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Evaluating a class of balanced q-series

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Abstract: By means of the modified Abel lemma on summation by parts, we examine a class of terminating balanced q-series. Two transformation formulae are established that contain ten summation formulae as consequences.

Key words: Abel's lemma on summation by parts, basic hypergeometric series, terminating balanced series q-Pfaff–Saalschütz theorem

1. Introduction and motivation

Let $\mathbb{N}$ be the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For an indeterminate x, the shifted factorial with the base q is defined by $(x;q)_0 = 1$ and

$$(x;q)_n \;=\; (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n \in \mathbb{N}.$$

Its quotient form will be abbreviated as follows:

$$\left[\begin{matrix}\alpha,\ \beta,\ \cdots,\ \gamma\\ A,\ B,\ \cdots,\ C\end{matrix}\,\middle|\, q\right]_n = \frac{(\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n\,(B;q)_n\cdots(C;q)_n}.$$

This paper will investigate the following balanced series:

$$\Omega_m^n(\lambda,x,y) = \sum_{k=-m}^{n}\left[\begin{matrix}x,y\\ q\lambda\end{matrix}\,\middle|\, q\right]_k \frac{(\lambda xy;q^3)_k}{(xy;q)_{2k}}q^k, \tag{1}$$

$$\omega_m^n(\lambda,x,y) = \sum_{k=-m}^{n}\left[\begin{matrix}\lambda\\ qx,qy\end{matrix}\,\middle|\, q\right]_k \frac{(qxy;q)_{2k}}{(q^3\lambda xy;q^3)_k}q^k. \tag{2}$$

By making the replacement $k \to -k$ on the summation index, we can check without difficulty that they satisfy the following reciprocal relation:

$$\Omega_m^n(\lambda,x,y) = \omega_n^m(1/\lambda,1/x,1/y). \tag{3}$$

In 1979, Andrews [1, Eq. 4.7] (see also Gessel and Stanton [10, Eq. 4.32]) found the following identity:

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^ny\\ q\end{matrix}\,\middle|\, q\right]_k \frac{(y;q^3)_k}{(y;q)_{2k}}q^k = \chi(n \equiv_3 0)\left[\begin{matrix}q,q^2\\ qy,q^2y\end{matrix}\,\middle|\, q^3\right]_{\lfloor\frac{n}{3}\rfloor} y^{\lfloor\frac{n}{3}\rfloor}, \tag{4}$$

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2000 *AMS Mathematics Subject Classification:* Primary 33D15, Secondary 05A30 where, for brevity, we use the notations $\lfloor x \rfloor$ for the integer part of a real number x and $i \equiv_m j$ for "i is congruent to j modulo m" as well as χ for the logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$ otherwise.

Observe that Andrews' identity (4) corresponds to only one of the four terminating cases of the Ω-series listed below (where $\delta = 0, 1$):

- $\lambda = 1$ and $x = q^{-n}$.
- $xy = q^{1+\delta}$ and $\lambda xy = q^{-3n}$.
- $x = q^{-n}$ and $y = q^{1+\delta+n}$.
- $\lambda = 1$ and $xy = q^{-3n}$.

It is natural to ascertain whether there are closed formulae corresponding to the other three terminating cases. This has been confirmed recently by Chen and Chu [5] through the inverse series relations due to Carlitz [4].

In this paper, an alternative approach will be presented. In the next section, the modified Abel lemma will be employed to establish two transformations that connect both sums $\Omega_m^n(\lambda, x, y)$ and $\omega_n^m(\lambda, x, y)$ to partial sums of balanced series. Then they will be utilized, in the third section, to recover ten summation formulae as consequences.

2. Main theorems and proofs

In order to make the paper self-contained, we record Abel's lemma on summation by parts (cf. Chu [7] and Chu and Wang [8]) as follows.

For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$\nabla \tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta \tau_k = \tau_k - \tau_{k+1}.$$

It should be pointed out that Δ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign.

Then Abel's lemma on summation by parts may be modified as follows:

$$\sum_{k=-m}^{n} B_k \nabla A_k = \big\{ A_n B_{n+1} - A_{-m-1} B_{-m} \big\} + \sum_{k=-m}^{n} A_k \Delta B_k. \tag{5}$$

In fact, it is almost trivial to check the following expression:

$$\sum_{k=-m}^{n} B_k \nabla A_k = \sum_{k=-m}^{n} B_k \big\{ A_k - A_{k-1} \big\} = \sum_{k=-m}^{n} A_k B_k - \sum_{k=-m}^{n} A_{k-1} B_k.$$

Replacing k by $k+1$ for the last sum, we can reformulate the equation as follows:

$$\begin{aligned} \sum_{k=-m}^{n} B_k \nabla A_k &= A_n B_{n+1} - A_{-m-1} B_{-m} + \sum_{k=-m}^{n} A_k \big\{ B_k - B_{k+1} \big\} \\ &= A_n B_{n+1} - A_{-m-1} B_{-m} + \sum_{k=-m}^{n} A_k \Delta B_k, \end{aligned}$$

which is exactly the equality stated in the modified Abel lemma. □

When applying the modified Abel lemma on summation by parts to deal with q-series, the crucial step lies in finding shifted factorial fraction pair $\{A_k, B_k\}$ so that their differences are expressible as ratios of linear factors. This has not been an easy task, even though it is routine to make factorizations once they are figured out.

For the difference pair given by

$$\mathcal{A}_k := \frac{(xy/q^2\lambda;q)_k(q^3\lambda xy;q^3)_k}{(xy;q)_{2k}} \quad \text{and} \quad \mathcal{B}_k := \left[\begin{matrix} x,\ y \\ q^3\lambda, xy/q^3\lambda \end{matrix} \,\Big|\, q\right]_k$$

it is not hard to check the differences

$$\begin{aligned}\nabla\mathcal{A}_k =& q^k\frac{(xy/q^3\lambda;q)_k(\lambda xy;q^3)_k}{(xy;q)_{2k}}\frac{(1-q^{k+1}\lambda)(1-q^{k+2}\lambda)}{(1-\lambda xy)(1-q^3\lambda/xy)},\\ \Delta\mathcal{B}_k =& q^k\left[\begin{matrix} x,\ y \\ q^4\lambda, xy/q^2\lambda \end{matrix} \,\Big|\, q\right]_k\frac{(1-q^3\lambda/x)(1-q^3\lambda/y)}{(1-q^3\lambda)(1-q^3\lambda/xy)},\end{aligned}$$

and to determine the boundary conditions

$$\begin{aligned}\mathcal{A}_n\mathcal{B}_{n+1} =& \frac{(1-x)(1-y)}{(1-q^3\lambda)(1-xy/q^3\lambda)}\left[\begin{matrix} qx,\ qy \\ q^4\lambda \end{matrix} \,\Big|\, q\right]_n\frac{(q^3\lambda xy;q^3)_n}{(xy;q)_{2n}},\\ \mathcal{A}_{-m-1}\mathcal{B}_{-m} =& \frac{(1-q/xy)(1-q^2/xy)}{(1-1/\lambda xy)(1-q^3\lambda/xy)}\left[\begin{matrix} q^{-2}/\lambda \\ q/x,\ q/y \end{matrix} \,\Big|\, q\right]_m\frac{(q^3/xy;q)_{2m}}{(q^3/\lambda xy;q^3)_m}.\end{aligned}$$

According to the modified Abel lemma on summation by parts, we can reformulate $\Omega_m(\lambda,x,y)$-sum as follows

$$\begin{aligned}\Omega_m^n(\lambda,x,y) &= \sum_{k=-m}^{n}\left[\begin{matrix} x, y \\ q\lambda \end{matrix} \,\Big|\, q\right]_k\frac{(\lambda xy;q^3)_k}{(xy;q)_{2k}}q^k = \frac{(1-\lambda xy)(1-q^3\lambda/xy)}{(1-q\lambda)(1-q^2\lambda)}\sum_{k=-m}^{n}\mathcal{B}_k\nabla\mathcal{A}_k\\ &= \frac{(1-\lambda xy)(1-q^3\lambda/xy)}{(1-q\lambda)(1-q^2\lambda)}\Big\{\mathcal{A}_n\mathcal{B}_{n+1}-\mathcal{A}_{-m-1}\mathcal{B}_{-m}+\sum_{k=-m}^{n}\mathcal{A}_k\Delta\mathcal{B}_k\Big\},\end{aligned}$$

which can be expressed as the following recurrence relation:

$$\begin{aligned}\Omega_m^n(\lambda,x,y) =& \Omega_m^n(q^3\lambda,x,y)\frac{(1-\lambda xy)(1-q^3\lambda/x)(1-q^3\lambda/y)}{(1-q\lambda)(1-q^2\lambda)(1-q^3\lambda)}\\ &+\frac{(\lambda xy)(1-q/xy)(1-q^2/xy)}{(1-q\lambda)(1-q^2\lambda)}\left[\begin{matrix} q^{-2}/\lambda \\ q/x,\ q/y \end{matrix} \,\Big|\, q\right]_m\frac{(q^3/xy;q)_{2m}}{(q^3/\lambda xy;q^3)_m}\\ &-\frac{(q^3\lambda/xy)(1-x)(1-y)(1-\lambda xy)}{(1-q\lambda)(1-q^2\lambda)(1-q^3\lambda)}\left[\begin{matrix} qx,\ qy \\ q^4\lambda \end{matrix} \,\Big|\, q\right]_n\frac{(q^3\lambda xy;q^3)_n}{(xy;q)_{2n}}.\end{aligned}$$

By iterating the last relation ℓ-times, we derive, after some simplification, the following transformation formula.

Theorem 1 (Transformation formula) *For the $\Omega_m^n(\lambda,x,y)$-sum defined by (1) and the balanced series $\Theta_\ell(\lambda,x,y)$ by*

$$\Theta_\ell(\lambda,x,y) = \sum_{k=0}^{\ell-1}q^{3k}\left[\begin{matrix} \lambda xy, q^3\lambda/x, q^3\lambda/y \\ q^3\lambda, q^4\lambda, q^5\lambda \end{matrix} \,\Big|\, q^3\right]_k \tag{6}$$

there holds the following transformation formula:

$$\begin{aligned}\Omega_m^n(\lambda,x,y)=&\Omega_m^n(q^{3\ell}\lambda,x,y)\left[\begin{matrix}\lambda xy,q^3\lambda/x,q^3\lambda/y\\ q\lambda,q^2\lambda,q^3\lambda\end{matrix}\,\Big|\,q^3\right]_\ell\\ &+(\lambda xy)\frac{\Theta_\ell(q^{-m}\lambda,q^{-m}x,q^{-m}y)}{(1-q\lambda)(1-q^2\lambda)}\left[\begin{matrix}q^{-2}/\lambda\\ q/x,q/y\end{matrix}\,\Big|\,q\right]_m\frac{(q/xy;q)_{2m+2}}{(q^3/\lambda xy;q^3)_m}\\ &-\Big(\frac{q^3\lambda}{xy}\Big)\frac{\Theta_\ell(q^{n+1}\lambda,q^{n+1}x,q^{n+1}y)}{(1-q\lambda)(1-q^2\lambda)}\left[\begin{matrix}x,\ y\\ q^3\lambda\end{matrix}\,\Big|\,q\right]_{n+1}\frac{(\lambda xy;q^3)_{n+1}}{(xy;q)_{2n}}.\end{aligned}$$

Performing the replacements $x\to 1/x$, $y\to 1/y$ and $\lambda\to q^{-3\ell}/\lambda$ in the last theorem, we can express the resulting transformation formula, in view of the reciprocal relation (3), in the following proposition.

Proposition 2 (Transformation formula) *For the $\omega_n^m(\lambda,x,y)$-sum defined by (2) and the balanced series $\theta_\ell(\lambda,x,y)$ by*

$$\theta_\ell(\lambda,x,y)=\sum_{k=0}^{\ell-1}q^{3k}\left[\begin{matrix}\lambda,q\lambda,q^2\lambda\\ q^3\lambda/x,q^3\lambda/y,q^3\lambda xy\end{matrix}\,\Big|\,q^3\right]_k \tag{7}$$

there holds the following transformation formula:

$$\begin{aligned}\omega_n^m(\lambda,x,y)=&\omega_n^m(q^{3\ell}\lambda,x,y)\left[\begin{matrix}\lambda,q\lambda,q^2\lambda\\ \lambda/x,\lambda/y,q^3\lambda xy\end{matrix}\,\Big|\,q^3\right]_\ell\\ &-\frac{1-\lambda}{\lambda}\frac{\theta_\ell(q^{m+1}\lambda,q^{m+1}x,q^{m+1}y)}{(1-x/\lambda)(1-y/\lambda)}\left[\begin{matrix}q\lambda\\ qx,qy\end{matrix}\,\Big|\,q\right]_m\frac{(qxy;q)_{2m+2}}{(q^3\lambda xy;q^3)_{m+1}}\\ &-(1-\lambda)\frac{\theta_\ell(q^{-n}\lambda,q^{-n}x,q^{-n}y)}{(1-\lambda/x)(1-\lambda/y)}\left[\begin{matrix}1/x,1/y\\ 1/\lambda\end{matrix}\,\Big|\,q\right]_{n+1}\frac{(1/\lambda xy;q^3)_n}{(1/xy;q)_{2n}}.\end{aligned}$$

The two transformation formulae shown in Theorem 1 and Proposition 2 are remarkably useful, which will be exemplified by the summation formulae recorded in the next section.

3. Closed formulae for terminating series

In this section, ten closed formulae for both sums $\Omega_m^n(\lambda,x,y)$ and $\omega_n^m(\lambda,x,y)$ will be deduced from Theorem 1 and Proposition 2 as consequences. Different proofs of them can be found in the work of Chen and Chu [5].

We begin with the case $m=0$, i.e. the $\Omega_0^n(\lambda,x,y)$-sum consisting of the terms with nonnegative indices. First let $\ell\to n+1$ and $\lambda\to 1$, $xy\to q^{-3n}$ in Theorem 1. The corresponding series $\Theta_\ell(\lambda,x,y)$ displayed in (6) can be evaluated by the q-Pfaff–Saalschutz theorem (cf. Bailey [2, §8.4] and Gasper and Rahman [9, II-12]):

$$\sum_{k=0}^{n}q^k\left[\begin{matrix}q^{-n},\ a,\ b\\ q,\ c,\ q^{1-n}ab/c\end{matrix}\,\Big|\,q\right]_k=\left[\begin{matrix}c/a,c/b\\ c,c/ab\end{matrix}\,\Big|\,q\right]_n. \tag{8}$$

The resulting formula is stated in the following corollary.

Corollary 3 ($\Omega_0^n(1,x,q^{-3n}/x)$: Chen and Chu [5, Theorem 26])

$$\sum_{k=0}^{n}\left[\begin{matrix}x,q^{-3n}/x\\ q\end{matrix}\,\Big|\,q\right]_k\frac{(q^{-3n};q^3)_k}{(q^{-3n};q)_{2k}}q^k=\frac{1}{x^n}\left[\begin{matrix}qx,q^2x\\ q,q^2\end{matrix}\,\Big|\,q^3\right]_n.$$

Then, letting $\ell = \lfloor \frac{n}{3} \rfloor + 1$ and $\lambda \to 1, x \to q^n$ in Proposition 2, the corresponding identity $\omega_n^0(1, q^n, q^{-n}/x) = \Omega_0^n(1, q^{-n}, q^n x)$ can be simplified into another formula.

Corollary 4 **($\Omega_0^n(1, q^{-n}, q^n x)$: Andrews [1] and Chu [6, Eq. 4.4d])**

$$\sum_{k=0}^{n} \begin{bmatrix} q^{-n}, q^n x \\ q \end{bmatrix} q \Bigg]_k \frac{(x;q^3)_k}{(x;q)_{2k}} q^k = \chi(n \equiv_3 0) \begin{bmatrix} q, q^2 \\ qx, q^2 x \end{bmatrix} q^3 \Bigg]_{\lfloor \frac{n}{3} \rfloor} x^{\lfloor \frac{n}{3} \rfloor}.$$

The next two formulae can be derived from Theorem 1 by letting $\ell \to n$ and $\lambda \to q^{-1-\delta-3n}$, $xy \to q^{1+\delta}$ with $\delta = 0,\ 1$, and then by making the shift $n \to n+1$.

Corollary 5 **($\Omega_0^n(q^{-3n-1}, x, q/x)$: Chu [6, Eq. 3.9a])**

$$\sum_{k=0}^{n} \left[\begin{matrix} x, q/x \\ q^{-3n} \end{matrix} \,\middle|\, q \right]_k \frac{(q^{-3n};q^3)_k}{(q;q)_{2k}} q^k = \left[\begin{matrix} qx, q^2/x \\ q, q^2 \end{matrix} \,\middle|\, q^3 \right]_n.$$

Corollary 6 **($\Omega_0^n(q^{-3n-2}, x, q^2/x)$: Chu [6, Eq. 3.9b])**

$$\sum_{k=0}^{n} \left[\begin{matrix} x, q^2/x \\ q^{-1-3n} \end{matrix} \,\middle|\, q \right]_k \frac{(q^{-3n};q^3)_k}{(q^2;q)_{2k}} q^k = \left[\begin{matrix} q^2 x, q^4/x \\ q^2, q^4 \end{matrix} \,\middle|\, q^3 \right]_n.$$

Finally in Theorem 1, making the replacements $x \to q^{-n}$ and $y \to q^{\varepsilon+n}$ with $\varepsilon = 1, 2$, we get the following reduced transformation:

$$\Omega_0^n(\lambda, q^{-n}, q^{\varepsilon+n}) = \Omega_0^n(q^{3\ell}\lambda, q^{-n}, q^{\varepsilon+n}) \left[\begin{matrix} q^{\varepsilon}\lambda, q^{3+n}\lambda, q^{3-\varepsilon-n}\lambda \\ q\lambda, q^2\lambda, q^3\lambda \end{matrix} \,\middle|\, q^3 \right]_\ell. \tag{9}$$

By letting $\ell \to \pm\infty$, we get two further expressions:

$$\Omega_0^n(\lambda, q^{-n}, q^{\varepsilon+n}) = \Omega_0^n(0, q^{-n}, q^{\varepsilon+n}) \left[\begin{matrix} q^{\varepsilon}\lambda, q^{3+n}\lambda, q^{3-\varepsilon-n}\lambda \\ q\lambda, q^2\lambda, q^3\lambda \end{matrix} \,\middle|\, q^3 \right]_\infty; \tag{10}$$

$$\Omega_0^n(\lambda, q^{-n}, q^{\varepsilon+n}) = \Omega_0^n(\infty, q^{-n}, q^{\varepsilon+n}) \left[\begin{matrix} 1/\lambda, q/\lambda, q^2/\lambda \\ q^{3-\varepsilon}/\lambda, q^{-n}/\lambda, q^{\varepsilon+n}/\lambda \end{matrix} \,\middle|\, q^3 \right]_\infty. \tag{11}$$

When $\lambda = q^{-\varepsilon-3n}$, the above series on the left can be evaluated by Corollaries 5 and 6 respectively for $\varepsilon = 1,\ 2$ as follows:

$$\Omega_0^n(q^{-\varepsilon-3n}, q^{-n}, q^{\varepsilon+n}) = \left[\begin{matrix} q^{\varepsilon-n}, q^{2\varepsilon+n} \\ q^{\varepsilon}, q^{2\varepsilon} \end{matrix} \,\middle|\, q^3 \right]_n.$$

Substituting this into (10) and (11), and then simplifying the results, we get the following interesting evaluations:

$$\Omega_0^n(0, q^{-n}, q^{\varepsilon+n}) = (-1)^{\lfloor \frac{n+1}{3} \rfloor} \frac{(1-q)\chi(n \not\equiv_3 \varepsilon)}{1-q^{1+n(\varepsilon-1)}} q^{\frac{4n\varepsilon-3n+n^2}{6}}, \tag{12}$$

$$\Omega_0^n(\infty, q^{-n}, q^{\varepsilon+n}) = (-1)^{\lfloor \frac{n+1}{3} \rfloor} \frac{(1-q)\chi(n \not\equiv_3 \varepsilon)}{1-q^{1+n(\varepsilon-1)}} q^{\frac{2n\varepsilon-3n-n^2}{6}}, \tag{13}$$

which can be reformulated as q-binomial identities:

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+k+\varepsilon-1\\2k+\varepsilon-1\end{bmatrix}q^{\binom{n-k}{2}}=(-1)^{\lfloor\frac{n+1}{3}\rfloor}\chi(n\not\equiv_3\varepsilon)q^{\frac{2n\varepsilon-3n+2n^2}{3}},$$

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+k+\varepsilon-1\\2k+\varepsilon-1\end{bmatrix}q^{\binom{n-k}{2}+k(\varepsilon-2+k)}=(-1)^{\lfloor\frac{n+1}{3}\rfloor}\chi(n\not\equiv_3\varepsilon)q^{\frac{n\varepsilon-3n+n^2}{3}}.$$

Combining (10) and (11) respectively with (12) and (13), we obtain two further summation formulae displayed in the next two corollaries.

Corollary 7 ($\Omega_0^n(\lambda,q^{-n},q^{1+n})$**: Chen and Chu [5, Theorem 20])**

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^{1+n}\\q\lambda\end{matrix}\,\middle|\,q\right]_k\frac{(q\lambda;q^3)_k}{(q;q)_{2k}}q^k=\begin{cases}\dfrac{(q/\lambda;q^3)_m}{(q^3\lambda;q^3)_m}(q\lambda)^m, & n=3m;\\ 0, & n=3m+1;\\ \dfrac{(1/\lambda;q^3)_{m+1}}{(q^2\lambda;q^3)_{m+1}}(q\lambda)^{m+1}, & n=3m+2.\end{cases}$$

Corollary 8 ($\Omega_0^n(\lambda,q^{-n},q^{2+n})$**: Chen and Chu [5, Theorem 21])**

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^{2+n}\\q\lambda\end{matrix}\,\middle|\,q\right]_k\frac{(q^2\lambda;q^3)_k}{(q;q)_{2k+1}}q^k=\begin{cases}\frac{(q^2\lambda)^m}{1-q^{3m+1}}\frac{(q^2/\lambda;q^3)_m}{(q^3\lambda;q^3)_m}, & n=3m;\\ \frac{-(q^2\lambda)^{m+1}}{q(1-q^{3m+2})}\frac{(1/\lambda;q^3)_{m+1}}{(q\lambda;q^3)_{m+1}}, & n=3m+1;\\ 0 & +n=3m+2.\end{cases}$$

Now we are going to derive four closed formulae from Proposition 2 by examining the $\omega_0^m(\lambda,x,y)$-sum (i.e. $\Omega_m^0(\lambda,x,y)$-sum consists of the terms with nonpositive subscripts). By letting $\ell\to\lfloor\frac{m}{3}\rfloor$ and $\lambda\to q^{-m}$, $x\to1$ in Proposition 2, we get the following summation formula.

Corollary 9 ($\omega_0^m(q^m,1,q/y)$**: Chen and Chu [5, Example 30])**

$$\sum_{k=0}^{m}\left[\begin{matrix}q^{-m}\\q,y\end{matrix}\,\middle|\,q\right]_k\frac{(y;q)_{2k}}{(q^{2-m}y;q^3)_k}q^k=\chi(m\equiv_3 0)\left[\begin{matrix}q,\quad q^2\\q/y,\,q^2y\end{matrix}\,\middle|\,q^3\right]_{\lfloor\frac{m}{3}\rfloor}.$$

Instead, letting first $\ell\to m$ and $\lambda\to q^{-m}$, $xy\to q^m$ in Proposition 2, and then evaluating the series in (7) by the q-Pfaff–Saalschutz theorem (8), we derive, after some simplification, another identity.

Corollary 10 ($\omega_0^m(q^{-m},q^m/y,y)$**: Chen and Chu [5, Example 33])**

$$\sum_{k=0}^{m}\left[\begin{matrix}q^{-m}\\qy,q^{1+m}/y\end{matrix}\,\middle|\,q\right]_k\frac{(q^{1+m};q)_{2k}}{(q^3;q^3)_k}q^k=\frac{(q^{1-m}/y;q^3)_m(q^{2-m}/y;q^3)_m}{(q^{-m}/y;q)_m(q^{1+m}/y;q)_m}.$$

The next identity results from the commom reversal of the two formulae displayed in Corollaries 5 and 6 (justified by reversing the summation index $k\to n-k$ there), even though it cannot be obtained directly as a particular case of Proposition 2.

Corollary 11 ($\omega_0^m(q^{m+1}, q^{-1-m}/y, y)$**: Chen and Chu** [5, **Example 31**])

$$\sum_{k=0}^{\lfloor m/2 \rfloor} \left[\begin{matrix} q^{1+m} \\ qy, q^{-m}/y \end{matrix} \,\middle|\, q\right]_k \frac{(q^{-m};q)_{2k}}{(q^3;q^3)_k} q^k = \frac{(q^{2-m}y;q^3)_m}{(qy;q)_m}.$$

Finally, letting $x \to 1$ and $y \to q^{-m-1}$ in Proposition 2, the corresponding transformation reduces to the equality

$$\omega_0^m(\lambda, 1, q^{-m-1}) = \omega_0^m(q^{3\ell}\lambda, 1, q^{-m-1}) \left[\begin{matrix} q\lambda, q^2\lambda \\ q^{1+m}\lambda, q^{2-m}\lambda \end{matrix} \,\middle|\, q^3\right]_\ell. \tag{14}$$

Further letting $\ell \to \pm\infty$, we derive two limiting relations:

$$\omega_0^m(\lambda, 1, q^{-m-1}) = \omega_0^m(0, 1, q^{-m-1}) \left[\begin{matrix} q\lambda, q^2\lambda \\ q^{1+m}\lambda, q^{2-m}\lambda \end{matrix} \,\middle|\, q^3\right]_\infty, \tag{15}$$

$$\omega_0^m(\lambda, 1, q^{-m-1}) = \omega_0^m(\infty, 1, q^{-m-1}) \left[\begin{matrix} q^{1+m}/\lambda, q^{2-m}/\lambda \\ q/\lambda, q^2/\lambda \end{matrix} \,\middle|\, q^3\right]_\infty. \tag{16}$$

When $\lambda = 1$, the first one (15) gives rise to the closed formula

$$\omega_0^m(0, 1, q^{-m-1}) = (-1)^{\lfloor \frac{m}{3} \rfloor} \chi(m \not\equiv_3 2) q^{\frac{m-m^2}{6}}, \tag{17}$$

which can be restated as the following q-binomial identity:

$$\sum_{k=0}^{\lfloor \frac{m}{3} \rfloor} (-1)^k \begin{bmatrix} m-k \\ k \end{bmatrix} q^{\binom{m-k}{2}+k^2} = (-1)^{\lfloor \frac{m}{3} \rfloor} \chi(m \not\equiv_3 2) q^{\frac{m^2-m}{3}}.$$

When $m = 3n$ and $1+3n$, the corresponding identities can be reformulated as finite forms of Euler's pentagonal theorem:

$$\sum_k (-1)^k \begin{bmatrix} 2n-k \\ n+k \end{bmatrix} q^{\frac{k(3k+1)}{2}} = 1 \quad \text{and} \quad \sum_k (-1)^k \begin{bmatrix} 1+2n-k \\ n+k \end{bmatrix} q^{\frac{k(3k+1)}{2}} = 1,$$

where the former can be found in the works of Berkovich and Garvan [3] and Warnaar [11, Eq. 3].

Substituting (17) into (15), we recover, after simplifications, the following identity.

Corollary 12 ($\omega_0^m(\lambda, 1, q^{-m-1})$**: Chen and Chu** [5, **Example 32**])

$$\sum_{k=0}^{\lfloor m/2 \rfloor} \left[\begin{matrix} \lambda \\ q, q^{-m} \end{matrix} \,\middle|\, q\right]_k \frac{(q^{-m};q)_{2k}}{(q^{2-m}\lambda;q^3)_k} q^k = \chi(m \not\equiv_3 2) \left[\begin{matrix} q^{2-m}/\lambda \\ q^{2-m}\lambda \end{matrix} \,\middle|\, q^3\right]_{\lfloor \frac{m}{3} \rfloor} \lambda^{\lfloor \frac{m}{3} \rfloor}.$$

By combining this with (16), we derive the following counterpart of the identity (17):

$$\omega_0^m(\infty, 1, q^{-m-1}) = (-1)^{\lfloor \frac{m}{3} \rfloor} \chi(m \not\equiv_3 2) q^{\frac{m^2-m}{6}}, \tag{18}$$

which can be expressed as another q-binomial identity:

$$\sum_{k \geq 0} (-1)^k \begin{bmatrix} m-k \\ k \end{bmatrix} q^{\binom{k}{2}} = (-1)^{\lfloor \frac{m}{3} \rfloor} \chi(m \not\equiv_3 2) q^{\frac{m^2-m}{6}}.$$

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