Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2018) 42: 2690 – 2698
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doi:10.3906/mat-1803-14

On the Diophantine equation $((c+1)m^2+1)^x + (cm^2-1)^y = (am)^z$

Elif KIZILDERE[1], **Takafumi MIYAZAKI**[2], **Gökhan SOYDAN**[1,*]
[1]Department of Mathematics, Faculty of Arts and Science, Bursa Uludağ University, Bursa, Turkey
[2]Division of Pure and Applied Science, Faculty of Science and Technology, Gunma University, Kiryu City, Gunma, Japan

Received: 03.03.2018 • **Accepted/Published Online:** 10.08.2018 • **Final Version:** 27.09.2018

Abstract: Suppose that c, m, and a are positive integers with $a \equiv 11, 13 \pmod{24}$. In this work, we prove that when $2c+1 = a^2$, the Diophantine equation in the title has only solution $(x,y,z) = (1,1,2)$ where $m \equiv \pm 1 \pmod{a}$ and $m > a^2$ in positive integers. The main tools of the proofs are elementary methods and Baker's theory.

Key words: Exponential Diophantine equation, Jacobi symbol, lower bound for linear forms in logarithms

1. Introduction

As usual, we denote the set of all integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. Suppose that a, b, c are pairwise coprime positive integers. Then we call the equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N} \tag{1}$$

an exponential Diophantine equation. Many authors have studied the above equation for given $a, b, c \in \mathbb{N}$.

Eq. (1) was first considered by Mahler [10]. He proved that Eq. (1) has finitely many solutions with $a, b, c > 1$. Since his method was based on a p-adic generalization of Thue–Siegel method, it was ineffective as it provided no bounds for the size of possible solutions. Later, Gel'fond [4] gave an effective result for solutions of (1). His method was based on Baker's theory, which uses linear forms in the logarithms of algebraic numbers.

Using elementary number theory methods such as congruences, the Jacobi symbol, and standard divisibility arguments in algebraic number theory involving ideals in quadratic (or cubic) number fields, the complete solutions of Eq. (1) where a, b, c are distinct primes ≤ 17 were determined by some authors (see [5, 14, 24]).

Consider Eq. (1) for Pythagorean numbers a, b, c. A famous unsolved problem concerning the exponential Diophantine equation (1) was suggested by Jeśmanowicz [6]. He conjectured that the unique solution in positive integers of Eq. (1) is only $(x,y,z) = (2,2,2)$, where a, b, c satisfy $a^2 + b^2 = c^2$, i.e. they are Pythagorean triples. This conjecture have been solved for many special cases. Different conjectures concerning Eq. (1) were identified and discussed. One of these conjectures, which is an extension of Jeśmanowicz's conjecture, was suggested by Terai. Terai conjectured that the unique solution of the Diophantine equation (1) is $(x,y,z) = (u,v,w)$ except for some (a,b,c) where $a,b,c,u,v,w \in \mathbb{N}$ are fixed, $u,v,w \geq 2$ and $\gcd(a,b) = 1$, and satisfying $a^u + b^v = c^w$ (see [2, 9, 11, 12, 19, 20]). The correctness of this conjecture for many special cases

*Correspondence: gsoydan@uludag.edu.tr
2010 *AMS Mathematics Subject Classification:* 11D61

has been proved. Nevertheless, it remains unsolved. Recently, a survey paper on the conjectures of Jeśmanowicz and Terai was published by Soydan et al. (see [17] for details about these conjectures).

Now take the Diophantine equation

$$(am^2+1)^x + (bm^2-1)^y = (cm)^z, \tag{2}$$

where a, b, m, c are given positive integers such that $a+b=c^2$. Some authors studied Eq. (2) for some special values (see [1, 18, 21, 22]).

Miyazaki and Terai [13] considered Eq. (2) where $c \equiv 3, 5 \pmod 8$ and $a=1$. They showed that the unique solution of Eq. (2) in positive integers is $(x,y,z)=(1,1,2)$ where $m \equiv \pm 1 \pmod c$, and they also found the exceptional case $(m,b,c)=(1,8,3)$, which gives the solutions $(x,y,z)=(5,2,4)$ and $(1,1,2)$.

Recently, Terai and Hibino [23] studied Eq. (2) where $a=p(p-3)$, $b=3p$, $c=p$, $m \geq 1$, and p prime. As $m \equiv 1 \pmod 4$, $m \not\equiv 0 \pmod 3$, $p \equiv 1 \pmod 4$, and $p < 3784$, they proved that the unique solution of Eq. (2) in positive integers is $(x,y,z)=(1,1,2)$. Later, Fu and Yang [3] considered Eq. (2) where $2 \mid a$, $2 \nmid c$ and $m>1$. They proved that the only solution of Eq. (2) in positive integers is $(x,y,z)=(1,1,2)$ where $c \mid m$ and $m > 36c^3 \log c$. Thirdly, Pan [15] considered Eq. (2) again where $a+b=c^2$, $2 \nmid c$, $m>1$ and $m \equiv \pm 1 \pmod c$. She proved that if $a \equiv 4, 5 \pmod 8$, $((a+1)/c) = -1$, and $m > 6c^2 \log c$, where $(*/*)$ is the Jacobi symbol, then Eq. (2) has no other solution than $(x,y,z)=(1,1,2)$ in positive integers.

Here, we are interested in the exponential Diophantine equation

$$((c+1)m^2+1)^x + (cm^2-1)^y = (am)^z \tag{3}$$

with $m>0$. The following result is the main theorem of this paper:

Theorem 1.1 (Main theorem) *Suppose that a and c are positive integers with $a \equiv 11, 13 \pmod{24}$ and $2c+1=a^2$. Then the unique solution of Eq. (3) in positive integers is $(x,y,z)=(1,1,2)$, where $m \equiv \pm 1 \pmod a$ and $m > a^2$.*

2. Preliminaries

In this section, under certain conditions, we give a useful result to find an upper bound for y, which is a solution in Pillai's equation $W^z - V^y = U$. The theory needed in this result is based on finding lower bounds for linear forms in logarithms of two algebraic numbers called Baker's theory. To do this, we first present some notations. For two real algebraic numbers β_1 and β_2 with $|\beta_1| \geq 1$ and $|\beta_2| \geq 1$, denote the field of rational numbers by $\mathbb{Q}$. We consider the linear form

$$\Omega = c_2 \log \beta_2 - c_1 \log \beta_1,$$

where c_1 and c_2 are positive integers. Let β be any nonzero algebraic number with minimal polynomial over $\mathbb{Z}$ being $a_0 \prod_{i=1}^{n}(X-\beta^{(i)})$, which is of degree n over $\mathbb{Q}$. We denote by

$$h(\beta) = \frac{1}{n}\left(\log|a_0| + \sum_{i=1}^{n} \log \max\left\{1, |\beta^{(i)}|\right\}\right)$$

its absolute logarithmic height, where $(\beta^{(i)})_{1 \leq i \leq n}$ are conjugates of β. Suppose that B_1 and B_2 are real numbers ≥ 1 such that

$$\log B_j \geq \max\{\, h(\beta_j),\, |\log \beta_j|/K,\, 1/K \,\} \quad (j=1,2),$$

where the number field $\mathbb{Q}(\beta_1, \beta_2)$ over $\mathbb{Q}$ has degree K. Define

$$d' = \frac{c_1}{K \log B_2} + \frac{c_2}{K \log B_1}.$$

To use Laurent's result in [8, Corollary 2], we take $m = 10$ and $C_2 = 25.2$.

Proposition 2.1 *(Laurent, [8]) Suppose that Ω is given as above and $\beta_1 > 1$ and $\beta_2 > 1$ are multiplicatively independent. Then*

$$\log|\Omega| \geq -25.2K^4(\max\{\log d' + 0.38,\ 10/K\})^2(\log B_1)(\log B_2).$$

3. Proof of Theorem 1.1

3.1. The case $m = 1$

Lemma 3.1 *Suppose that $a \equiv 11, 13 \pmod{24}$ with $a > 0$. Then the unique solution in positive integers of the equation*

$$(c+2)^x + (c-1)^y = a^z$$

is $(x, y, z) = (1, 1, 2)$ where $2c + 1 = a^2$.

Proof Consider Eq. (3) with $m = 1$. Recall that $2c + 1 = a^2$. Then Eq. (3) becomes

$$[\,(a^2+3)/2\,]^x + [\,(a^2-3)/2\,]^y = a^z. \tag{4}$$

Taking this equation modulo $(a-1)/2$ and $(a+1)/2$, one sees that

$$2^x + (-1)^y \equiv 1 \mod (a-1)/2,$$
$$2^x + (-1)^y \equiv (-1)^z \mod (a+1)/2,$$

respectively. In particular, since $a > 5$ and $a \equiv 3, 5 \pmod 8$, it follows from the above congruences that z is even and y is odd. Put $z = 2Z$. Therefore, Eq. (4) becomes

$$(c+2)^x + (c-1)^y = (2c+1)^Z.$$

Since $4 \mid c$, one takes this equation modulo 4 to find that $2^x \equiv 2 \pmod 4$, so $x = 1$, and thereby

$$c + 2 + (c-1)^y = (2c+1)^Z. \tag{5}$$

It is clear that $Z \leq y < 2Z$. Let $y > 1$ and let us get a contradiction. We get $y \geq 3$ and $y < 2Z$. If $1+y = 2Z$, then Eq. (5) yields $(2c+1)^{\frac{1+y}{2}} - (c-1)^y = c + 2 > 0$, and thereby $\frac{3}{2} < \frac{2y}{1+y} < \frac{\log(2c+1)}{\log(c-1)} \leq \frac{\log 17}{\log 7}$, which is absurd. Thus, $1 + y < 2Z$. On the other hand, by taking Eq. (5) modulo c^2, one has

$$c + 2 + (-1 + cy) \equiv 1 + 2cZ \pmod{c^2},$$

so

$$1 + y \equiv 2Z \pmod c.$$

Since $1 + y < 2Z$, this congruence gives

$$c \leq 2Z - (1 + y).$$

Otherwise, using Proposition 2.1, one can get

$$y < 2521 \log(2c+1).$$

To show this, by (5) we are first interested in the following linear form in two logarithms:

$$\Omega = Z\log(2c+1) - y\log(c-1) \quad (>0).$$

As $\log(1+k) < k$ where $k>0$, one gets

$$0 < \Omega = \log\left(\frac{(2c+1)^Z}{(c-1)^y}\right) = \log\left(1 + \frac{c+2}{(c-1)^y}\right) < \frac{c+2}{(c-1)^y}.$$

Hence, we have

$$\log \Omega < \log(c+2) - y\log(c-1). \tag{6}$$

Otherwise, using Proposition 2.1, we want to find a lower bound for Ω. By Proposition 2.1, we obtain the following inequality:

$$\log\Omega \geq -25.2(\max\{\log d' + 0.38, 10\})^2 \log(c-1)\log(2c+1), \tag{7}$$

where $d' = \frac{y}{\log(2c+1)} + \frac{Z}{\log(c-1)}$.

It can be seen that $(c-1)^{y+1} > (2c+1)^Z$. Indeed,

$$\begin{aligned}(c-1)^{y+1} - (2c+1)^Z &= (c-1)((2c+1)^Z - (c+2)) - (2c+1)^Z \\ &= (c-2)(2c+1)^Z - (c-1)(c+2) \\ &\geq c^2 - 4c > 0,\end{aligned}$$

since $c \equiv 12 \pmod{24}$. Hence, $d' < \frac{2y+1}{\log(2c+1)}$.

Set $T = \frac{y}{\log(2c+1)}$. Using inequalities (6) and (7), we obtain

$$\begin{aligned} y\log(c-1) < \log(c+2) &+ 25.2(\max\{\log(2T + \frac{1}{\log(2c+1)}) + 0.38, 10\})^2 \\ &\times \log(c-1)\log(2c+1),\end{aligned}$$

so

$$T < 1 + 25.2(\max\{\log(2T+1) + 0.38, 10\})^2$$

as $\log(2c+1) \geq \log(25) > 3$. Therefore, we get $T < 2521$.

These inequalities together with the fact that $Z \leq y$ give that

$$c \leq 2Z - (1+y) < y < 2521\log(2c+1).$$

Thus, $c \leq 27518$. Finally, using the PARI/GP program [16], we found that Eq. (5) has no solution where $c \leq 27518$. Hence, the proof is completed. □

3.2. The case $m \geq 2$

Suppose that (x,y,z) satisfies (3). By the conclusion of the previous section, assuming $m \geq 2$, we want to fix parities for x,y,z. Using $m \equiv \pm 1 \pmod{a}$ and $a \equiv 11,\, 13 \pmod{24}$ with $m > a^2$, we prove the following.

Lemma 3.2 *Suppose that* $(x,y,z) = (1,1,2)$ *is a solution for Eq.* (3). *Then* z *is even and* y *and* x *are both odd.*

Proof Suppose that (x,y,z) is a solution for (3) and all of our conditions are satisfied.

Now it follows from $1+2c = a^2$ that $cm^2 - 1 = ((\frac{a^2-1}{2})m^2 - 1) > am$. Hence, $z \geq 2$ from (3). Considering (3) modulo m^2 requires that $1 + (-1)^y \equiv 0 \pmod{m^2}$. Hence, y is odd when $m \geq 2$. Using $1 + 2c = a^2$ and $m \equiv \pm 1 \pmod{a}$, (3) carries away to

$$(-c)^x \equiv -c^y \pmod{a},$$

so $\left(\frac{-c}{a}\right)^x = \left(\frac{-c}{a}\right)^y$. Then y and x are the same parities. This means that x is odd.

We now prove that $\left(\frac{m}{cm^2-1}\right) = 1$ and $\left(\frac{a}{cm^2-1}\right) = -1$. Note that $cm^2 - 1 \equiv 3 \pmod{8}$ and $c \equiv 12 \pmod{24}$. Write $m = 2^{\gamma} r$ with $\gamma \geq 0$ and r odd. Then

$$\left(\frac{m}{cm^2-1}\right) = \left(\frac{2}{cm^2-1}\right)^{\gamma}\left(\frac{r}{cm^2-1}\right) = \left(\frac{r}{cm^2-1}\right)$$

$$= \left(\frac{r}{cm^2-1}\right) = 1.$$

If $a \equiv 11 \pmod{24}$, then

$$\left(\frac{a}{cm^2-1}\right) = -\left(\frac{cm^2-1}{a}\right) = -\left(\frac{c-1}{a}\right) = \left(\frac{c+2}{a}\right)$$

$$= \left(\frac{3}{a}\right)\left(\frac{c+1}{a}\right) = -1,$$

since $\left(\frac{3}{a}\right) = 1, \left(\frac{2}{a}\right) = -1$ and $\left(\frac{2c+2}{a}\right) = \left(\frac{2}{a}\right)\left(\frac{c+1}{a}\right) = 1$.

If $a \equiv 13 \pmod{24}$, then

$$\left(\frac{a}{cm^2-1}\right) = \left(\frac{cm^2-1}{a}\right) = \left(\frac{c-1}{a}\right) = \left(\frac{c+2}{a}\right)$$

$$= \left(\frac{3}{a}\right)\left(\frac{c+1}{a}\right) = -1,$$

since $\left(\frac{3}{a}\right) = 1, \left(\frac{2}{a}\right) = -1$ and $\left(\frac{2c+2}{a}\right) = \left(\frac{2}{a}\right)\left(\frac{c+1}{a}\right) = 1$.

Thus,

$$\left(\frac{am}{cm^2-1}\right) = \left(\frac{a}{cm^2-1}\right)\left(\frac{m}{cm^2-1}\right) = (-1)\cdot 1 = -1.$$

Since $1+2c=a^2$,

$$\left(\frac{(c+1)m^2+1}{cm^2-1}\right)=\left(\frac{(c+1)m^2+cm^2}{cm^2-1}\right)=\left(\frac{a^2m^2}{cm^2-1}\right)=1.$$

In view of these, we have that z is even from (3). □

Considering (3) modulo m^3, (3) has only solution $(x,y,z)=(1,1,2)$ in positive integers with m even, as will be shown.

Lemma 3.3 *The unique solution of Eq.* (3) *in positive integers is* $(x,y,z)=(1,1,2)$ *where* m *is even.*

Proof Since $z\le 2$, then from (3), one gets $(x,y,z)=(1,1,2)$. Therefore, we can assume that $z\ge 3$. By Lemma 3.2, we know that y and x are both odd.

Considering (3) modulo m^3 one gets

$$(c+1)m^2x+1+cm^2y-1\equiv 0 \pmod{m^3},$$

so

$$(c+1)x+cy\equiv 0 \pmod{m},$$

but the fact that m is even, c is even, and x is odd contradicts the above congruence. The proof is thus completed. □

Lemma 3.4 *As* m *is odd, we have* $x=1$ *in* (3).

Proof By Lemma 3.2, we get that z is even while y is odd. Assume that $x\ge 2$. Considering (3) modulo 4 results that

$$(-1)^y\equiv 1 \pmod{4}.$$

Hence, we obtain that y is even, which contradicts Lemma 3.2. Therefore, we get $x=1$. □

3.3. Pillai's equation $W^z-V^y=U$

By Lemmas 3.2 and 3.4, we get that y is odd and $x=1$. Since we get $z=2$ from (3) where $y=1$, then we may assume that $y\ge 3$. Therefore, our result is induced to the solution of Pillai's equation

$$W^z-V^y=U \tag{8}$$

with $y\ge 3$, where $U=(c+1)m^2+1$, $V=cm^2-1$, and $W=am$.

We first desire to get a lower bound for the exponent y.

Lemma 3.5 *In* (8), $y\ge (m^2-1)/c-1$.

Proof If $y\ge 3$, by (8) we obtain

$$(am)^z=(c+1)m^2+1+(cm^2-1)^y\ge (c+1)m^2+1+(cm^2-1)^3>(am)^3.$$

Therefore, $z\ge 4$. Considering (8) modulo m^4 implies that

$$(c+1)m^2+1+cm^2y-1\equiv 0 \pmod{m^4},$$

so $c+1+cy\equiv 0 \pmod{m^2}$. Hereby we get our argument. □

We next desire to get an upper bound for the exponent y.

Lemma 3.6 *In* (8), $y < 2521 \log W$.

Proof (8) implies that we are interested in the linear form in two logarithms as follows:

$$\Gamma = z \log W - y \log V \quad (> 0).$$

As $\log(1+l) < l$ for $l > 0$, we get

$$0 < \Gamma = \log\left(\frac{W^z}{V^y}\right) = \log\left(1 + \frac{U}{V^y}\right) < \frac{U}{V^y}. \tag{9}$$

Hence,

$$\log \Gamma < \log U - y \log V. \tag{10}$$

Otherwise, we find a lower bound for Γ with Proposition 2.1. We first obtain

$$\log \Gamma \geq -25.2(\max\{\log d' + 0.38, 10\})^2 (\log V)(\log W) \tag{11}$$

with $d' = y/\log W + z/\log V$.

Note that $V^{y+1} > W^z$. In fact,

$$\begin{aligned} V^{y+1} - W^z &= V(W^z - U) - W^z = (V-1)W^z - UV \\ &\geq (cm^2 - 2)(2c+1)m^2 - ((c+1)m^2)(cm^2 - 1) > 0. \end{aligned}$$

Therefore, $d' < (2y+1)/\log W$.

Set $R = y/\log W$. Using (10) and (11) leads to

$$y \log V < \log U + 25.2(\max\{\log(2R + 1/\log W) + 0.38, 10\})^2 (\log V)(\log W).$$

Hence, since $\log W = \log(am) \geq \log 33 > 3$ and $U < V$, then we obtain

$$R < 1 + 25.2(\max\{\log(2R + 1/3) + 0.38, 10\})^2,$$

which implies that $R < 2521$. The proof is completed. □

Now we can prove Theorem 1.1. By Lemmas 3.5 and 3.6 we obtain

$$2\left(\frac{m^2-1}{a^2-1}\right) - 1 < 2521 \log(am).$$

If $m > a^2$ then we get

$$2m + 1 < 2521 \log(am).$$

It follows that $m < 18586$. Hence, we obtain $a \leq 136$.

By (9), we get the inequality

$$\left|\frac{\log V}{\log W} - \frac{z}{y}\right| < \frac{U}{yV^y \log W},$$

and hence $\left|\frac{\log V}{\log W} - \frac{z}{y}\right| < \frac{1}{2y^2}$, since $y \geq 3$. Thus, $\frac{z}{y}$ is one of the convergents in the simple continued fraction expansion of $\frac{\log V}{\log W}$.

Otherwise, if $\frac{d_r}{e_r}$ is the rth such convergent, then

$$\left|\frac{\log V}{\log W} - \frac{d_r}{e_r}\right| > \frac{1}{(u_{r+1}+2)e_r^2}$$

where the $(r+1)$st partial quotient of $\frac{\log V}{\log W}$ is u_{r+1} (see Khinchin [7]). Put $\frac{z}{y} = \frac{d_r}{e_r}$. Note that $e_r \leq y$. It follows that

$$u_{r+1} > \frac{V^y \log W}{Uy} - 2 \geq \frac{V^{e_r} \log W}{Ue_r} - 2. \tag{12}$$

Finally, using the PARI/GP program, we see that for each $a \leq 136$ with $a \equiv 11, 13 \pmod{24}$ and for each r with $e_r < 2521 \log(am)$ for $3 \leq m \leq 18585$, (12) is not satisfied. Thus, the proof of Theorem 1.1 is completed.

Acknowledgments

The first and third authors were supported by the project 117F287 under the program 3001 of the Scientific and Technological Research Council of Turkey (TÜBİTAK). The second author was supported by Grant-in-Aid for Young Scientists (B) (No. 16K17557).

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