \\ j \end{bmatrix}$ arranged upward in the jth column

$$H_n = \begin{pmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 2 \end{bmatrix} & \cdots & \begin{bmatrix} n-2 \\ n-2 \end{bmatrix} & \begin{bmatrix} n-1 \\ n-1 \end{bmatrix} \\ 1 & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \cdots & \begin{bmatrix} n-2 \\ n-3 \end{bmatrix} & \begin{bmatrix} n-1 \\ n-2 \end{bmatrix} \\ 0 & 1 & \begin{bmatrix} 2 \\ 0 \end{bmatrix} & \cdots & \begin{bmatrix} n-2 \\ n-4 \end{bmatrix} & \begin{bmatrix} n-1 \\ n-3 \end{bmatrix} \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \begin{bmatrix} n-2 \\ 0 \end{bmatrix} & \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \\ 0 & 0 & 0 & \cdots & 1 & \begin{bmatrix} n-1 \\ 0 \end{bmatrix} \end{pmatrix}.$$

Let

$$\pi_n(x) := \frac{\beta_n(x)}{n+1}, \tag{19}$$

be the monic polynomial of degree n.

Theorem 11 *For $n \geq 0$, we have*

$$\pi_0(x) = 1; \ \pi_{n+1}(x) = x\pi_n(x) - \sum_{j=1}^{n} \frac{2^{j+1}}{j+1} \binom{n}{j} |B_{j+1}| \pi_{n-j}(x). \tag{20}$$

where B_n denote the Bernoulli numbers.

Proof By using generating function techniques, we can verify (20) directly. From (19) and (6), we have

$$\sum_{n\geq 0}\left(x\pi_n\left(x\right)-\sum_{j=1}^{n}\frac{2^{j+1}}{j+1}\binom{n}{j}\left|B_{j+1}\right|\pi_{n-j}\left(x\right)\right)\frac{z^n}{n!}=x\sum_{n\geq 0}\pi_n\left(x\right)\frac{z^n}{n!}-\sum_{j\geq 1}\frac{2^{j+1}}{(j+1)!}\left|B_{j+1}\right|\sum_{n\geq 0}\pi_{n-j}\left(x\right)\frac{z^n}{(n-j)!}$$
$$=\frac{1}{z}\left(x-\frac{1}{z}\sum_{j\geq 2}\frac{2^j}{j!}\left|B_j\right|z^j\right)\sum_{n\geq 1}\beta_{n-1}\left(x\right)\frac{z^n}{n!}.$$

Since

$$\cot\left(z\right)-\frac{1}{z}=-\sum_{j\geq 2}\frac{2^j}{j!}\left|B_j\right|z^{j-1},$$

and

$$\sum_{n\geq 1}\beta_{n-1}\left(x\right)\frac{z^n}{n!}=\int e^{xz}\left(\cos z+x\sin z\right)dz$$
$$=e^{xz}\sin z.$$

We get

$$\sum_{n\geq 0}\left(x\pi_n\left(x\right)-\sum_{j=1}^{n}\frac{2^{j+1}}{j+1}\binom{n}{j}\left|B_{j+1}\right|\pi_{n-j}\left(x\right)\right)\frac{z^n}{n!}=\frac{e^{xz}}{z^2}\left(\left(xz-1\right)\sin z+z\cos z\right).$$

On the other hand, we have

$$\sum_{n\geq 0}\pi_{n+1}\left(x\right)\frac{z^n}{n!}=\sum_{n\geq 0}\frac{\beta_{n+1}\left(x\right)}{n+2}\frac{z^n}{n!}$$
$$=\sum_{n\geq 0}\left(n+1\right)\beta_{n+1}\left(x\right)\frac{z^n}{(n+2)!}$$
$$=\sum_{n\geq 1}\left(n-1\right)\beta_{n-1}\left(x\right)\frac{z^{n-2}}{n!}$$
$$=\frac{1}{z}\sum_{n\geq 0}\beta_n\left(x\right)\frac{z^n}{n!}-\frac{1}{z^2}\sum_{n\geq 1}\beta_{n-1}\left(x\right)\frac{z^n}{n!}$$
$$=\frac{1}{z}e^{xz}\left(\cos z+x\sin z\right)-\frac{1}{z^2}e^{xz}\sin z$$
$$=\frac{1}{z^2}e^{xz}\left(z\cos z+\left(zx-1\right)\sin z\right).$$

The theorem is verified. □

Now, using the fact that $B_{2n+1}=0$ for $n>1$, we can write

$$\begin{bmatrix}n\\2j\end{bmatrix}=0 \text{ and } \begin{bmatrix}n\\2j+1\end{bmatrix}=\frac{2^{2j+2}}{2j+2}\binom{n}{2j+1}\left|B_{2j+1}\right|.$$

Then the $n \times n$ Hessenberg matrix H_n takes the form

$$H_n = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{2}{15} & 0 & \frac{16}{63} & \cdots & \frac{2^n}{n} \left| B_n \right| \\ 1 & 0 & \frac{2}{3} & 0 & \frac{8}{15} & 0 & \cdots & 2^{n-1} \left| B_{n-1} \right| \\ 0 & 1 & 0 & 1 & 0 & \frac{32}{21} & \cdots & (n-1)\, 2^{n-3} \left| B_{n-2} \right| \\ 0 & 0 & 1 & 0 & \frac{4}{3} & 0 & \cdots & (n-1)\,(n-2)\, \frac{2^{n-4}}{3} \left| B_{n-3} \right| \\ 0 & 0 & 0 & 1 & 0 & \frac{5}{3} & \cdots & (n-1)\,(n-2)\,(n-3)\, \frac{2^{n-7}}{3} \left| B_{n-4} \right| \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{3}\,(n-1) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

in which the eigenvalues are $\lambda_k = \cot\left(\frac{k\pi}{n+1}\right)$, for $k = 1, \ldots, n$.

It is convenient to define a companion sequence $\alpha_n(x)$ of $\beta_n(x)$ by

$$\begin{aligned} \alpha_n(x) &= \mathrm{Re}((x+i)^n) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} \qquad (21) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos\left(\frac{k\pi}{2}\right) x^{n-k}, \end{aligned}$$

where $\mathrm{Re}(z)$ denotes the real part of z. By direct computation from (21), we find

$$\begin{aligned} &\alpha_0(x) = 1, \\ &\alpha_1(x) = x, \\ &\alpha_2(x) = x^2 - 1, \\ &\alpha_3(x) = x^3 - 3x, \\ &\alpha_4(x) = x^4 - 6x^2 + 1, \\ &\alpha_5(x) = x^5 - 10x^3 + 5x. \end{aligned}$$

Similarly, we obtain

Theorem 12

1. *The ordinary generating function of* $\alpha_n(x)$ *is given by*

$$\sum_{n \geq 0} \alpha_n(x)\, z^n = \frac{1 - xz}{1 - 2xz + (1 + x^2)\, z^2}. \qquad (22)$$

2. *The exponential generating function of* $\alpha_n(x)$ *is given by*

$$\sum_{n \geq 0} \alpha_n(x)\, \frac{z^n}{n!} = \cos(z) e^{xz}. \qquad (23)$$

3. *The* $\alpha_n(x)$ *satisfy the following three-term recurrence relation:*

$$\alpha_{n+1}(x) = 2x\alpha_n(x) - \left(1 + x^2\right) \alpha_{n-1}(x),$$

with initial conditions $\alpha_0(x) = 1$ *and* $\alpha_1(x) = x$.

4. We have

$$\alpha_n(x) = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} 0 & -\left(1+x^2\right) \\ 1 & 2x \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

$$= \begin{vmatrix} x & -\left(1+x^2\right) & 0 & \cdots & 0 \\ -1 & 2x & -\left(1+x^2\right) & & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ & & \ddots & \ddots & -\left(1+x^2\right) \\ 0 & \cdots & 0 & -1 & 2x \end{vmatrix} \tag{25}$$

$$= \left(\sqrt{1+x^2}\right)^n T_n\left(\frac{x}{\sqrt{1+x^2}}\right), \tag{26}$$

where T_n is the nth Chebyshev polynomial of the first kind defined by

$$T_n(x) = \cos(n\theta) \quad \textit{when} \quad x = \cos\theta.$$

5. The following result holds true

$$\alpha_n(-x) = (-1)^n \alpha_n(x). \tag{27}$$

6. We have

$$\frac{d}{dx}\alpha_n(x) = n\alpha_{n-1}(x). \tag{28}$$

7. $\alpha_n(x)$ satisfies the linear second order ODE

$$\left(1+x^2\right)\alpha_n''(x) - 2(n-1)x\alpha_n'(x) + n(n-1)\alpha_n(x) = 0 \tag{29}$$

8. The roots of $\alpha_n(x)$ of degree $n \geq 1$ have n simple zeros in $\mathbb{R}$ at

$$x_k = \cot\left(\frac{(2k-1)\pi}{2n}\right), \ \textit{for each}\ k = 1, \ldots, n. \tag{30}$$

9. For $n \geq 0$, we have

$$\alpha_0(x) = 1;\ \alpha_{n+1}(x) = x\alpha_n(x) - \sum_{j=1}^{n} \frac{2^{j+1}(2^{j+1}-1)}{j+1}\binom{n}{j}\left|B_{j+1}\right|\alpha_{n-j}(x). \tag{31}$$

Theorem 13 *For all $n \geq 1$, we have*

$$\alpha_n(x) = \beta_n(x) - x\beta_{n-1}(x)$$
$$\beta_n(x) = x\left(1+x^2\right)\alpha_{n-1}(x) - \left(x^2-1\right)\alpha_n(x).$$

Proof Since

$$\alpha_n(x) = \frac{(x+i)^n + (x-i)^n}{2} \tag{32}$$

and

$$\beta_n(x) = \frac{(x+i)^{n+1} - (x-i)^{n+1}}{2i}, \tag{33}$$

we get the desired result. □

In the same manner, we can prove the Turán's inequalities for $\alpha_n(x)$ and $\beta_n(x)$.

Theorem 14 *Turán's inequalities for $\alpha_n(x)$ and $\beta_n(x)$ are*

$$\alpha_n^2(x) - \alpha_{n-1}(x)\alpha_{n+1}(x) = (x^2+1)^{n-1} > 0, \quad \textit{for } n \geq 1$$
$$\beta_n^2(x) - \beta_{n-1}(x)\beta_{n+1}(x) = (x^2+1)^{n} > 0, \quad \textit{for } n \geq 0.$$

3. Connection with other sequences

It is well known that $\tan(n\arctan(x))$ is a rational function and is equal to the following identity: [5]

$$\tan(n\arctan(x)) = \frac{1}{i}\frac{(1+ix)^n - (1-ix)^n}{(1+ix)^n + (1-ix)^n}.$$

It follows from (32) and (33) that for all $n \geq 1$ we have

$$\tan(n\arctan(x)) = \begin{cases} -\dfrac{\beta_{n-1}(x)}{\alpha_n(x)}, & n \text{ even} \\ \dfrac{\alpha_n(x)}{\beta_{n-1}(x)}, & n \text{ odd} \end{cases}$$
$$= \begin{cases} x - (1+x^2)\dfrac{\alpha_{n-1}(x)}{\alpha_n(x)}, & n \text{ even} \\ \dfrac{\beta_n(x)}{\beta_{n-1}(x)} - x, & n \text{ odd} \end{cases}.$$

3.1. Fibonacci polynomial

Let $h(x)$ be a polynomial with real coefficients. The link between Fibonacci polynomials and Chebyshev polynomials of the second kind is given by

$$F_{n,h}(x) = i^{n-1}U_{n-1}\left(\frac{h(x)}{2i}\right);$$

now using (16) we get

$$F_{n,h}(x) = \left(\frac{i}{2}\right)^{n-1}\left(\sqrt{h^2(x)+4}\right)^{n-1}\beta_{n-1}\left(\frac{-ih(x)}{\sqrt{h^2(x)+4}}\right)$$
$$= \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n+1}{2k+1}h^{n-2k}(x)\left(h^2(x)+4\right)^k \tag{34}$$

3.2. Lucas polynomial

In the same manner, Lucas polynomials and Chebyshev polynomials of the first kind are related by

$$L_{n,h}\left(x\right) = 2i^{n} T_n\left(\frac{h\left(x\right)}{2i}\right),$$

Using (26), we get

$$\begin{aligned} L_{n,h}\left(x\right) &= \frac{i^n}{2^{n-1}}\left(\sqrt{h^2\left(x\right)+4}\right)^n \alpha_n\left(\frac{-ih\left(x\right)}{\sqrt{h^2\left(x\right)+4}}\right) \\ &= \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k} h^{n-2k}\left(x\right)\left(h^2\left(x\right)+4\right)^k \end{aligned} \tag{35}$$

Note that the above formulas (34) and (35) are given in [15] and they generalize the Catalan formulas for Fibonacci and Lucas numbers (see Koshy [13] page 162).

3.3. Matching polynomial

The matching polynomial [9] is a well-known polynomial in graph theory and is defined by

$$M_G(x) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k m(G,k) x^{n-2k}.$$

We know from Hosoya in [12] about transformation of a matching polynomial into typical orthogonal polynomials by

$$M_{P_n}(x) = U_n(x/2),$$
$$M_{C_n}(x) = 2T_n(x/2),$$

where P_n and C_n are the path and the cycle graph, respectively.

Now, by using (16) and (26) with an appropriate change of variables, we get

$$M_{P_n}(x) = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n+1}{2k+1}x^{n-2k}\left(4-x^2\right)^k, \tag{36}$$

$$M_{C_n}(x) = \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2K}x^{n-2k}\left(4-x^2\right)^k. \tag{37}$$

4. Conclusion

In our present investigation, we have studied polynomials induced from the higher-order derivatives of $\arctan(x)$. We have derived some explicit formula for higher-order derivatives of the inverse tangent function, generating functions, recurrence relations, and some particular properties for these polynomials. As a consequence, we have established connections to Chebyshev, Fibonacci, Lucas, and matching polynomials. We did not examine the orthogonality of $\alpha_n(x)$ and $\beta_n(x)$ polynomials. We think that these polynomials are a nice example for Sobolev orthogonal polynomials.

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References

[1] Adamchik VS. On the Hurwitz function for rational arguments. Appl Math Comput 2007; 187: 3–12.

[2] Adegoke K, Layeni O. The higher derivatives of the inverse tangent function and rapidly convergent BBP-type formulas for pi. Appl Math E-Notes 2010; 10: 70-75.

[3] Adegoke K, Layeni O. The higher derivatives of the inverse tangent function and rapidly convergent BBP-type formulas. arXiv 2016; 1603.08540V1.

[4] Al-Hassan Q. An inverse eigenvalue problem for general tridiagonal matrices. Int J Contemp Math Sci 2009; 4: 625-634.

[5] Beeler M, Gosper RW, Schroeppel R. HAKMEM. MIT AI Memo 239, Feb. 29: 1972.

[6] Boyadzhiev KN. Derivative polynomials for tanh, tan, sech and sec in explicit form. Fibonacci Quart 2007; 45: 291-303.

[7] Brychkov YA. Handbook of Special Functions. Derivatives, Integrals, Series and Other Formulas. Boca Raton, FL, USA: CRC Press, 2008.

[8] Cvijović D. Derivative polynomials and closed-form higher derivative formulae. Appl Math Comput 2009; 215: 3002-3006.

[9] Farrel EJ. An introduction to matching polynomials. J Comb Theory B 1979; 27: 75-86.

[10] Gautschi W, Orthogonal Polynomials: Computation and Approximation. Oxford, UK: Oxford University Press, 2004.

[11] Hobson EW, Hudson HP, Singh AN, Kempe AB. Squaring the Circle and other monographs. New York, NY, USA: Chelsea Publishing Company, 1953.

[12] Hosoya H. Topological Index and Some Counting Polynomials for Characterizing the Topological Structure and Properties of Molecular Graphs, Research of Pattern Formation. Tokyo, Japan: KTK Scientific Publishers, 1994, pp. 63-75.

[13] Koshy T. Fibonacci and Lucas Numbers with Applications. New York: Wiley, 2001.

[14] Lampret V. The higher derivatives of the inverse tangent function revisited. Appl Math E-Notes 2011; 11: 224-231.

[15] Nalli A, Haukkanen P. On generalized Fibonacci and Lucas polynomials. Chaos Soliton Fract 2009; 42: 3179-3186.

[16] Qi F. Derivatives of tangent function and tangent numbers. Appl Math Comput 2015; 268: 844–858.

[17] Qi F, Zheng MM. Explicit expressions for a family of the Bell polynomials and applications. Appl Math Comput 2015; 258: 597-607.

[18] Vein R, Dale P. Determinants and Their Applications in Mathematical Physics. Applied Mathematical Sciences. New York, NY, USA: Springer, 1999.

[19] Xu AM, Cen ZD. Closed formulas for computing higher-order derivatives of functions involving exponential functions. Appl Math Comput 2015; 270: 136-141.